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On the relations between different flatness based design methods for tracking controllers

Felix Antritter

***Abstract*— Flatness based tracking controllers (see e.g. [1], [2], [3], [4]) are a very important tool for nonlinear controller design. Such tracking controllers achieve an exact linearization of the tracking error dynamics. In [5] a different flatness based tracking controller has been proposed based on the notion of feedforward linearization. This contribution clarifies the relations between these two flatness based tracking controllers and the tracking controller design for flat systems which has been proposed in [6], [7]. This approach uses a differential operator representation of the linearized tracking error dynamics and yields a linear time varying feedback. Similarities and differences of the different controllers are pointed out and are illustrated for a magnetic levitation system.**

I. INTRODUCTION

Flatness based tracking controllers (see e.g. [1], [2], [3], [4]) are a very important tool for nonlinear controller design. To improve the robustness of flatness based tracking controllers against parameter variations, in [5] another flatness based tracking controller has been proposed which is based on the notion of feedforward linearization. Additionally, in some cases not all states have to be available for measurement to achieve a stabilizing feedback with this tracking controller. In [6], [7] the stabilization of trajectories for flat systems using only linear feedback has been proposed. This feedback is derived from a time varying differential operator representation (see e.g. [8], [9]) of the linearized tracking error dynamics. Based on the differential operator representation also a linear dynamic output feedback can be constructed to estimate the feedback when not all states are available for measurement. In this contribution the relations between these three flatness based tracking controller design strategies are investigated for the case of SISO systems. The paper is organized as follows: Section II shortly reviews the relevant relations for flat SISO systems, which are used in this contribution. In Section III the flatness based tracking controller design with exact linearization of the tracking error dynamics is recalled. In Section IV the two approaches of [5] and [7] are introduced which make both use of a linearization of the tracking error dynamics. Then, in Section V the relations between the different tracking controllers are discussed by analyzing the resulting feedback laws. In Section VI the construction of a dynamic output feedback using a nonlinear tracking observer as well as the construction of a linear dynamic output feedback based on a differential operator representation are recalled. Finally, in Section VII the results are illustrated for the case of a magnetic levitation system. The robustness properties of flatness based tracking controllers are an important question which has been raised, e.g., in [5]. Therefore, by means of the investigated example system, the robustness properties of the different controllers are discussed and it is shown that the construction of an output feedback can be more relevant with respect to robustness than the used feedback law.

Automatisierungs- und Regelungstechnik EIT 8.1, Universität der Bundeswehr München, Werner-Heisenberg-Weg 39, D–85579 Neubiberg, Germany felix.antritter@unibw.de

II. FLAT SYSTEMS

For a unified presentation of the different controllers the following relations are used: For a nonlinear SISO system

$$\dot{x} = f(x,u) \tag{1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ the flatness property implies the existence of a flat output $y_f \in \mathbb{R}$, such that

$$\begin{aligned} y_f &= h_f(x) && (2)\\ x &= \psi_x(y_f, \dot{y}_f, \ldots, y_f^{(n-1)}) && (3)\\ u &= \psi_u(y_f, \dot{y}_f, \ldots, y_f^{(n)}) && (4) \end{aligned}$$

holds, with h_f, ψ_x, ψ_u smooth at least on an open subset of $\mathbb{R}^n$, $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively. Introducing the new coordinates

$$\zeta = (\zeta_1, \ldots, \zeta_n) = (y_f, \dot{y}_f, \ldots, y_f^{(n-1)}) \, , \tag{5}$$

the flat system (1) can be transformed via the well defined diffeomorphism

$$\zeta = \Phi(x) \tag{6}$$

into controller normal form

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, \qquad i = 1, 2, \ldots n-1 \\ \dot{\zeta}_n &= \alpha(\zeta, u) \, . \end{aligned} \tag{7}$$

Setting $v = y_f^{(n)}$ yields

$$u = \psi_u(\zeta, v) \tag{8}$$

in view of (4) and (5). In [5] it has been shown that

$$\alpha(\zeta, \psi_u(\zeta, v)) = v \tag{9}$$

holds and thus by application of the feedback law (8), system (1) is diffeomorphic to the Brunovský normal form

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, \qquad i = 1, 2, \ldots n-1 \\ \dot{\zeta}_n &= v \end{aligned} \tag{10}$$

with the new input v. From these relations it becomes clear that (4) is an exact linearizing feedback law.

