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Control of Uncertain Systems with Guaranteed Performance

Jianming Lian, Scott D. Sudhoff and Stanislaw H. Żak
School of Electrical and Computer Engineering
Purdue University, West Lafayette, IN 47907-2035, USA
jlian@purdue.edu, sudhoff@ecn.purdue.edu, zak@purdue.edu

Abstract— **Approximation-free controllers are proposed for the output tracking control of a class of multi-input multi-output uncertain systems. The proposed state feedback and output feedback controllers incorporate high-gain observers to estimate the system uncertainty. The tracking error of the closed-loop system is guaranteed to be semi-globally uniformly ultimately bounded. The proposed control architecture is simple without any approximation components for unknown system dynamics, and, therefore, any robustifying components for the compensation of approximation errors. Simulations performed on two benchmark problems illustrate the effectiveness of the proposed output feedback controller.**

I. Introduction

Various adaptive control strategies have been proposed for feedback linearizable uncertain systems including single-input single-output (SISO) systems [1]–[4] and multi-input multi-output (MIMO) systems [5]–[8]. Adaptive controllers often involve certain types of function approximators to approximate unknown system dynamics in the controller implementation. However, the approximation error, resulting from function approximation, and the disturbance, internal or external, may deteriorate the controller performance or even destabilize the closed-loop system. Hence, in order to ensure guaranteed controller performance, various robustifying components have been incorporated into the design of adaptive controller, which results in adaptive robust controllers. On the other hand, most of the proposed controllers (see, for example, [1], [2], [4]–[6], [8]) are state feedback controllers, which require the availability of the controlled system's states. However, it is common in practice that only the system outputs are available. Hence, output feedback controllers utilizing state observers have been developed to overcome the limitation associated with state feedback controllers. In particular, high-gain observers for system state estimation were incorporated into the construction of output feedback controllers in [3], [7], [9].

In this paper, we apply the method of "perturbation estimation," presented in [10], to the output tracking control of the class of feedback linearizable MIMO uncertain systems considered in [5]–[8]. In [10], a high-gain state and perturbation observer was used in the decentralized control of a class of interconnected SISO uncertain systems. Here, we propose approximation-free state feedback and output feedback controllers that employ high-gain observers to estimate the system uncertainty. The system uncertainty is the combination of unknown system dynamics and disturbance. Our proposed controllers do not have any fuzzy or neural components to approximate unknown system dynamics, so we do not invert any approximated matrix in the controller implementation. At the same time, robustifying components are not required for the compensation of approximation errors either. Hence, the advantage of our proposed approximation-free tracking controllers over those in [5]–[8] is the much simpler control architecture which leads to the easier controller implementation. Moreover, we take into account disturbance which is assumed to be zero in [5], [7], [8], when we develop our tracking control strategy. Although Chang [6] also dealt with disturbance, only state feedback controller was proposed. The tracking error of the closed-loop system driven by our proposed tracking controllers is guaranteed to be semi-globally uniformly ultimately bounded. We can specify the upper bound on the norm of the steady state tracking error using the controller's design parameters.

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II. System Description and Problem Statement

The class of multi-input multi-output (MIMO) nonlinear systems we consider in this paper is modeled by

$$y_i^{(n_i)} = f_i(\boldsymbol{x}) + \sum_{j=1}^{p} g_{ij}(\boldsymbol{x})u_j + d_i, \quad i = 1, \ldots, p \qquad (1)$$

where y_i is the system output, u_j is the system input, d_i models the disturbance, and $\boldsymbol{x} = [\boldsymbol{x}_1^\top \ \cdots \ \boldsymbol{x}_p^\top]^\top \in \mathbb{R}^n$ is the system state vector with $\boldsymbol{x}_i = [y_i \ \cdots \ y_i^{(n_i-1)}]^\top$ and $n = \sum_{i=1}^{p} n_i$. Let $\boldsymbol{u} = [u_1 \ \cdots \ u_p]^\top$, $\boldsymbol{y} = [y_1 \ \cdots \ y_p]^\top$, $\boldsymbol{d} = [d_1 \ \cdots \ d_p]^\top$, $\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}) \ \cdots \ f_p(\boldsymbol{x})]^\top$, $\boldsymbol{G}(\boldsymbol{x}) = [g_{ij}(\boldsymbol{x})]_{p\times p}$. Let $(\boldsymbol{A}_i, \boldsymbol{b}_i)$ be the canonical controllable pair that represents chains of n_i integrators, and let $\boldsymbol{c}_i = [1 \ \boldsymbol{0}_{n_i-1}^\top]$ where $\boldsymbol{0}_{n_i-1}$ is the (n_i-1)-dimensional zero vector. We can represent (1) in a state-space model as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\left(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{d}\right), \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}, \end{aligned} \qquad (2)$$

