Discrete-time Mixed $\mathscr{H}_2/\mathscr{H}_\infty$ Nonlinear Filtering

M. D. S. Aliyu and E. K. Boukas

Abstract— **In this paper, we consider the discrete-time mixed $\mathscr{H}_2/\mathscr{H}_\infty$ filtering problem for affine nonlinear systems. Sufficient conditions for the solvability of this problem with a finite-dimensional filter are given in terms of a pair of coupled discrete-time Hamilton-Jacobi-Isaac's equations (DHJIE) with some side conditions. Both the finite-horizon and infinite-horizon problems are discussed for the case when the initial condition of the plant is unknown. Moreover, sufficient conditions for approximate solvability of the problem are also derived. These solutions are especially useful for computational purposes, considering the difficulty of solving the coupled DHJIEs.**

Keywords: ℓ_2, ℓ_∞-norms, two-person nonzero-sum differential game, discrete-time coupled Hamilton-Jacobi-Isaac's equations, Taylor-series approximation.

I. INTRODUCTION

Many authors have considered $\mathscr{H}_\infty$ filtering techniques [7], [15] for linear systems. While the Kalman-filter is a minimum-variance estimator, and is the best unbiased linear optimal filter [1], [2], [6] for all Gaussian noise processes, the $\mathscr{H}_\infty$ filter is derived from a completely deterministic setting, and is the optimal worst-case filter for all bounded energy or power noise signals. Thus, it is natural that other researchers have sort to extend the capabilities of the linear $\mathscr{H}_\infty$ filter to nonlinear systems [18], [20]. Moreover, previous statistical nonlinear filtering techniques developed using minimum-variance [4] as well as maximum-likelihood [14] criteria are infinite-dimensional and too complicated to solve the filter differential equations. On the other hand, the nonlinear $\mathscr{H}_\infty$ filter is easy to derive, and relies on finding a smooth solution to a Hamilton-Jacobi-Isaac's (HJI) partial-differential equation (PDE) or HJIE in short, which can be found using polynomial approximations or other methods.

Subsequently, other researchers have considered a mixed $\mathscr{H}_2/\mathscr{H}_\infty$ criterion for designing a filter that enjoys the advantages of both the Kalman-filter and the $\mathscr{H}_\infty$ filter [5], [9], [16], [19], [23], and has been demonstrated by countless authors to be superior to the individual approaches. In particular, the paper [19] considers a differential game approach to the problem, which is a very attractive and transparent approach. However, to the best of our knowledge, the nonlinear and in particular, discrete-time problem has not been considered by any authors. We propose to discuss this problem in this paper. We shall use the differential-games approach.

Department of Mechanical Engineering, Ecole Polytechnique de Montreal, P. O. Box 6079, Centre-Ville, H3C 3A7, Montreal, Canada dikko@mathstat.concordia.ca, el-kebir.boukas@polymtl.ca

The paper is organized as follows. In section 2, we give problem definition and preliminaries. In section 3, we give sufficient conditions for the solvability of the finite-horizon mixed problem, while in section 4, we discuss the infinite-horizon problem. In section 5, we consider an approximate approach for solving the problem. Finally, in section 6 we give conclusions.

The notation is fairly standard except where otherwise stated. Moreover, $\|(.)\|$, will denote the Euclidean vector norm on $\Re^n$ and $\mathbf{Z}$ will denote the set of integers. Other notations will be defined appropriately.

II. PROBLEM DEFINITION AND PRELIMINARIES

We consider an affine causal discrete-time state-space system with zero input defined on a smooth n-dimensional manifold $\mathscr{X} \subseteq \Re^n$ in local coordinates $x = (x_1, \ldots, x_n)$:

$$\Sigma_0^{da} : \left\{ \begin{array}{rcl} x_{k+1} & = & f(x_k) + g_1(x_k)w_k;\ x(k_0) = x_0 \\ z_k & = & h_1(x_k) \\ y_k & = & h_2(x_k) + k_{21}(x_k)w_k \end{array} \right. \tag{1}$$

where $x \in \mathscr{X}$ is the state vector, $w \in \mathscr{W}$ is the disturbance or noise signal, which belongs to the set $\mathscr{W} \subset \Re^r$ of admissible disturbances or noise signals, the output $y \in \Re^m$ is the measured output of the system, while $z \in \Re^s$ is the output to be estimated. The functions $f : \mathscr{X} \to \mathscr{X}$, $g_1 : \mathscr{X} \to \mathscr{M}^{n\times r}(\mathscr{X})$, $h_1 : \mathscr{X} \to \Re^s$, $h_2 : \mathscr{X} \to \Re^m$, and $k_{21} : \mathscr{X} \to \mathscr{M}^{m\times r}(\mathscr{X})$ are real C^∞ functions of x, where $\mathscr{M}^{i\times j}$ is the ring of $i \times j$ matrices over $\mathscr{X}$. Furthermore, we assume without any loss of generality that the system (1) has a unique equilibrium-point at $x = 0$ such that $f(0) = 0$, $h_1(0) = h_2(0) = 0$.

In the standard mixed $\mathscr{H}_2/\mathscr{H}_\infty$ filtering problem the noise signal $w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$, is comprised of two parts: (i) a bounded power signal or ℓ_2 signal $w_1 \in \mathscr{P}'$, the space of bounded power signals, and (ii) a bounded spectral signal (e.g. a white Gaussian-noise signal) $w_0 \in \mathscr{S}'$ the space of bounded spectral signals.