For flat systems a feedforward controller can be derived very naturally from the differential parameterization. When assigning for the flat output a sufficiently smooth reference trajectory

$$y_f^* : I \to \mathbb{R}, \quad I = [t_0, t_e] \tag{11}$$

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a feedforward controller is obtained by inserting y_f^* into the differential parameterization (4)

$$u^* = \psi_u(y_f^*, \dot{y}_f^*, \ldots, y_f^{*(n)}) \tag{12}$$

In case of consistent initial conditions, i.e.

$$x(t_0) = \psi_x(y_f^*(t_0), \dot{y}_f^*(t_0), \ldots, y_f^{*(n-1)}(t_0)) \tag{13}$$

and in absence of disturbances the feedforward controller (12) leads to exact tracking of the reference trajectory (11), due to relations (7), (9) which yield

$$\dot{\zeta}_n = \alpha(\zeta^*, \psi_u(\zeta^*, \dot{\zeta}_n^*) = \dot{\zeta}_n^* \tag{14}$$

The corresponding trajectory in the original coordinates is obtained to

$$x^* = \psi_x(y_f^*, \dot{y}_f^*, \ldots, y_f^{*(n-1)}) \tag{15}$$

III. Tracking Controller design with Exact Feedback Linearization

To stabilize the tracking of the reference trajectory y_f^*, the tracking error e is introduced as

$$\begin{aligned} e &= (e_1, e_2, \ldots, e_n) = (y_f - y_f^*, \dot{y}_f - \dot{y}_f^*, \ldots, y_f^{(n-1)} - y_f^{*(n-1)}) \\ &= (\zeta_1 - \zeta_1^*, \zeta_2 - \zeta_2^*, \ldots, \zeta_n - \zeta_n^*) \end{aligned} \tag{16}$$

In view of (10) the tracking error dynamics are given by

$$\begin{aligned} \dot{e}_i &= e_{i+1}, \quad i = 1, 2, \ldots, n-1 \\ \dot{e}_n &= \alpha(\zeta, u) - \dot{\zeta}_n^* \end{aligned} \tag{17}$$

When setting the new input v in (8) to $v = v_{el}$, i.e.

$$u_{el} = \psi_u(\zeta, v_{el}) \tag{18}$$

where

$$v_{el} = \dot{\zeta}_n^* + \Delta v_{el} = \dot{\zeta}_n^* - \sum_{i=1}^{n} \tilde{a}_{i-1} e_i = \dot{\zeta}_n^* - \sum_{i=0}^{n-1} \tilde{a}_i e_1^{(i)} \tag{19}$$

it follows with (17) and (9) that the tracking error obeys the linear differential equation

$$0 = e_1^{(n)} + \sum_{i=0}^{n-1} \tilde{a}_i e_1^{(i)} \tag{20}$$

The $\tilde{a}_i$ are usually chosen as the lower order coefficients of some monic Hurwitz polynomial to assure stability.

IV. Tracking controller design based on a linearization about the reference trajectory

The other two flatness based control methodologies discussed in this paper do not achieve an exact linearization of the tracking error dynamics but rather use a linearization about the reference trajectory. A linearization of the nonlinear tracking error dynamics (17) at $e = 0$ (i.e. $\zeta = \zeta^*$ in view of (16)) and $u = u^*$ yields

$$\Delta\dot{e} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{c_0}(t) & a_{c_1}(t) & a_{c_2}(t) & \cdots & a_{c_{n-1}}(t) \end{bmatrix} \Delta e + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_c(t) \end{bmatrix} \Delta u \tag{21}$$

where

$$a_{c_{i-1}}(t) = \left.\frac{\partial\ \alpha(\zeta, u)}{\partial\ \zeta_i}\right|_{\substack{\zeta=\zeta^* \\ u=u^*}}, \; i = 0, 1, \ldots, n-1 \tag{22}$$

$$b_c(t) = \left.\frac{\partial\ \alpha(\zeta, u)}{\partial\ u}\right|_{\substack{\zeta=\zeta^* \\ u=u^*}} \tag{23}$$

Note that due to the flatness property of (1)

$$b_c(t) \neq 0 \quad \forall t \in I \tag{24}$$

is assured (see e.g. [4]). From the structure of (21) it follows furthermore that

$$\Delta e = (\Delta e_1, \Delta e_2, \ldots, \Delta e_n) = (\Delta e_1, \Delta\dot{e}_1, \ldots, \Delta e_1^{(n-1)}) \tag{25}$$