where $\boldsymbol{A} = \mathrm{diag}[\boldsymbol{A}_1 \ \cdots \ \boldsymbol{A}_p]$, $\boldsymbol{B} = \mathrm{diag}[\boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_p]$, $\boldsymbol{C} = \mathrm{diag}[\boldsymbol{c}_1 \ \cdots \ \boldsymbol{c}_p]$. We assume that $f_i(\boldsymbol{x})$ and $g_{ij}(\boldsymbol{x})$ are unknown continuous or bounded functions, and $\boldsymbol{G}(\boldsymbol{x})$ is definite. Without loss of generality, we assume that $\boldsymbol{G}(\boldsymbol{x})$ is positive definite. We also assume that d_i is bounded.

Our objective in this paper is to develop a tracking control strategy that forces the i-th output y_i of the uncertain system (1) to track a given reference signal y_{di} that has bounded derivatives up to the n_i-th order. We define the desired system state vector as $\boldsymbol{x}_d = [\boldsymbol{x}_{d1}^\top \ \cdots \ \boldsymbol{x}_{dp}^\top]^\top \in \boldsymbol{\Omega}_{x_d}$ with $\boldsymbol{x}_{di} = [y_{di} \ \cdots \ y_{di}^{(n_i-1)}]^\top$, where $\boldsymbol{\Omega}_{x_d}$ is a given compact subset of $\mathbb{R}^n$. Let $e_i = y_i - y_{di}$ denote the i-th output tracking error and let $\boldsymbol{e}_i = [e_i \ \cdots \ e_i^{(n_i-1)}]^\top$. The system tracking error is defined as $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}_d$. Let $\boldsymbol{y}^{(n)} = [y_1^{(n_1)} \ \cdots \ y_p^{(n_p)}]^\top$ and $\boldsymbol{y}_d^{(n)} = [y_{d1}^{(n_1)} \ \cdots \ y_{dp}^{(n_p)}]^\top$. Then the tracking error dynamics can be modeled as

$$\begin{aligned} \dot{\boldsymbol{e}} &= \boldsymbol{A}\boldsymbol{e} + \boldsymbol{B}\left(\boldsymbol{y}^{(n)} - \boldsymbol{y}_d^{(n)}\right) \\ &= \boldsymbol{A}\boldsymbol{e} + \boldsymbol{B}\left(\boldsymbol{f} + \boldsymbol{G}\boldsymbol{u} - \boldsymbol{y}_d^{(n)} + \boldsymbol{d}\right). \end{aligned} \tag{3}$$

If $\boldsymbol{f}$, $\boldsymbol{G}$ and $\boldsymbol{d}$ were known to us and all the system states were measurable for feedback implementation, we could have used the following ideal control strategy,

$$\boldsymbol{u} = \boldsymbol{G}^{-1}\left(-\boldsymbol{f} - \boldsymbol{K}\boldsymbol{e} + \boldsymbol{y}_d^{(n)} - \boldsymbol{d}\right), \tag{4}$$