The objective is to synthesize a filter, $\mathscr{F}_k$, for estimating the state x_k or a function of it $z_k = h_1(x_k)$ from observations of y_k up to time k, over a finite horizon $[0, K]$, i.e., from

$$\mathsf{Y}_k \stackrel{\Delta}{=} \{y_i : i \le k\}, \quad k \in [k_0, K]$$

such that the gains (or induced norm) from the input w_0 to the error or penalty variable $\breve{z}$ (to be defined later) defined as the ℓ_2-norm of the composite system $\mathscr{F} \circ \Sigma^{da}$, i.e., from

$$\|\mathscr{F}_k \circ \Sigma^{da}\|_{\ell_2} \stackrel{\Delta}{=} \sup_{0 \neq w_0 \in \mathscr{S}'} \frac{\|\breve{z}_k\|_{\mathscr{P}'}}{\|w_{0,k}\|_{\mathscr{S}'}}, \tag{2}$$

is minimized, while the induced norm from w_1 to $\breve{z}$ defined as the ℓ_∞-norm of $\mathscr{F} \circ \Sigma^{da}$, i.e.,

$$\|\mathscr{F}_k \circ \Sigma^{da}\|_{\ell_\infty} \stackrel{\Delta}{=} \sup_{0 \neq w_1 \in \mathscr{P}'} \frac{\|\breve{z}_k\|_2}{\|w_{1,k}\|_2} \tag{3}$$

is renderred less or equal to some given positive number γ, where $\mathscr{P}'$ and $\mathscr{S}'$ are the spaces of bounded power and bounded spectral signals respectively.

For nonlinear systems, the above $\mathscr{H}_\infty -$norm constraint is interpreted as the ℓ_2-gain constraint and is represented as

$$\sum_{k=k_0}^{K} \|\breve{z}_k\|^2 \leq \gamma^2 \sum_{k_0}^{K} \|w_k\|^2, \;\; K > k_0 \in \mathbf{Z}, \tag{4}$$

for all $w_k \in \mathscr{W}$ and for all $x^0 \in O \subset \mathscr{X}$.

In this paper, we do not solve the above problem. Instead, we solve an associated mixed $\mathscr{H}_2/\mathscr{H}_\infty$ filtering/estimation problem involving a single noise signal $w \in \ell_2[0,\infty)$. This problem can be defined as follows.

Definition 2.1: (Discrete-time Mixed $\mathscr{H}_2/\mathscr{H}_\infty$ (Sub-optimal) Nonlinear Filtering Problem (DMH2HINLFP)). Given the plant Σ^{da} and a number $\gamma^\star > 0$, find an *admissible* (to be defined later) filter $\mathscr{F}_k : \mathscr{Y} \to \mathscr{X}$ such that

$$\hat{x}_{k+1} = \mathscr{F}_k(\mathsf{Y}_k)$$

and $\|\breve{z}\|_{\ell_2}$ is minimized while the constraint (4) is satisfied for all $\gamma \geq \gamma^\star$ for all $w \in \mathscr{W} \subseteq \ell_2[k_0,\infty)O$, and for all $x^0 \in O$.

Remark 2.1: The problem defined above is the *finite-horizon* filtering/estimation problem. We have the infinite-horizon problem if we let $K \to \infty$.

III. Solution to the *DMH2HINLFP*

To solve the above problem, we consider the following class of estimators Σ^{daf}:

$$\left\{ \begin{array}{rcl} \hat{x}_{k+1} & = & f(\hat{x}_k) + L(\hat{x}_k,k)[y_k - h_2(\hat{x}_k)], \\ & & \hat{x}(k_0) = \hat{x}^0 \\ \hat{z}_k & = & h_1(\hat{x}_k) \end{array} \right. \tag{5}$$

where $\hat{x}_k \in \mathscr{X}$ is the estimated state, $L(.,.) \in \mathscr{M}^{n\times m}(\mathscr{X} \times \mathbf{Z})$ is the error-gain matrix which is smooth and has to be determined, and $\hat{z} \in \Re^s$ is the estimated output of the filter. We can now define the estimation error or penalty variable, $\breve{z}$, which has to be controlled as:

$$\breve{z} := z_k - \hat{z}_k = h_1(x_k) - h_1(\hat{x}_k).$$

Then we combine the plant (1) and estimator (5) dynamics to obtain the following augmented system:

$$\left. \begin{array}{rcl} \breve{x}_{k+1} & = & \breve{f}(\breve{x}_k) + \breve{g}(\breve{x}_k)w_k, \; \breve{x}(k_0) = (x^{0^T} \; \hat{x}^{0^T})^T \\ \breve{z}_k & = & \breve{h}(\breve{x}_k) \end{array} \right\}, \tag{6}$$

where

$$\breve{x}_k = \begin{pmatrix} x_k \\ \hat{x}_k \end{pmatrix}, \;\; \breve{g}(\breve{x}) = \begin{pmatrix} g_1(x_k) \\ L(\hat{x}_k,k)k_{21}(x_k) \end{pmatrix},$$

$$\breve{f}(\breve{x}) = \begin{pmatrix} f(x_k) \\ f(\hat{x}_k) + L(\hat{x}_k,k)(h_2(x_k) - h_2(\hat{x}_k)) \end{pmatrix},$$

$$\breve{h}(\breve{x}_k) = h_1(x_k) - h_1(\hat{x}_k).$$

A filter is also required to be stable, so that trajectories do not blow-up for example in an open-loop system. Thus, we define the *"admissibility"* of a filter as follows.