A. Tracking Controller Design based on a Differential Operator Representation

In this section the flatness based tracking controller design as elaborated in [7] is introduced, which uses a time varying differential operator representation (see e.g. [10], [11], [12] for the time invariant case and [8], [9] for the time varying case). A differential operator representation of the linearized tracking error dynamics can be derived from (21) by solving the last row for Δu and taking into account (25)

$$\underbrace{\frac{1}{b_c(t)}\left(D^n - \sum_{i=0}^{n-1} a_{c_i}(t) D^i\right)}_{n(D,t)} \Delta e_1 = \Delta u \tag{26}$$

where the differential operator $D = \frac{d}{dt}$ was used. Relation (26) can be restated using the time varying polynomial differential operator $n(D,t)$

$$n(D,t)\Delta e_1 = \Delta u \tag{27}$$

For the controller design, the differential operator $n(D,t)$ is split up using the highest column degree matrix $\Gamma_c[n(D,t)]$ (see [12]) in the following way

$$n(D,t) = \Gamma_c[n(D,t)] D^n + n_R(D,t) \tag{28}$$

where in view of (26)

$$\Gamma_c[n(D,t)] = \frac{1}{b_c(t)} \tag{29}$$

$$n_R(D,t) = -\frac{1}{b_c(t)} \sum_{i=0}^{n-1} a_{c_i}(t) D^i \tag{30}$$

Taking into account (26), the additional control action

$$\Delta u_{dor} = -\Gamma_c[n(D,t)]\, \bar{n}_R(D)\Delta e_1 + n_R(D,t)\Delta e_1 \tag{31}$$

achieves the linear time invariant dynamics

$$\bar{n}(D)\Delta e_1 = (D^n + \sum_{i=0}^{n-1} \tilde{a}_i D^i)\Delta e_1 = 0 \tag{32}$$

for the linearized tracking error, when $\bar{n}_R(D,t)$ is chosen according to

$$\bar{n}_R(D) = \sum_{i=0}^{n-1} \tilde{a}_i D^i \tag{33}$$

The $\tilde{a}_i$ are again chosen such that $\bar{n}(D)$ is a monic Hurwitz polynomial. The resulting tracking controller can be summarized as

$$u_{dor} = u^* + \Delta u_{dor} \tag{34}$$

B. Tracking Controller Design based on Exact Feedforward Linearization

For the tracking controller design based on exact feedforward linearization (see e.g. [5]) the feedback

$$u_{fl} = \psi_u(\zeta^*, v_{fl}) \tag{35}$$

is used, where it is proposed to assign for v_{fl}

$$v_{fl} = \dot{\zeta}_n^* + \Delta v_{fl} = \dot{\zeta}_n^* + \sum_{i=1}^{n} \lambda_i e_i \tag{36}$$

For the design of the parameters λ_i the tracking error dynamics of the controlled system

$$\begin{aligned} \dot{e}_i &= e_{i+1}, \quad i = 1,2,\ldots,n-1 \\ \dot{e}_n &= \alpha(\zeta^* + e, \psi_u(\zeta^*, \dot{\zeta}_n^* + \sum_{i=1}^{n} \lambda_{i-1} e_i)) - \dot{\zeta}_n^* \end{aligned} \tag{37}$$

are linearized at $e = 0$. In contrast to [5], this is done by showing that when the feedback law (35)–(36) is applied, Δu in (21) is given by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial u_{fl}}{\partial v} \frac{\partial v_{fl}}{\partial e_i} \Delta e_i = \frac{1}{b_c(t)} \sum_{i=1}^{n} \lambda_i \Delta e_i \tag{38}$$

Thus, when assigning

$$\lambda_i = -a_{c_{i-1}}(t) - \tilde{a}_{i-1}, \quad i = 1,2,\ldots,n \tag{39}$$

in (36), the linearized tracking error Δe_1 satisfies the linear differential equation (32), in view of (21), (25) and (38). This choice of the controller parameters to achieve time invariant linearized tracking error dynamics has been mentioned explicitely in [5], although also time invariant parameters λ_i have been considered.