where $\boldsymbol{K} = \mathrm{diag}[\boldsymbol{k}_1 \ \ldots \ \boldsymbol{k}_p]$ is selected such that $\boldsymbol{A}_{mi} = \boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_i$ is Hurwitz. Thus, there exists $\boldsymbol{P}_{mi} = \boldsymbol{P}_{mi}^\top > 0$ such that $\boldsymbol{A}_{mi}^\top\boldsymbol{P}_{mi} + \boldsymbol{P}_{mi}\boldsymbol{A}_{mi} = -2\boldsymbol{Q}_{mi}$ for any $\boldsymbol{Q}_{mi} = \boldsymbol{Q}_{mi}^\top > 0$. Let $\boldsymbol{A}_m = \mathrm{diag}[\boldsymbol{A}_{m1} \ \cdots \ \boldsymbol{A}_{mp}]$, $\boldsymbol{P}_m = \mathrm{diag}[\boldsymbol{P}_{m1} \ \cdots \ \boldsymbol{P}_{mp}]$ and $\boldsymbol{Q}_m = \mathrm{diag}[\boldsymbol{Q}_{m1} \ \cdots \ \boldsymbol{Q}_{mp}]$. It follows that $\boldsymbol{A}_m$ is Hurwitz and $\boldsymbol{A}_m^\top\boldsymbol{P}_m + \boldsymbol{P}_m\boldsymbol{A}_m = -2\boldsymbol{Q}_m$. Substituting the ideal controller (4) into (3), we obtain the tracking error dynamics, $\dot{\boldsymbol{e}} = \boldsymbol{A}_m\boldsymbol{e}$, which implies that $\lim_{t\to\infty}\boldsymbol{e}(t) = \boldsymbol{0}$. However, $\boldsymbol{f}$, $\boldsymbol{G}$ and $\boldsymbol{d}$ are unknown. Furthermore, in most cases only the system outputs $\boldsymbol{y}_i$ are available. Thus, we first develop an approximation-free state feedback based tracking control strategy. The resulting tracking controller approximates the ideal controller (4) to achieve the control objective.

III. State Feedback Controller Development

We first assume that the system states are available to us. We define the uncertainty $\boldsymbol{\psi}$ of the system (2) in the same way as the perturbation term introduced in [10], that is,

$$\boldsymbol{\psi} = \boldsymbol{f} + (\boldsymbol{G} - \boldsymbol{G}_0)\boldsymbol{u} + \boldsymbol{d}, \tag{5}$$

where $\boldsymbol{G}_0 = g_0\boldsymbol{I}_p$ is a chosen diagonal control gain matrix and g_0 is a chosen constant. This uncertainty represents the combined effect of unknown nonlinearities and disturbance. We assume that $\boldsymbol{\psi}$ is bounded. In the following analysis, we also assume that $\dot{\boldsymbol{\psi}} = [\dot{\psi}_1 \ \cdots \ \dot{\psi}_p]^\top$ is bounded, that is, the components $\dot{\psi}_i$ are bounded.

Remark 1: The constant g_0 must be chosen to be positive (or negative) if $\boldsymbol{G}$ is positive (or negative) definite.

We now investigate the following state feedback based tracking control strategy,

$$\boldsymbol{u}_s\left(\boldsymbol{e}, \hat{\boldsymbol{\psi}}, \boldsymbol{y}_d^{(n)}\right) = \boldsymbol{G}_0^{-1}\left(-\boldsymbol{K}\boldsymbol{e} - \hat{\boldsymbol{\psi}} + \boldsymbol{y}_d^{(n)}\right) \tag{6}$$

with the estimated system uncertainty $\hat{\boldsymbol{\psi}}$ provided by the following high-gain uncertainty observer,

$$\left.\begin{aligned} \hat{e}_i^{(n_i)} &= \hat{\psi}_i + \frac{\alpha_{i1}^s}{\varepsilon_s}\left(e_i^{(n_i-1)} - \hat{e}_i^{(n_i-1)}\right) + g_0u_i - y_{di}^{(n_i)} \\ \dot{\hat{\psi}}_i &= \frac{\alpha_{i2}^s}{\varepsilon_s^2}\left(e_i^{(n_i-1)} - \hat{e}_i^{(n_i-1)}\right), \end{aligned}\right\} \tag{7}$$

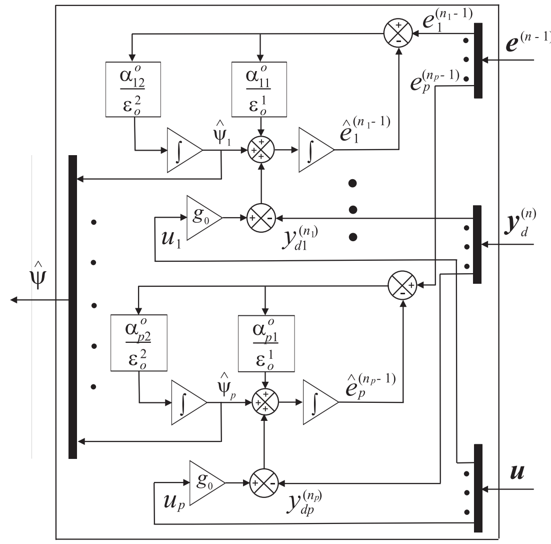

Fig. 1. High-gain uncertainty observer.