Definition 3.1: A filter $\mathscr{F}$ is admissible if it is asymptotically (or exponentially) stable for any given initial condition $x(k_0)$ of the plant Σ^{da}, and with $w = 0$

$$\lim_{k\to\infty} \breve{z}_k = 0.$$

To solve the problem, we formulate it as a two-player nonzero-sum differential game with the two cost functionals:

$$J_1(L_k, w_k) = \frac{1}{2} \sum_{k=k_0}^{K} \{\gamma^2 \|w_k\|^2 - \|\breve{z}_k\|^2\} \tag{7}$$

$$J_2(L_k, w_k) = \frac{1}{2} \sum_{k_0}^{K} \|\breve{z}_k\|^2 \tag{8}$$

where $L_k := L(x_k,k)$. The first functional is associated with the $\mathscr{H}_\infty$ constraint criterion, while the second functional is related to the output energy of the system or $\mathscr{H}_2$ criterion. By making $J_1 \geq 0$, then the $\mathscr{H}_\infty$ constraint $\|\Sigma^a\|_{\mathscr{H}_\infty} \leq \gamma$ is satisfied. A Nash-equilibrium solution to the above game is said to exist and is admissible if we can find a pair $(L_k^\star,\, w_k^\star)$ such that

$$\begin{array}{rcl} J_1(L_k^\star, w_k^\star) & \leq & J_1(L_k^\star, w_k) \;\; \forall w_k \in \mathscr{W}, \quad (9) \\ J_2(L_k^\star, w_k^\star) & \leq & J_2(L_k, w_k^\star) \;\; \forall L_k \in \mathscr{M}^{n\times m}. \quad (10) \end{array}$$

Sufficient conditions for the solvability of the above game are well-known [3] and are given in terms of the pair of discrete-time Hamilton-Jacobi-Isaacs equations (DHJIE):

$$Y(\breve{x},k) = \inf_{w_k \in \mathscr{W}} \Big\{ \frac{1}{2}[\gamma^2[\|w_k\|^2 - \|\breve{z}(\breve{x})\|^2] + Y(\breve{x}_{k+1},k+1) \Big\},\; Y(\breve{x},K+1) = 0, \tag{11}$$

$$\begin{array}{rcl} V(\breve{x},k) & = & \inf_{L_k \in \mathscr{M}^{n\times m}} \left\{ \frac{1}{2} \|\breve{z}(\breve{x})\|^2 + V(\breve{x}_{k+1},k+1) \right\}, \\ & & V(\breve{x},K+1) = 0, \end{array} \tag{12}$$

$\forall \breve{x} \in N \times N$, $k = k_0, \ldots, K$, and where $\breve{x} = \breve{x}_k$.

Thus, to solve the problem at hand, we define the Hamiltonian functions $H_i : (\mathscr{X} \times \mathscr{X}) \times \mathscr{W} \times \mathscr{M}^{n\times m} \times \Re \to \Re$, $i = 1,2$ associated with the cost functionals (7), (8):

$$\begin{array}{rcl} H_1(\breve{x}, w_k, L_k, Y) & = & Y(\breve{f}(\breve{x}) + \breve{g}(\breve{x})w_k, k+1) - Y(\breve{x},k) \\ & & + \frac{1}{2}\gamma^2 \|w_k\|^2 - \frac{1}{2}\|\breve{z}_k\|^2 \quad (13) \\ H_2(\breve{x}, w_k, L_k, V) & = & V(\breve{f}(\breve{x}) + \breve{g}(\breve{x})w_k, k+1) - V(\breve{x},k) \\ & & + \frac{1}{2}\|\breve{z}_k\|^2 \quad (14) \end{array}$$

The following theorem then presents sufficient conditions for the solvability of the *DMH2HINLFP* on a finite-horizon, and is the main result of this paper.

Theorem 3.1: Consider the nonlinear system (1) and the *DMH2HINLFP* for this system. Suppose the function h_1 is one-to-one (or injective) and the plant Σ^a (or the vector-field f) is locally asymptotically stable about the equilibrium point

$x=0$. Further, suppose there exists a pair of C^2 (with respect to both arguments) negative and positive-definite functions $Y,V : N\times N\times\mathfrak{R}\to\mathfrak{R}$ respectively, locally defined in a neighborhood $N\times N\subset\mathscr{X}\times\mathscr{X}$ of the origin $\breve{x}=0$, and a smooth matrix function $L : N\times\mathbf{Z}\to\mathscr{M}^{n\times m}$ satisfying the following pair of coupled DHJIEs:

$$Y(\breve{x},k)=Y(\breve{f}^\star(\breve{x})+\breve{g}^\star(\breve{x})w_k^\star,k+1)+\frac{1}{2}\gamma^2\|w_k^\star\|^2-\frac{1}{2}\|\breve{z}_k\|^2,\ Y(\breve{x},K+1)=0,\ \breve{x}\in N\times N \tag{15}$$

$$\begin{aligned}V(\breve{x},k) &= V(\breve{f}^\star(\breve{x})+\breve{g}^\star(\breve{x})w_k^\star,k+1)+\frac{1}{2}\|\breve{z}_k\|^2,\\ &\quad V(\breve{x},K+1)=0,\ \ \breve{x}\in N\times N\end{aligned} \tag{16}$$

together with the side-conditions

$$w^\star := -\frac{1}{\gamma^2}\breve{g}^T(\breve{x})\frac{\partial^T Y(\lambda,k+1)}{\partial\lambda}\bigg|_{\lambda=\breve{f}(\breve{x})+\breve{g}(\breve{x})w^\star}, \tag{17}$$

$$L^\star(\hat{x}) = \arg\min_{L}\{H_2(\breve{x},w^\star,L,V)\}, \tag{18}$$

$$\frac{\partial^2 H_1}{\partial w^2}(\breve{x},w,L^\star,Y)\bigg|_{\breve{x}=0}>0, \tag{19}$$

$$\frac{\partial^2 H_2}{\partial L^2}(\breve{x},w^\star,L,V)\bigg|_{\breve{x}=0}>0, \tag{20}$$

where

$$\breve{f}^\star(\breve{x})=\breve{f}(\breve{x})\big|_{L=L^\star},\ \ \breve{g}^\star(\breve{x})=\breve{g}(\breve{x})\big|_{L=L^\star}.$$

Then:

(i) there exists a unique Nash-equilibrium solution $(w^\star,L^\star)$ for the game (7), (8), (1) locally in N;

(ii) the augmented system (6) is locally dissipative with respect to the supply rate $s(w_k,\breve{z}_k)=\frac{1}{2}(\gamma^2\|w_k\|^2-\|\breve{z}_k\|^2)$ and hence has ℓ_2-gain from w to $\breve{z}$ less or equal to γ [12];

(iii) the optimal costs or performance objectives of the game are $J_1^\star(L^\star,w^\star)=Y(\breve{x}^0,k_0)$ and $J_2^\star(L^\star,w^\star)=V(\breve{x}^0,k_0)$;

(iv) the filter Σ^{daf} with the gain matrix $L(\hat{x}_k,k)$ satisfying (18) solves the finite-horizon $DMH2HINLFP$ for the system locally in N.