To derive (38) it is realized that

$$e = 0 \quad \Leftrightarrow \quad \zeta = \zeta^* \tag{40}$$

holds and clearly

$$\frac{\partial \psi_u(\zeta, v)}{\partial v}\Big|_{\zeta=\zeta^*} = \frac{\partial \psi_u(\zeta^*, v)}{\partial v} \tag{41}$$

Together with (9), it follows that

$$\frac{\partial \alpha(\zeta, \psi_u(\zeta, v))}{\partial v}\Big|_{\zeta=\zeta^*} = \frac{\partial \alpha(\zeta^*, \psi_u(\zeta^*, v))}{\partial v} = 1 \tag{42}$$

Having

$$v_{el}\big|_{\zeta=\zeta^*} = v_{fl}\big|_{\zeta=\zeta^*} = \dot{\zeta}_n^* \tag{43}$$

it follows that

$$u_{el}|_{\zeta=\zeta^*} = u_{fl}|_{\zeta=\zeta^*} = u^* \tag{44}$$

(see (18)–(19), (35)–(36)). This yields with (42)

$$\frac{\partial \alpha(\zeta, u)}{\partial u}\Big|_{\substack{\zeta=\zeta^* \\ u=u^*}} \frac{\partial \psi_u(\zeta, v)}{\partial v}\Big|_{\zeta=\zeta^*} = 1 \tag{45}$$

and thus, in view of (23)

$$\frac{\partial \psi_u(\zeta, v)}{\partial v}\Big|_{\zeta=\zeta^*} = \left(\frac{\partial \alpha(\zeta, u)}{\partial u}\Big|_{\substack{\zeta=\zeta^* \\ u=u^*}} \right)^{-1} = \frac{1}{b_c(t)} \tag{46}$$

So, with (41)

$$\frac{\partial \psi_u(\zeta^*, v)}{\partial v} = \frac{1}{b_c(t)} \tag{47}$$

holds and finally with (36)

$$\frac{\partial v_{fl}}{\partial e_i} = \lambda_i \tag{48}$$

is obtained. As a consequence, with

$$\begin{aligned} \Delta u &= \frac{\partial u_{fl}}{\partial e}\Big|_{e=0} \Delta e = \sum_{i=1}^{n} \frac{\partial u_{fl}}{\partial v} \frac{\partial v_{fl}}{\partial e_i} \Delta e_i \\ &= \sum_{i=1}^{n} \frac{\partial \psi_u(\zeta^*, v)}{\partial v}\Big|_{v=v_{fl}} \frac{\partial v_{fl}}{\partial e_i} \Delta e_i \end{aligned} \tag{49}$$

and (47)–(48), it can be concluded that for the application of the feedback law (35)–(36) relation (38) holds.

V. Comparison of the resulting feedback laws

The considerations in Sections III and IV showed that there are strong relations between the three different tracking controller design methods. This will be made more precise by analyzing the resulting feedback laws. It will be shown that all three feedback laws are identical when their taylor series expansions with respect to the tracking error about the reference trajectory are truncated after the first order. More explicitely it will be shown that

$$u_{dor} = u^* + \nabla_e u_{el}\big|_{e=0} \cdot \Delta e = u^* + \nabla_e u_{fl}\big|_{e=0} \cdot \Delta e \tag{50}$$

holds, with the differential operator

$$\nabla_e = \left[\frac{d}{de_1}, \frac{d}{de_2}, \ldots, \frac{d}{de_n} \right] \tag{51}$$

Relation (50) means that the feedback law (34) is a first order approximation of the nonlinear feedback laws (18) and (35) in view of (44).

A. Feedback law achieving exact linearization of the tracking error dynamics

For the investigation of the feedback law (18)–(19) it is realized at first that for the linearizing feedback (8) it holds in view of (9) that

$$\frac{d\alpha(\zeta, \psi_u(\zeta, v))}{d\zeta_i}\Big|_{\zeta=\zeta^*} = \frac{\partial \alpha}{\partial \zeta_i}\Big|_{\substack{\zeta=\zeta^* \\ u=\psi_u}} + \frac{\partial \alpha}{\partial u}\Big|_{\substack{\zeta=\zeta^* \\ u=\psi_u}} \frac{\partial \psi_u}{\partial \zeta_i}\Big|_{\zeta=\zeta^*} = 0 \tag{52}$$