where ε_s is a design parameter such that $0 < \varepsilon_s < 1$ and α_{ij}^s, $j = 1, 2$, is selected so that the roots of the polynomial equation, $s^2 + \alpha_{i1}^s s + \alpha_{i2}^s = 0$, have negative real parts. In Fig. 1, we show the structure of the above high-gain uncertainty observer, where $\boldsymbol{e}^{(n-1)} = [e_1^{(n_1-1)} \ \cdots \ e_p^{(n_p-1)}]^\top$. We note that the above observer is based on the high-gain state and perturbation observer proposed in [10]. To facilitate the stability analysis of the closed-loop system, we cast the control problem into a standard singular perturbation form. Let $\boldsymbol{\zeta}_s = [\boldsymbol{\zeta}_{s1}^\top \ \cdots \ \boldsymbol{\zeta}_{sp}^\top]^\top$ with $\boldsymbol{\zeta}_{si} = [\zeta_{si1} \ \zeta_{si2}]^\top$, where

$$\zeta_{si1} = \frac{e_i^{(n_i-1)} - \hat{e}_i^{(n_i-1)}}{\varepsilon_s} \quad \text{and} \quad \zeta_{si2} = \psi_i - \hat{\psi}_i. \tag{8}$$

Let $\bar{\boldsymbol{e}} = [\bar{\boldsymbol{e}}_1^\top \ \cdots \ \bar{\boldsymbol{e}}_p^\top]^\top$ with $\bar{\boldsymbol{e}}_i = [\boldsymbol{e}_i^\top \ \psi_i]^\top$, $\boldsymbol{e}_y = [e_1 \ \cdots \ e_p]^\top$, and let $\boldsymbol{0}_{n_i}$ be the n_i-dimensional zero vector. Then it follows from (3) that

$$\begin{aligned} \dot{\bar{\boldsymbol{e}}} &= \bar{\boldsymbol{A}}\bar{\boldsymbol{e}} + \bar{\boldsymbol{B}}_1\left(\boldsymbol{G}_0\boldsymbol{u} - \boldsymbol{y}_d^{(n)}\right) + \bar{\boldsymbol{B}}_2\dot{\boldsymbol{\psi}}, \\ \boldsymbol{e}_y &= \bar{\boldsymbol{C}}\bar{\boldsymbol{e}}, \end{aligned} \tag{9}$$

where $\bar{\boldsymbol{A}} = \mathrm{diag}[\bar{\boldsymbol{A}}_1 \ \cdots \ \bar{\boldsymbol{A}}_p]$, $\bar{\boldsymbol{B}}_1 = \mathrm{diag}[\bar{\boldsymbol{b}}_{11} \ \cdots \ \bar{\boldsymbol{b}}_{1p}]$, $\bar{\boldsymbol{B}}_2 = \mathrm{diag}[\bar{\boldsymbol{b}}_{21} \ \cdots \ \bar{\boldsymbol{b}}_{2p}]$ and $\bar{\boldsymbol{C}} = \mathrm{diag}[\bar{\boldsymbol{c}}_1 \ \cdots \ \bar{\boldsymbol{c}}_p]$ with

$$\bar{\boldsymbol{A}}_i = \begin{bmatrix} \boldsymbol{0}_{n_i} & \boldsymbol{I}_{n_i} \\ 0 & \boldsymbol{0}_{n_i}^\top \end{bmatrix}, \quad \bar{\boldsymbol{b}}_{1i} = \begin{bmatrix} \boldsymbol{b}_i \\ 0 \end{bmatrix}, \quad \bar{\boldsymbol{b}}_{2i} = \begin{bmatrix} \boldsymbol{0}_{n_i} \\ 1 \end{bmatrix},$$

and $\bar{\boldsymbol{c}}_i = [1 \ \boldsymbol{0}_{n_i}^\top]$. Then combining (7), (8) and (9) gives