Proof: Assume there exist definite solutions Y,V to the DHJIEs (15)-(16), and (i) consider the Hamiltonian function $H_1(.,.,.,.)$ first. We can apply the necessary condition for optimality, i.e.,

$$\frac{\partial^T H_1}{\partial w}\bigg|_{w=w^\star} = \breve{g}^T(\breve{x})\frac{\partial^T Y(\lambda,k+1)}{\partial\lambda}\bigg|_{\lambda=\breve{f}(\breve{x})+\breve{g}(\breve{x})w^\star}+\gamma^2 w^\star=0, \tag{21}$$

to get

$$w^\star := -\frac{1}{\gamma^2}\breve{g}^T(\breve{x})\frac{\partial^T Y(\lambda,k+1)}{\partial\lambda}\bigg|_{\lambda=\breve{f}(\breve{x})+\breve{g}(\breve{x})w^\star} := \alpha_0(\breve{x},w^\star). \tag{22}$$

Thus, $w^\star$ is expressed implicitly. Moreover, since

$$\frac{\partial^2 H_1}{\partial w^2}=\breve{g}^T(\breve{x})\frac{\partial^2 Y(\lambda,k+1)}{\partial\lambda^2}\bigg|_{\lambda=\breve{f}(\breve{x})+\breve{g}(\breve{x})w^\star}\breve{g}(\breve{x})+\gamma^2 I$$

is nonsingular about $(\breve{x},w)=(0,0)$, the equation (22) has a unique solution $\alpha_0(\breve{x})$, $\alpha_0(0)=0$ in the neighborhood $N\times W$ of $(x,w)=(0,0)$ by the implicit function theorem [17].

Now substitute $w^\star$ in the expression for $H_2(.,.,.,.)$ (14), to get

$$\begin{aligned}H_2(\breve{x},w^\star,L_k,V) &= V(\breve{f}(\breve{x})+\breve{g}(\breve{x})\alpha_1(\breve{x}),k+1)-V(\breve{x},k)\\ &\quad+\frac{1}{2}\|\breve{z}_k\|^2\end{aligned}$$

and let

$$L_k^\star=\arg\min_{L_k}\{H_2(\breve{x},w^\star,L_k,V)\},$$

Then by Taylor's theorem, we can expand $H_2(.,w^\star,.,.)$ about $L_k^\star$ as [21]

$$\begin{aligned}&H_2(\breve{x},w^\star,L_k,Y)=H_2(\breve{x},w^\star,L_k^\star,Y_{\breve{x}}^T)+\\ &\frac{1}{2}Tr\Big\{[I_n\otimes(L_k-L_k^\star)^T]\frac{\partial^2 H_2}{\partial L_k^2}(w^\star,L_k)\times\\ &[I_m\otimes(L_k-L_k^\star)^T]\Big\}+O(\|L_k-L_k^\star\|^3).\end{aligned}$$

Thus, taking $L_k^\star$ as in (18) and if the condition (20) holds, then $H_2(.,.,w^\star,.)$ is minimized and the Nash-equilibrium condition

$$H_2(w^\star,L_k^\star)\le H_2(w^\star,L_k)\ \ \forall L_k\in\mathscr{M}^{n\times m},k=k_0,\dots,K$$

is satisfied. Moreover, substituting $(w^\star,L^\star)$ in (12) gives the DHJIE (16).

Now substitute $L_k^\star$ as given by (18) in the expression for $H_1(.,.,.,.)$ and expand it in Taylor's series about $w^\star$ to obtain:

$$\begin{aligned}H_1(\breve{x},w,L_k^\star,Y) &= Y(\breve{f}^\star(\breve{x})+\breve{g}^\star(\breve{x})w,k+1)-Y(x,k)+\\ &\quad\frac{1}{2}\gamma^2\|w\|^2-\frac{1}{2}\|\breve{z}_k\|^2\\ &= H_1(\breve{x},w^\star,L_k^\star,Y)+\\ &\quad\frac{1}{2}(w-w^\star)^T\frac{\partial^2 H_2}{\partial w^2}(w,L_k^\star)(w-w^\star)+\\ &\quad O(\|w-w^\star\|^3).\end{aligned}$$

Substituting now $w=w^\star$ as given in (22), and if the condition (19) is satisfied, we see that the second Nash-equilibrium condition

$$H_1(w^\star,L_k^\star)\le H_1(w,L_k^\star),\ \ \forall w\in\mathscr{W}$$

is also satisfied. Thus, the pair $(w^\star,L^\star)$ constitute a Nash-equilibrium solution to the two-player nonzero-sum dynamic game. Moreover, substituting $(w^\star,L^\star)$ in (11) gives the DHJIE (15).