Togehter with (23)–(24) it can be concluded that the partial derivative of the feedback law (8) with respect to ζ_i satifies

$$\frac{\partial \psi_u(\zeta, v)}{\partial \zeta_i}\Big|_{\zeta=\zeta^*} = -\left(\frac{\partial \alpha}{\partial u}\Big|_{\substack{\zeta=\zeta^* \\ u=\psi_u}} \right)^{-1} \frac{\partial \alpha}{\partial \zeta_i}\Big|_{\substack{\zeta=\zeta^* \\ u=\psi_u}} \tag{53}$$

At $e = 0$ relations (40)–(45) are valid and thus

$$\begin{aligned} \frac{d\psi_u(\zeta, v)}{de_i}\Big|_{e=0} &= \frac{d\psi_u(\zeta^* + e, v)}{de_i}\Big|_{e=0} \\ &= \frac{\partial \psi_u(\zeta^* + e, v)}{\partial e_i}\Big|_{e=0} + \frac{\partial \psi_u}{\partial v}\Big|_{e=0} \frac{\partial v}{\partial e_i}\Big|_{e=0} \\ &= \frac{\partial \psi_u(\zeta, v)}{\partial \zeta_i}\Big|_{\zeta=\zeta^*} + \frac{\partial \psi_u}{\partial v}\Big|_{\zeta=\zeta^*} \frac{\partial v}{\partial e_i}\Big|_{e=0} \\ &= -\left(\frac{\partial \alpha}{\partial u}\Big|_{\substack{\zeta=\zeta^* \\ u=u^*}} \right)^{-1} \frac{\partial \alpha}{\partial \zeta_i}\Big|_{\substack{\zeta=\zeta^* \\ u=u^*}} + \frac{\partial \psi_u}{\partial v}\Big|_{\zeta=\zeta^*} \frac{\partial v}{\partial e_i} \end{aligned} \tag{54}$$

With

$$\frac{\partial v_{el}}{\partial e_i} = -\tilde{a}_{i-1} \tag{55}$$

(see (19)) and (22)–(23),(46), (53), (55) it follows that

$$\frac{du_{el}}{de_i} = \frac{1}{b_c(t)}(-a_{c_{i-1}}(t) - \tilde{a}_{i-1}) \tag{56}$$

Taking into account (51) $\nabla_e u_{el}$ evaluates to

$$\nabla_e u_{el} = \left[\frac{du_{el}}{de_1}, \ldots, \frac{du_{el}}{de_n}\right] \tag{57}$$

This yields with (56)

$$\nabla_e u_{el} = \frac{1}{b_c(t)}\left[-\tilde{a}_0 - a_{c_0}(t), \ldots, -\tilde{a}_{n-1} - a_{c_{n-1}}(t)\right] \tag{58}$$

Thus, a taylor series expansion at $e = 0$ for the feedback law (18) reads with (44)

$$u_{el} = \psi_u(\zeta, v_{el}) = u^* + \nabla_e u_{el} \cdot \Delta e + \ldots \tag{59}$$

B. Feedback law for exact feedforward linearization

For the determination of the first order taylor series expansion of the feedback law (35)–(36) it is realized that

$$\frac{d\psi_u(\zeta^*, v_{fl})}{de_i} = \left.\frac{\partial \psi_u(\zeta^*, v)}{\partial v}\right|_{\substack{\zeta=\zeta^* \\ v=\zeta_n^*}} \frac{\partial v_{fl}}{\partial e_i} = \frac{1}{b_c(t)}\lambda_i \tag{60}$$

(see also (43), (47)–(49)). Thus, choosing the controller parameters according to (39) yields together with (51)

$$\begin{aligned}\nabla_e u_{fl} &= \left[\frac{du_{fl}}{de_1}, \ldots, \frac{du_{fl}}{de_n}\right] \\ &= \frac{1}{b_c(t)}\left[-a_{c_0}(t) - \tilde{a}_0, \ldots, -a_{c_{n-1}}(t) - \tilde{a}_{n-1}\right]\end{aligned} \tag{61}$$

As a consequence, a taylor series expansion of u_{fl} at $e = 0$ has in view of (44) the form

$$u_{fl} = \psi_u(\zeta^*, v_{fl}) = u^* + \nabla_e u_{fl} \cdot \Delta e + \ldots \tag{62}$$

C. Feedback law derived from the differential operator representation

The feedback law (34) consiststs only of the feedforward controller signal and a linear error feedback. It follows with (29)–(31) and (25) that

$$u_{dor} = u^* + \frac{1}{b_c(t)}\sum_{i=1}^{n}(-a_{c_{i-1}}(t) + \tilde{a}_{i-1})\Delta e_i \tag{63}$$

In view of (58), (61) this can be expressed as

$$u_{dor} = u^* + \nabla_e u_{el} \cdot \Delta e = u^* + \nabla_e u_{fl} \cdot \Delta e \tag{64}$$

i.e., u_{dor} is, for the assumed choices of the controller parameters, a first order approximation of the feedback laws u_{el} and u_{fl}.