$$\varepsilon_s\dot{\boldsymbol{\zeta}}_s = \boldsymbol{A}_\alpha^s\boldsymbol{\zeta}_s + \varepsilon_s\boldsymbol{B}_\alpha^s\dot{\boldsymbol{\psi}}, \tag{10}$$

where $\boldsymbol{A}_\alpha^s = \mathrm{diag}[\boldsymbol{A}_{\alpha1}^s \ \cdots \ \boldsymbol{A}_{\alpha p}^s]$ with

$$\boldsymbol{A}_{\alpha i}^s = \begin{bmatrix} -\alpha_{i1}^s & 1 \\ -\alpha_{i2}^s & 0 \end{bmatrix}$$

and $\boldsymbol{B}_\alpha^s = \mathrm{diag}[\boldsymbol{b}_{\alpha1}^s \ \cdots \ \boldsymbol{b}_{\alpha p}^s]$ with $\boldsymbol{b}_{\alpha i}^s = [0 \ 1]^\top$.

We assume that $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, where $\boldsymbol{\Omega}_{e_0}$ is a compact set that contains all possible initial tracking errors. Let $c_{e_0} = \max_{\boldsymbol{e}\in\boldsymbol{\Omega}_{e_0}} \frac{1}{2}\boldsymbol{e}^\top\boldsymbol{P}_m\boldsymbol{e}$. We then choose c_e such that

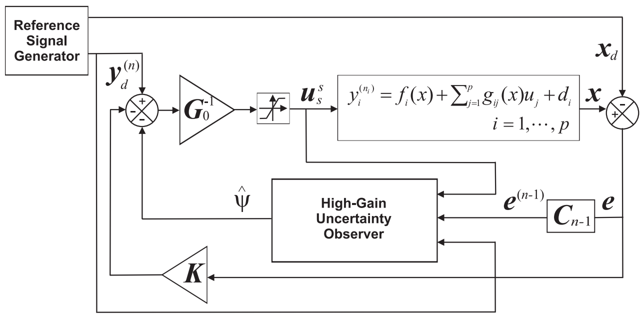

Fig. 2. The closed-loop system driven by the state feedback controller, where $\boldsymbol{C}_{n-1} = \mathrm{diag}[\boldsymbol{c}_{n_1-1} \cdots \boldsymbol{c}_{n_p-1}]$ with $\boldsymbol{c}_{n_i-1} = [\boldsymbol{0}_{n_i-1}^\top \; 1]$.

$c_e > c_{e_0}$, and define $\boldsymbol{\Omega}_e = \{\boldsymbol{e} : \frac{1}{2}\boldsymbol{e}^\top \boldsymbol{P}_m \boldsymbol{e} \le c_e\}$. Because $\boldsymbol{\psi}$ is bounded, we assume that $\boldsymbol{\psi} \in \boldsymbol{\Omega}_\psi$, where $\boldsymbol{\Omega}_\psi = \{\boldsymbol{\psi} : \|\boldsymbol{\psi}\| \le c_\psi\}$ and $\| \bullet \|$ denotes the standard Euclidean norm. We also assume that $\boldsymbol{y}_d^{(n)} \in \boldsymbol{\Omega}_{y_d}$, where $\boldsymbol{\Omega}_{y_d}$ is a compact subset of $\mathbb{R}^p$. In order to eliminate the peaking phenomena that accompany a high-gain observer [11], we introduce the saturation of the control input. Let $S_i \ge \max|u_{si}(\boldsymbol{e}_i, \hat{\psi}_i, y_{di}^{(n_i)})|$, $i = 1, \ldots, p$, where the maximum is taken over $\boldsymbol{e} \in \boldsymbol{\Omega}_e$, $\boldsymbol{y}_d^{(n)} \in \boldsymbol{\Omega}_{y_d}$ and $\hat{\boldsymbol{\psi}} \in \boldsymbol{\Omega}_{\psi_1}$, where $\boldsymbol{\Omega}_{\psi_1} = \{\boldsymbol{\psi} : \|\boldsymbol{\psi}\| \le c_{\psi_1}\}$ with $c_{\psi_1} > c_\psi$. The resulting saturated state feedback controller is

$$\boldsymbol{u}_s^s\left(\boldsymbol{e}, \hat{\boldsymbol{\psi}}, \boldsymbol{y}_d^{(n)}\right) = \left[u_{s1}^s \; \cdots \; u_{sp}^s\right]^\top, \tag{11}$$

where

$$u_{si}^s = S_i \,\mathrm{sat}\left(\frac{u_{si}\left(\boldsymbol{e}_i, \hat{\psi}_i, y_{di}^{(n_i)}\right)}{S_i}\right)$$

and $\mathrm{sat}(\bullet)$ is the saturation function. A block diagram of the closed-loop system is given in Fig. 2.