(ii) The Nash-equilibrium condition

$$\begin{aligned}H_1(\breve{x},w,L_k^\star,Y) &\ge H_1(\breve{x},w^\star,L_k^\star,Y)=0\\ \Leftrightarrow Y(\breve{x},k)-Y(\breve{x}_{k+1},k+1) &\le \frac{1}{2}\gamma^2\|w\|^2-\frac{1}{2}\|\breve{z}_k\|^2\\ \Leftrightarrow \breve{Y}(\breve{x}_{k+1},k+1)-\breve{Y}(\breve{x},k) &\le \frac{1}{2}\gamma^2\|w\|^2-\frac{1}{2}\|\breve{z}_k\|^2,\end{aligned} \tag{23}$$

$\forall\breve{x}\in U$, $\forall w\in\mathscr{W}$, for some positive-definite function $\breve{Y}=-Y>0$. Summing now from $k=k_0$ to $k=K$ we get the dissipation inequality [12]

$$\breve{Y}(\breve{x}_{K+1},K+1)-\breve{Y}(\breve{x}_{k_0},k_0)\leq\sum_{k_0}^{K}\frac{1}{2}\gamma^2\|w_k\|^2-\frac{1}{2}\|\breve{z}_k\|^2. \quad (24)$$

Hence the system has ℓ_2-gain from w to $\breve{z}$ less or equal to γ.
(iii) Consider the cost functional $J_1(L_k,w_k)$ first, and rewrite it as

$$\begin{aligned} & J_1(L_k,w_k)+Y(\breve{x}_{K+1},K+1)-Y(\breve{x}(k_0),k_0) \\ = & \sum_{k=k_0}^{K}\left\{\frac{1}{2}\gamma^2\|w_k\|^2-\frac{1}{2}\|\breve{z}_k\|^2+Y(\breve{x}_{k+1},k+1)-Y(\breve{x}_k,k)\right\} \\ = & \sum_{k=k_0}^{K}H_1(\breve{x},w_k,L,Y)\end{aligned}$$

Substitute now $(L_k^\star,w_k^\star)$ respectively and use the DHJIE (15) to get $H_1(\breve{x},w_k^\star,L_k^\star,Y)=0$ and the result.

Similarly, consider the cost functional $J_2(L,w)$ and rewrite it as

$$\begin{aligned} & J_2(L,w)+V(\breve{x}_{K+1},K+1)-V(\breve{x}_{k_0},k_0) \\ = & \sum_{k=k_0}^{K}\left\{\frac{1}{2}\|\breve{z}_k\|^2+V(\breve{x}_{k+1},k+1)-V(\breve{x}_k,k)\right\} \\ = & \sum_{k=k_0}^{K}H_2(\breve{x},w_k,L_k,V).\end{aligned}$$

Again, since $V(\breve{x}_{K+1},K+1)=0$, substituting $(L_k^\star,w_k^\star)$ and using the DHJIE (16) the result also follows.
(iv) Notice that the inequality (23) implies that with $\breve{w}_k=0$,

$$\breve{Y}(\breve{x}_{k+1},k+1)-\breve{Y}(\breve{x}_k,k)\leq-\frac{1}{2}\|\breve{z}_k\|^2,\ \forall\breve{x}\in\Upsilon \quad (25)$$

and since $\breve{Y}$ is positive-definite, by Lyapunov's theorem, the augmented system is locally stable. Furthermore, for any trajectory of the system $\breve{x}_k$ such that $\breve{Y}(\breve{x}_{k+1},k+1)-\breve{Y}(\breve{x}_k,k)=0$ for all $k\geq k_c>k_0$, it implies that $z_k=0$, which in-turn implies $h_1(x_k)=h_1(\hat{x}_k)$, and $x_k=\hat{x}_k\,\forall k\geq k_c$ since h_1 is injective. This further implies that $h_2(x_k)=h_2(\hat{x}_k)\,\forall k\geq k_c$ and it is a trajectory of the free system:

$$\breve{x}_{k+1}=\begin{pmatrix} f(x_k) \\ f(\hat{x}_k)\end{pmatrix}.$$

By asymptotic stability of the vector-field f about the equilibrium-point $x=0$, we have internal stability of the augmented system, and $z_k=0$. On the other hand, if this condition is eliminated, and strict inequality holds in (25), then since $\breve{Y}(\breve{x},t)>0$, there exist constants κ_1, κ_2, κ_3 such that

$$\begin{gathered} 0\leq\kappa_1\|\breve{x}\|^2\leq\breve{Y}(\breve{x}_k,k)\leq\kappa_2\|\breve{x}\|^2 \\ \breve{Y}(\breve{x}_{k+1},k+1)-Y(\breve{x}_k,k)<-\tfrac{1}{2}\|z_k\|^2\leq-\kappa_3\|\breve{x}\|^2\end{gathered}$$

locally about $\breve{x}=0$ for all $\breve{x}\in\Upsilon_1\subset\Upsilon$ by Taylor's theorem. Again, by Lyapunov's theorem [17], we have exponential stability of the equilibrium point $\breve{x}=0$, i.e., $\breve{x}_k\to0$ exponentially, and equivalently $z_k\to0$ exponentially as $k\to K$. Hence Σ^{daf} is admissible. Finally, combining (i)-(iii), (iv) follows. □

IV. Infinite-Horizon Case

In this section, we discuss the infinite-horizon filtering problem, in which case we let $K\to\infty$. Since we are interested in finding a time-invariant gain for the filter, we seek time-independent functions $Y,V:\widetilde{N}\times\widetilde{N}\to\Re$ locally defined in a neighborhood $\widetilde{N}\times\widetilde{N}\subset\mathscr{X}\times\mathscr{X}$ of $(x,\hat{x})=(0,0)$ such that the following DHJIEs:

$$\begin{aligned} & Y(\breve{f}^\star(\breve{x})+\breve{g}^\star(\breve{x})\tilde{w}^\star)-Y(\breve{x})+\frac{1}{2}\gamma^2\|\tilde{w}^\star\|^2-\frac{1}{2}\|\breve{z}\|^2=0, \\ & \quad Y(0)=0,\ \ x,\hat{x}\in\widehat{N} \quad (26) \\ & V(\breve{f}^\star(\breve{x})+\breve{g}^\star(\breve{x})\tilde{w}^\star)-V(\breve{x})+\frac{1}{2}\|\breve{z}\|^2=0, \\ & \quad V(0)=0,\ x,\hat{x}\in\widehat{N} \quad (27)\end{aligned}$$