VI. Output Feedback

For the implementation of the feedback laws (18), (34), (35), in general, all states have to be available for measurement in view of the state transformation (6). If only the output

$$y = h(x) \tag{65}$$

with $y \in \mathbb{R}$ is available for measurement, the feedback laws have to be estimated. This can be done by estimating the states of system (1) using a nonlinear tracking observer (see Section VI-A). On the other hand, the linear feedback law (31) can be estimated directly, instead of estimating the states of (1), using a reduced order linear dynamic output feedback (Section VI-B).

A. Nonlinear Tracking Observer

A nonlinear tracking observer with time varying observer gain $L(t)$ is given by (see [13])

$$\dot{\hat{x}} = f(\hat{x}, u) + L(t)(y - h(\hat{x})) \tag{66}$$

The observer (66) basically consists of a model of the plant and a feedback of the difference of the measured output and the estimated output. The time varying observer gain $L(t)$ is designed such that the linearization of the estimation error dynamics about the reference trajectory y_f^* which results to

$$\Delta\dot{\hat{x}} - \Delta\dot{x} = (A(t) - L(t)C(t))(\Delta\hat{x} - \Delta x) \tag{67}$$

with

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^* \\ u=u^*}}, \qquad C(t) = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x^* \\ u=u^*}} \tag{68}$$

is asymptotically stable. This is done by assigning stable time invariant eigenvalues to the observer normal form of (67) (see e.g. [13]).

B. Linear Dynamic Output feedback

For the estimation of the control signal based on the differential operator representation of the linearized error dynamics, (27) is extended with an output equation. To this end it is realized that the output (65) can also be expressed as a function of the ζ coordinates according to

$$y = \tilde{h}(\zeta) = h \circ \Phi^{-1}(\zeta) \tag{69}$$

The linearization of the nonlinear output equation (69) yields

$$\Delta y = C_c(t)\Delta e = [c_{c_0}, c_{c_1}, \ldots, c_{c_{n-1}}]\Delta e \tag{70}$$

(note that $\Delta\zeta = \Delta e$) where

$$C_c(t) = \left.\frac{\partial \tilde{h}(\zeta)}{\partial \zeta}\right|_{\substack{\zeta=\zeta^* \\ u=u^*}} \tag{71}$$

With (25) this can be expressed as

$$\Delta y = \left(\sum_{i=0}^{n-1} c_{c_i}(t) D^i\right)\Delta e_1 = z(D,t)\Delta e_1 \tag{72}$$

To summarize, the linearized error dynamics with the output (70) are given by

$$n(D,t)\Delta e_1 = \Delta u \tag{73}$$

$$\Delta y = z(D,t)\Delta e_1 \tag{74}$$

A dynamic output feedback for the time varying differential operator representation (73)–(74) is given by (for details see [6], [7])

$$\Delta(D)\Delta\hat{u}_{dor} = z_u(D,t)\Delta u + z_y(D,t)\Delta y \quad (75)$$

$$\hat{u}_{dor} = u^* + \Delta\hat{u}_{dor} \quad (76)$$

where Δu is the actual input to the plant, which can deviate from Δu_{dor} e.g. in the case of input saturations (see e.g. [14]). The differential operators $z_u(D,t)$ and $z_y(D,t)$ are determined from the Diophantine equation

$$z_u(D,t)n(D,t) + z_y(D,t)z(D,t) = \Delta(D)(n(D,t) - \tilde{n}(D,t)) \quad (77)$$