To proceed, we need the following two definitions. Let $\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0))$ denote the tracking error $\boldsymbol{e}$ at time t subject to the initial condition $\boldsymbol{e}(t_0)$.

Definition 1: The solutions $\boldsymbol{e}$ to the tracking error dynamics (3) are T-uniformly bounded if for a given real number $d > 0$, there exists a positive real number $b = b(d)$ such that if $\|\boldsymbol{e}(t_0 + T; t_0, \boldsymbol{e}(t_0))\| \le d$, where $T \ge 0$, then $\|\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0))\| \le b$ for $t \ge t_0 + T$.

Remark 2: In the above definition, the trajectory of the tracking error $\boldsymbol{e}$ emanates from $\boldsymbol{e}(t_0)$. We may not know anything about the trajectory in the time interval $[t_0 \; t_0 + T]$. But for $t \ge t_0 + T$, the trajectory must be confined in the ball of radius $b(d)$ if $\|\boldsymbol{e}(t_0 + T; t_0, \boldsymbol{e}(t_0))\| \le d$.

Definition 2: The solutions $\boldsymbol{e}$ to the tracking error dynamics (3) are semi-globally uniformly ultimately bounded (SGUUB) with respect to a closed ball $\bar{B}$ if for any given compact set $\boldsymbol{\Omega}_{e_0}$, there exists a finite time T such that if $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, then $\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0)) \in \bar{B}$ for $t \ge t_0 + T$.

Proposition 1: If $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, then there exist a constant ε_s^* $(0 < \varepsilon_s^* < 1)$ and a finite time T_{s1} such that if $\varepsilon_s < \varepsilon_s^*$, then $\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0)) \in \boldsymbol{\Omega}_e$ for $t \ge t_0$ and the tracking error $\boldsymbol{e}$ is T_{s1}-uniformly bounded.

Proof: See [12]. ∎

Theorem 1: For the MIMO uncertain system described in (1) with $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$ and the state feedback based tracking control strategy (6), (7), (11), there exist a constant ε_s^* $(0 < \varepsilon_s^* < 1)$ and a finite time T_{s1} such that if $\varepsilon_s < \varepsilon_s^*$, then the tracking error $\boldsymbol{e}$ is semi-globally uniformly ultimately bounded with respect to any ball of radius greater than $\beta_s = \sqrt{\lambda_{\max}(\boldsymbol{P}_m)/\lambda_{\min}(\boldsymbol{P}_m)} R_s$, where $R_s = r_s \varepsilon_s / \lambda_{\min}(\boldsymbol{Q}_m)$ for some $r_s > 0$. That is, if $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, then for a given $b_s > \beta_s$, $\|\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0))\| \le b_s$ for $t \ge t_0 + T_{s1} + T_s$, where

$$T_s = \begin{cases} 0 & d_s \le \tilde{R}_s \\ \frac{\lambda_{\max}(\boldsymbol{P}_m) d_s^2 - \lambda_{\min}(\boldsymbol{P}_m)\tilde{R}_s^2}{2\lambda_{\min}(\boldsymbol{Q}_m)\tilde{R}_s(\tilde{R}_s - R_s)} & d_s > \tilde{R}_s, \end{cases}$$

with

$$d_s = \sqrt{\frac{2c_e}{\lambda_{\min}(\boldsymbol{P}_m)}} \quad \text{and} \quad \tilde{R}_s = \sqrt{\frac{\lambda_{\min}(\boldsymbol{P}_m)}{\lambda_{\max}(\boldsymbol{P}_m)}} b_s.$$