are satisfied together with the side-conditions:

$$\tilde{w}^\star:=-\frac{1}{\gamma^2}\breve{g}^T(\breve{x})\frac{\partial^TY(\lambda)}{\partial\lambda}\bigg|_{\lambda=\breve{f}(\breve{x})+\breve{g}(\breve{x})\tilde{w}^\star}:=\alpha_2(\breve{x},w^\star) \quad (28)$$

$$\tilde{L}^\star(\hat{x})=\arg\min_{\tilde{L}}\left\{\widetilde{H}_2(\breve{x},\tilde{w}^\star,\tilde{L},V)\right\},\ \ \breve{x}\in\widetilde{N}\times\widetilde{N}. \quad (29)$$

$$\frac{\partial^2\widetilde{H}_1}{\partial w^2}(\breve{x},w,\tilde{L}^\star,Y)\bigg|_{\breve{x}=0}>0 \quad (30)$$

$$\frac{\partial^2\widetilde{H}_2}{\partial\tilde{L}^2}(\breve{x},\tilde{w}^\star,\tilde{L},V)\bigg|_{\breve{x}=0}>0 \quad (31)$$

where

$$\breve{f}^\star(\breve{x})=\breve{f}(\breve{x})\big|_{\tilde{L}=\tilde{L}^\star},\ \ \breve{g}^\star(\breve{x})=\breve{g}(\breve{x})|_{\tilde{L}=\tilde{L}^\star}$$

$$\begin{aligned}\widetilde{H}_1(\breve{x},w,\tilde{L},Y) &= Y(\breve{f}(\breve{x})+\breve{g}(\breve{x})w)-Y(\breve{x})+\frac{1}{2}\gamma^2\|w\|^2 \\ & \quad -\frac{1}{2}\|\breve{z}_k\|^2 \\ \widetilde{H}_2(\breve{x},w,\tilde{L},V) &= V(\breve{f}(\breve{x})+\breve{g}(\breve{x})w)-V(\breve{x})+\frac{1}{2}\|\breve{z}_k\|^2\end{aligned}$$

where $\tilde{L}^\star$ is the asymptotic value of $L_k^\star$. Again here, since the estimation is carried over an infinite-horizon, it is necessary to ensure that the augmented system (6) is stable with $w=0$. Hence we require the following definition.

Definition 4.1: The pair $\{f,h\}$ is said to be locally zero-state detectable if there exists a neighborhood $\mathscr{O}$ of $x=0$ such that, if x_k is a trajectory of $x_{k+1}=f(x_k)$ satisfying $x(k_0)\in\mathscr{O}$, then $h(x_k)$ is defined for all $k\geq k_0$ and $h(x_k)=0$, for all $k\geq k_s$, implies $\lim_{k\to\infty}x_k=0$.

The following proposition can now be proven along the same lines as Theorem 3.1.

Proposition 4.1: Consider the nonlinear system (1) and the infinite-horizon *DMH2HINLFP* for this system. Suppose the function h_1 is one-to-one (or injective) and the plant Σ^{da} is locally zero-state detectable. Further, suppose there exists a pair of C^2 locally defined negative and positive-definite functions $Y,V:\widetilde{N}\times\widetilde{N}\to\Re$ respectively, and a smooth matrix function $\tilde{L}:\widetilde{N}\to\mathscr{M}^{n\times m}$ satisfying the pair of coupled DHJIEs (26), (27) together with (28)-(31). Then: (i) there exists locally a unique Nash-equilibrium solution $(\tilde{w}^\star,\tilde{L}^\star)$ for the game; (ii) the augmented system (6) is dissipative with respect to the supply rate $s(w,\breve{z})=\frac{1}{2}(\gamma^2\|w\|^2-\|\breve{z}\|^2)$

and hence has ℓ_2-gain from $\breve{w}$ to $\breve{z}$ less or equal to γ; (iii) the optimal costs or performance objectives of the game are $J_1^\star(\tilde{L}^\star,\hat{w}^\star) = Y(\breve{x}^0)$ and $J_2^\star(\tilde{L}^\star,\hat{w}^\star) = V(\breve{x}^0)$; (iv) the filter Σ^{daf} with the gain matrix $L(\hat{x}) = \hat{L}^\star(\hat{x})$ satisfying (29) solves the infinite-horizon *DMH2HINLFP* locally in $\widetilde{N}$.

Furthermore, we also have the follwing result.

Proposition 4.2: Consider the nonlinear system (6) and the infinite-horizon *MH2HINLFP* for this system. Suppose the following hold:

(a1) the function h_1 is one-to-one (or injective);
(a2) $\{f, h_1\}$ is locally zero-state detectable;
(a3) there exists a pair of C^1 negative and positive-definite functions $Y, V : \widetilde{N}\times\widetilde{N} \to \Re$ respectively, locally defined in a neighborhood $\widetilde{N}\times\widetilde{N} \subset \mathscr{X}\times\mathscr{X}$ of the origin $\breve{x}=0$, and a smooth matrix function $\tilde{L} : \widetilde{N} \to \mathscr{M}^{n\times m}$ satisfying the pair of coupled DHJIEs (26), (27) together with (28)-(31).

Then:

(i) $\breve{f}^\star$ is locally asymptotically stable;
(ii) $\breve{f} + \breve{g}w^\star$ is locally asymptotically stable so that $\tilde{w}^\star$ is an optimal control for the system $\dot{\breve{x}} = \breve{f}(\breve{x}) + \breve{g}(\breve{x})u$.