Relation (77) is the central design equation for a linear dynamic output feedback base on a differential operator representation, where $\Delta(D)$ specifies the desired time invariant estimation error dynamics between the estimated control signal $\Delta\hat{u}_{dor}$ and the desired input signal Δu_{dor} according to

$$\Delta(D)(\Delta\hat{u}_{dor} - \Delta u_{dor}) = 0 \quad (78)$$

The coefficients of $\Delta(D)$ are derived from a monic Hurwitz polynomial. The differential operator $\tilde{n}(D,t)$ in (77) specifies the linearized tracking error dynamics (32) as it is related to $\bar{n}(D,t)$ by

$$\tilde{n}(D,t) = \Gamma_c[n(D,t)]\bar{n}(D) \quad (79)$$

Remark 1 *Based on the solvability conditions in [9] it can be deduced that for single-input systems with a measured output $y \in \mathbb{R}^m$, the degree d of $\Delta(D)$, i.e. the order of the output feedback (75)–(76), satifies $d \leq n - m$ (see [15], [14] for an applicational verification, where also the extension of this approach to non-flat systems with feedforward controller design according to [16] is illustrated for systems with strong accessibility). The low order of the dynamic output feedback results from the fact that directly the additional control input Δu_{dor} is estimated instead of all the states of (1) as is done by the tracking observer (66).*

VII. Application to a magnetic levitation system

In this section the above derived results are discussed for a model of a magnetic levitation system, which, on the one hand, is of interest for industrial applications (see e.g. [17]) on the other hand, it has a simple enough structure to illustrate nicely the differences of the investigated approaches. Integral error feedback is not considered here as it would conceal, to the author's opinion, the robustness properties. The system equations for the magnetic levitation system are given by [5], [17]

$$\dot{x}_1 = x_2 \quad (80)$$

$$\dot{x}_2 = \frac{k}{m}\frac{u^2}{(s_0 - x_1)^2} - g \quad (81)$$

where x_1 is the load position and x_2 its velocity. The input u is the current for the electromagnet. The constant $k = 58.042 \cdot 10^{-6}\,\frac{\mathrm{kgm}^3}{\mathrm{A}^2\mathrm{s}^2}$ depends on several setup parameters, $m = 0.0844\,\mathrm{kg}$ is the mass of the load, $s_0 = 0.0011\,\mathrm{m}$ is the minimum distance of the load to the magnet and $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the gravity acceleration constant. System (80) is already in controller normal form (see (7)). Therefore, a flat output is given by $y_f = x_1$. Consequently, the differential parameterization of the states and of the input results to

$$(x_1, x_2) = (y_f, \dot{y}_f), \qquad u = (s_0 - y_f)\sqrt{\frac{m}{k}(\ddot{y}_f + g)} \quad (82)$$

A reference trajectory has been planned which lifts up the load from an initial position x_0 to a final position x_e with

$$x_0 = (-4\,\mathrm{mm}, 0\,\frac{\mathrm{mm}}{\mathrm{s}}), \qquad x_e = (-2\,\mathrm{mm}, 0\,\frac{\mathrm{mm}}{\mathrm{s}}) \quad (83)$$

From (80) it can be deduced that the linearization of the tracking error dynamics about the reference trajectory has the structure

$$\Delta\dot{e} = \begin{bmatrix} 0 & 1 \\ a_{c_0}(t) & 0 \end{bmatrix}\Delta e + \begin{bmatrix} 0 \\ b_c(t) \end{bmatrix}\Delta u. \quad (84)$$

The feedback laws for exact linearization, feedforward linearization and differential operator representation for the derived choice of parameters result to (see (18)–(19), (35)–(36), (39), (29)–(34))

$$u_{el} = (s_0 - y_f)\sqrt{\frac{m}{k_c}((\ddot{y}_f^* - \tilde{a}_1 e_2 - \tilde{a}_0 e_1) + g)} \quad (85)$$

$$u_{fl} = (s_0 - y_f^*)\sqrt{\frac{m}{k_c}((\ddot{y}_f^* - \tilde{a}_1 e_2 - (a_{c_0}(t) + \tilde{a}_0)e_1) + g)} \quad (86)$$

$$u_{dor} = u^* + \frac{1}{b_c(t)}\left(-\tilde{a}_1 e_2 - (a_{c_0}(t) + \tilde{a}_0)e_1\right) \quad (87)$$