Proof: See [12]. ∎

The ball given in Theorem 1,

$$\left\{\boldsymbol{e} : \|\boldsymbol{e}\| \le \sqrt{\frac{\lambda_{\max}(\boldsymbol{P}_m) r_s^2 \varepsilon_s^2}{\lambda_{\min}(\boldsymbol{P}_m)\lambda_{\min}^2(\boldsymbol{Q}_m)}}\right\},$$

represents the trade-off between the magnitude of the steady-state tracking error and the peaking phenomenon caused by the high-gain uncertainty observer. That is, the smaller the ε_s, the better tracking performance but at the cost of larger peaking. We can refer to the above set as the uncertainty ball because this is the smallest ball containing the origin such that the tracking error is guaranteed to enter this ball and stay there thereafter. Thus the volume of this ball can be viewed as the performance measure of the tracking controller.

IV. Output Feedback Controller Construction

We now assume that only the system outputs are available to us. Then we use the developed state feedback controller (6) to construct an output feedback controller as follows,

$$\boldsymbol{u}_o\left(\hat{\boldsymbol{e}}, \hat{\boldsymbol{\psi}}, \boldsymbol{y}_d^{(n)}\right) = \boldsymbol{G}_0^{-1}\left(-\boldsymbol{K}\hat{\boldsymbol{e}} - \hat{\boldsymbol{\psi}} + \boldsymbol{y}_d^{(n)}\right), \tag{12}$$

with the estimated tracking error $\hat{\boldsymbol{e}}$ and the estimated system uncertainty $\hat{\boldsymbol{\psi}}$ provided by the following high-gain tracking error and uncertainty observer adapted from [10],

$$\left.\begin{aligned} \hat{e}_i^{(k)} &= \hat{e}_i^{(k+1)} + \tfrac{\alpha_{ik}^o}{\varepsilon_o^k}(e_i - \hat{e}_i), \; 1 \le k \le n_i - 1 \\ \hat{e}_i^{(n_i)} &= \hat{\psi}_i + \tfrac{\alpha_{in_i}^o}{\varepsilon_o^{n_i}}(e_i - \hat{e}_i) + g_0 u_i - y_{di}^{(n_i)} \\ \dot{\hat{\psi}}_i &= \tfrac{\alpha_{i(n_i+1)}^o}{\varepsilon_o^{n_i+1}}(e_i - \hat{e}_i), \end{aligned}\right\} \tag{13}$$

where ε_o is a design parameter such that $0 < \varepsilon_o < 1$ and α_{ik}^o, $k = 1, \ldots, n_i + 1$, are selected so that the roots of the polynomial equation, $s^{n_i+1} + \alpha_{i1}^o s^{n_i} + \cdots + \alpha_{in_i}^o s + \alpha_{i(n_i+1)}^o = 0$, have negative real parts. In Fig. 3, we show the structure of the above high-gain tracking error and uncertainty observer. As in Section III, we cast the problem into a standard singular perturbation form. Let $\boldsymbol{\zeta}_o = [\boldsymbol{\zeta}_{o1}^\top \; \cdots \; \boldsymbol{\zeta}_{op}^\top]^\top$ with $\boldsymbol{\zeta}_{oi} = [\zeta_{oi1} \; \cdots \; \zeta_{oi(n_i+1)}]^\top$, where

$$\left.\begin{aligned} \zeta_{oik} &= \tfrac{e_i^{(k-1)} - \hat{e}_i^{(k-1)}}{\varepsilon_o^{n_i+1-k}}, \quad k = 1, \ldots, n_i \\ \zeta_{oi(n_i+1)} &= \psi_i - \hat{\psi}_i. \end{aligned}\right\} \tag{14}$$

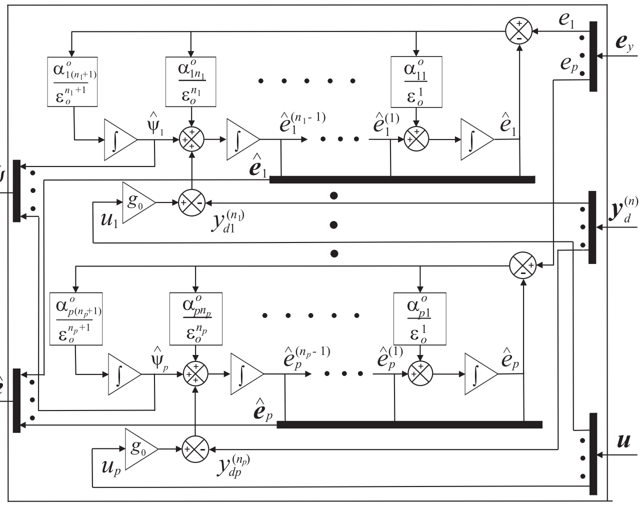

Fig. 3. High-gain tracking error and uncertainty observer.