Proof (i) As in the proof of item (i) of Theorem 3.1,

$$H_1(w, L^\star) \geq H_1(\tilde{w}^\star, L^\star) = 0, \ \ \forall w \in \mathscr{W}$$
$$\Leftrightarrow Y(\breve{f}^\star(\breve{x}) + \breve{g}^\star(\breve{x})w) - Y(\breve{x}) + \tfrac{1}{2}\gamma^2\|w\|^2 - \tfrac{1}{2}\|\breve{z}\|^2 \geq 0.$$

Set now $w = 0$ to get

$$\breve{Y}(\breve{f}^\star(\breve{x})) - \breve{Y}(\breve{x}) \leq -\frac{1}{2}\|\breve{z}\|^2 \leq 0$$

and thus, $f^\star$ is locally stable. By assumptions (a1) and (a2), $\{f, h_1\}$ zero-state detectable $\Rightarrow \{\breve{f}, \breve{h}\}$ zero-state detectable. The result then follows from Lyapunov's Theorem and the LaSalle's invariance principle by the zero-state detectability of $\{\breve{f}, \breve{h}\}$.

(ii) This follows from the DHJIE (27). Rewriting it as

$$V(\breve{f}^\star(\breve{x}) + \breve{g}^\star(\breve{x})\tilde{w}^\star) - V(\breve{x}) = -\frac{1}{2}\|\breve{z}\|^2 = 0,$$

we have that the closed-loop system $\breve{f}^\star(\breve{x}) + \breve{g}^\star(\breve{x})\tilde{w}^\star$ is locally stable. Finally, the result again follows by the zero-state detectability of $\{\breve{f}, \breve{h}\}$ and LaSalle's invariance-principle. □

V. Approximate and Explicit Solutions (Quasi-Linearization)

In this section, we discuss how the *DMH2HINLFP* can be solved approximately to obtain explicit solutions [8]. We consider the infinite-horizon problem for this purpose, and for simplicity, we make the following assumption on the system matrices.

Assumption 5.1: The system matrices are such that

$$\begin{aligned} k_{21}(x)g_1^T(x) &= 0 \\ k_{21}(x)k_{21}^T(x) &= I \end{aligned}$$

Using a Taylor-series approximation we can aproximate the functions Y, V in (26), (27) respectively with their first-order approximations about $\widehat{f}(\breve{x})$ in a neighborhood $\widehat{N}\times\widehat{N}$ of the origin to obtain the following explicit result.

Theorem 5.1: Consider the nonlinear system (1) and the infinite-horizon *DMH2HINLFP* for this system. Suppose the function h_1 is one-to-one (or injective) and the plant Σ^a is locally zero-state detectable. Further, suppose there exists a pair of C^1 locally defined negative and positive-definite functions $\tilde{Y}, \tilde{V} : \widehat{N}\times\widehat{N} \to \Re$ respectively, and a smooth matrix function $\hat{L} : \widehat{N} \to \mathscr{M}^{n\times m}$ satisfying the pair of coupled DHJIEs:

$$\begin{aligned}
&\tilde{Y}(\widehat{f}(\breve{x})) - \tilde{Y}(\breve{x}) - \frac{1}{2\gamma^2}\tilde{Y}_x(\widehat{f}(\breve{x}))g_1(x)g_1^T(x)\tilde{Y}_x^T(\widehat{f}(\breve{x})) - \\
&\quad \frac{1}{2\gamma^2}\tilde{Y}_{\hat{x}}(\widehat{f}(\breve{x}))\hat{L}(\hat{x})\hat{L}^T(\hat{x})\tilde{Y}_{\hat{x}}^T(\widehat{f}(\breve{x})) - \\
&\quad\quad \frac{1}{\gamma^2}\tilde{Y}_{\hat{x}}(\widehat{f}(\breve{x}))\hat{L}(\hat{x})\hat{L}^T(\hat{x})\tilde{V}_{\hat{x}}^T(\widehat{f}(\breve{x})) - \\
&\frac{1}{2}(h_1(x) - h_1(\hat{x}))^T(h_1(x) - h_1(\hat{x})) = 0, \ \tilde{Y}(0) = 0, \quad (32) \\
&\tilde{V}(\widehat{f}(\breve{x})) - \tilde{V}(\breve{x}) - \gamma^2(h_2(x) - h_2(\hat{x}))^T(h_2(x) - h_2(\hat{x})) - \\
&-\frac{1}{\gamma^2}\tilde{V}_x(\widehat{f}(\breve{x}))g_1(x)g_1^T(x)\tilde{Y}_x^T(\widehat{f}(\breve{x})) - \\
&\frac{1}{\gamma^2}\tilde{V}_{\hat{x}}(\widehat{f}(\breve{x}))\hat{L}(\hat{x})\hat{L}^T(\hat{x})\tilde{Y}_{\hat{x}}^T(\widehat{f}(\breve{x})) + \\
&\frac{1}{2}(h_1(x) - h_1(\hat{x}))^T(h_1(x) - h_1(\hat{x})) = 0, \ \tilde{V}(0) = 0, \quad (33)
\end{aligned}$$

together with the side-condition

$$\tilde{V}_{\hat{x}}(\widehat{f}(\breve{x}))\hat{L}^\star(\hat{x}) = -\gamma^2(h_2(x) - h_2(\hat{x}))^T, \ \ x, \hat{x} \in \widehat{N}. \qquad (34)$$

Then:

(i) there exists locally in $\widehat{N}$ a unique Nash-equilibrium solution $(\hat{w}^\star, \hat{L}^\star)$ for the dynamic game corresponding to (7), (8), (6);
(ii) the augmented system (6) is locally dissipative in $\widehat{N}$ with respect to the supply rate $s(w, \breve{z}) = \frac{1}{2}(\gamma^2\|w\|^2 - \|\breve{z}\|^2)$ and hence has ℓ_2-gain from w to $\breve{z}$ less or equal to γ;
(iii) the optimal costs or performance objectives of the game are approximately $J_1^\star(\hat{L}^\star,\hat{w}^\star) = \tilde{Y}(\breve{x}_0)$ and $J_2^\star(\hat{L}^\star,\hat{w}^\star) = \tilde{V}(\breve{x}_0)$;
(iv) the filter Σ^{daf} with the gain matrix $\hat{L}(\hat{x}) = \hat{L}^\star(\hat{x})$ satisfying (34) solves the infinite-horizon *DMH2HINLFP* for the system locally in $\widehat{N}$.