(where the identity $e = \Delta e$ was used in (87)). The parameter k has been replaced in the feedback laws (85)–(87) by k_c to indicate that, in case of parameter uncertainty, k_c and k do not necessarily have to be equal. The $\tilde{a}_i$ are chosen for all feedback laws according to the specification

$$(s+p)^2 \stackrel{!}{=} s^2 + \tilde{a}_1 s + \tilde{a}_0 \quad (88)$$

with $p = 100$. In Figure 1 the dependence of the feedback laws (85)–(87) on the error e_1 is shown for the operation point x_e, when $e_2 = 0$. The feedback laws (85), (87) for exact linearization and differential operator representation do almost coincide whereas the feedback law (86) for the feedforward linearization shows the strongest curvature. However, it can be seen that, as predicted by relation (50), the slopes at $e_1 = 0$ are equal. If $e_1 = 0$ and only e_2 is varied the controller signals are almost coincident for all feedback laws. In Figure 1 also the exact linearizing feedback law is shown for different values of the parameter k_c. The effects of the different feedback laws can be observed in Figure 2. It shows the resulting trajectories, for the case when the plant parameter k deviates from the value of k_c, which has been used for the controller design. If $k > k_c$ a steady state deviation with $e_1 > 0$ results at the final operation point x_e. When looking at Figure 1 it can be observed that in this case $u_{fl} < u_{el}, u_{dor}$. This explains the slightly lower deviation for the feedforward linearization in this case. With similar arguments the slightly bigger steady state deviation for the feedforward linearization can be explained for the case when $k < k_c$. Although the feedback laws seem to exhibit quite different properties, the resulting trajectories in Figure 2 do not deviate strongly.

When only x_1 is available for measurement, output feedback has to be used. In Figure 3 the resulting trajectories for this case are shown. For the estimation of the feedback laws (85), (86) the state x_2 (i.e. $\dot{y}_f$) has been replaced by the estimate $\hat{x}_2$ of a nonlinear tracking observer of the kind (66), which is of order two. It should be mentioned at this point that even with the considerations in [5] it is necessary to estimate x_2 in order to achieve an asymptotically stable tracking error dynamics using the feedback law (86). For the feedback law u_{dor} an estimate using a first order linear output feedback according to (75)–(76) has been used. The root of $\Delta(D)$ in (78) and the poles in the observer normal form of (67) have all been set to -200. It can be observed that when output feedback is used, the resulting trajectories deviate by far more from the reference trajectory than in the case of state feedback. The lower order of the dynamic output feedback which was derived on the basis of the differential operator representation seems to yield, at least for the considered example, better robustness with respect to parameter variations, although the feedback law u_{dor} is purely linear. A more detailled investigation of the robustness properties could be done using the methods proposed in [18] which has been extended to investigate controllers with output feedback in [19].

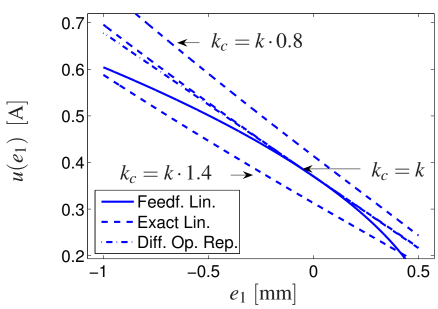

Fig. 1. Dependence of the feedback laws on e_1

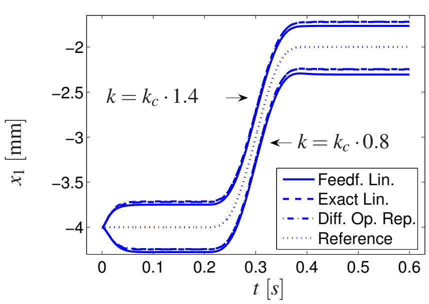

Fig. 2. Resulting trajectory for deviations of k

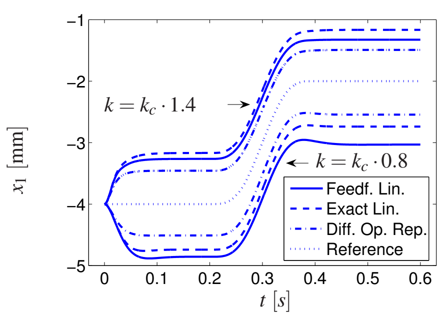

Fig. 3. Trajectories when output feedback is used

VIII. Conclusions

This contribution pointed out the differences and relations between three different flatness based tracking controllers. These results may clearly be extended to MIMO systems based on the results in [20]. It has furthermore been shown that in the case when not all states can be measured the implementation of an output feedback may have more influence on the robustness properties of the resulting controller, than the used feedback law.

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