Proof: Part (i) can be shown by repalcing $H_1(.,.,.,.)$, $H_2(.,.,.,.)$ with their Taylor-series approximation $\widehat{H}_1(.,.,.,.)$, $\widehat{H}_2(.,.,.,.)$ about $\widehat{f}(\breve{x})$ respectively. In addition, substituting $(\hat{L}^\star, \hat{w}^\star)$ in the DHJIEs (26), (27) we have the DHJIEs (32), (33) respectively.

(ii) Consider the time-variation of $\tilde{Y}$ along the trajectories of the system (6) with $\hat{L} = \hat{L}^\star$:

$$\begin{aligned}
\tilde{Y}(\breve{x}_{k+1}) &= \tilde{Y}(\breve{f}^\star(x) + \breve{g}^\star(x)w) \quad \forall \breve{x} \in \widehat{N},\ \forall w \in \mathscr{W} \\
&\approx \tilde{Y}(\widehat{f}(\breve{x})) + \tilde{Y}_x(\widehat{f}(\breve{x}))g_1(x)w + \\
&\quad \tilde{Y}_{\hat{x}}(\widehat{f}(\breve{x}))[\hat{L}^\star(\hat{x})(h_2(x) - h_2(\hat{x}) + k_{21}(x)w)] \\
&= \tilde{Y}(\widehat{f}(\breve{x})) - \frac{1}{\gamma^2}\tilde{Y}_{\hat{x}}(\widehat{f}(\breve{x}))\hat{L}^\star(\hat{x})\hat{L}^{\star^T}(\hat{x})\tilde{V}_{\hat{x}}^T(\widehat{f}(\breve{x})) \\
&\quad -\frac{1}{2\gamma^2}\tilde{Y}_x(\widehat{f}(\breve{x}))g_1(x)g_1^T(x)\tilde{Y}_x^T(\widehat{f}(\breve{x})) + \\
&\quad \frac{\gamma^2}{2}\Big\| w + \frac{1}{\gamma^2}g_1^T(x)\tilde{Y}_x^T(\widehat{f}(\breve{x})) + \\
&\quad \frac{1}{\gamma^2}k_{21}^T(x)\hat{L}^{\star^T}(\hat{x})\tilde{Y}_{\hat{x}}^T(\widehat{f}(\breve{x}))\Big\|^2 - \frac{\gamma^2}{2}\|w\|^2 - \\
&\quad \frac{1}{2\gamma^2}\tilde{Y}_{\hat{x}}(\widehat{f}(\breve{x}))\hat{L}^\star(\hat{x})\hat{L}^{\star^T}(\hat{x})\tilde{Y}_{\hat{x}}^T(\widehat{f}(\breve{x})) \\
&= \tilde{Y}(\breve{x}) - \frac{\gamma^2}{2}\|w\|^2 + \frac{\gamma^2}{2}\Big\| w + \frac{1}{\gamma^2}g_1^T(x)\tilde{Y}_x^T(\widehat{f}(\breve{x})) \\
&\quad + \frac{1}{\gamma^2}k_{21}^T(x)\hat{L}^{\star^T}(\hat{x})\tilde{Y}_{\hat{x}}^T(\widehat{f}(\breve{x}))\Big\|^2 + \frac{1}{2}\|\breve{z}\|^2 \\
&\geq \tilde{Y}(\breve{x}) + \frac{1}{2}\|\breve{z}\|^2 - \frac{\gamma^2}{2}\|w\|^2 \quad \forall \breve{x} \in \widehat{N},\ \forall w \in \mathscr{W}
\end{aligned}$$

where use has been made of the first-order Taylor approximation, the coupling condition (34), and the DHJIE (32) in the above manipulations. The last inequality further implies that

$$\widetilde{Y}(\breve{x}_{k+1}) - \widetilde{Y}(\breve{x}) \leq \frac{\gamma^2}{2}\|w\|^2 - \frac{1}{2}\|\breve{z}\|^2 \quad \forall \breve{x} \in \widehat{N},\ \forall w \in \mathscr{W}$$

for some $\widetilde{Y} = -\tilde{Y} > 0$, which is the infinitesimal dissipation inequality [13]. Therefore, the system has ℓ_2-gain $\leq \gamma$. The proof of asymptotic stability can now be pursued along the same lines as in Proposition 4.1. The proofs of items (iii)-(iv) are similar to those in Theorem 3.1.□

Remark 5.1: The result of the above theorem are highly beneficial in many sense. First and foremost is the benefit of the explicit solutions for computational purposes. Secondly, the approximation is reasonably accurate, as it captures a great deal of the dynamics of the system. Thirdly, it simplifies the solution as it does away with extra sufficient conditions (see e.g. the conditions (19), (20) in Theorem 3.1, and Proposition 4.1). Fourthly, it affords us the ability to develop an iterative procedure for solving the coupled DHJIEs.

Remark 5.2: The converse of the above theorem regarding the existence of the solutions to the DHJIEs (32), (33) can be proven, and similarly the approximate analog of Proposition 4.2 can also be proven.

VI. Conclusion

In this paper, we have considered the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ filtering problem for discrete-time affine nonlinear systems. Sufficient conditions for the solvability of this problem have been given implicitly in terms of a pair of coupled DHJIEs. Both the finite-horizon and infinite-horizon problems have been discussed.

An explicit solution to the problem using first-order Taylor-series approximation has also been presented. Moreover, the resulting coupled DHJIEs are computationally more amenable to iterative solutions. Future work will concentrate in developing computational algorithms.

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