Then it follows from (13), (14) and (9) that

$$\varepsilon_o \dot{\boldsymbol{\zeta}}_o = \boldsymbol{A}_\alpha^o \boldsymbol{\zeta}_o + \varepsilon_o \boldsymbol{B}_\alpha^o \dot{\psi}, \tag{15}$$

where $\boldsymbol{A}_\alpha^o = \mathrm{diag}[\boldsymbol{A}_{\alpha 1}^o \ \cdots \ \boldsymbol{A}_{\alpha p}^o]$ with

$$\boldsymbol{A}_{\alpha i}^o = \begin{bmatrix} -\alpha_{i1}^o & 1 & 0 & \cdots & 0 \\ -\alpha_{i2}^o & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{in_i}^o & 0 & 0 & \cdots & 1 \\ -\alpha_{i(n_i+1)}^o & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $\boldsymbol{B}_\alpha^o = \mathrm{diag}[\boldsymbol{b}_{\alpha 1}^o \ \cdots \ \boldsymbol{b}_{\alpha p}^o]$ with $\boldsymbol{b}_{\alpha i}^0 = [\boldsymbol{0}_{n_i}^\top \ 1]^\top$.

We still assume that $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, and introduce the saturation of the control input in order to eliminate the peaking phenomena associated with the high-gin tracking error and uncertainty observer. Let $S_i \geq \max|u_{oi}(\hat{\boldsymbol{e}}_i, \hat{\psi}_i, y_{di}^{(n_i)})|$, $i = 1, \ldots, p$, where the maximum is taken over $\hat{\boldsymbol{e}} \in \boldsymbol{\Omega}_{e_1}$, $\boldsymbol{y}_d^{(n)} \in \boldsymbol{\Omega}_{y_d}$ and $\hat{\boldsymbol{\psi}} \in \boldsymbol{\Omega}_{\psi_1}$, where $\boldsymbol{\Omega}_{e_1} = \{\boldsymbol{e} : \frac{1}{2}\boldsymbol{e}^\top \boldsymbol{P}_m \boldsymbol{e} \leq c_{e_1}\}$ with $c_{e_1} > c_e$ and $\boldsymbol{\Omega}_{\psi_1} = \{\boldsymbol{\psi} : \|\boldsymbol{\psi}\| \leq c_{\psi_1}\}$ with $c_{\psi_1} > c_\psi$. The resulting saturated output feedback controller has the form as

$$\boldsymbol{u}_o^s(\hat{\boldsymbol{e}}, \hat{\boldsymbol{\psi}}, \boldsymbol{y}_d^{(n)}) = \left[u_{o1}^s \ \cdots \ u_{op}^s\right]^\top, \tag{16}$$

where

$$u_{oi}^s = S_i \,\mathrm{sat}\left(\frac{u_{oi}\left(\hat{\boldsymbol{e}}_i, \hat{\psi}_i, y_{di}^{(n_i)}\right)}{S_i}\right).$$

A block diagram of the closed-loop system is given in Fig. 4.

Proposition 2: If $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, then there exist a constant ε_o^* $(0 < \varepsilon_o^* < 1)$ and a finite time T_{o1} such that if $\varepsilon_o < \varepsilon_o^*$, then $\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0)) \in \boldsymbol{\Omega}_e$ for $t \geq t_0$ and the tracking error $\boldsymbol{e}$ is T_{o1}-uniformly bounded.

Proof: See [12]. ■

Theorem 2: For the MIMO uncertain system modeled by (1) with $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$ and the output feedback based tracking control strategy (12), (13), (16), there exist a constant ε_o^* $(0 < \varepsilon_o^* < 1)$ and a finite time T_{o1} such that if $\varepsilon < \varepsilon_o^*$, then the tracking error $\boldsymbol{e}$ is semi-globally uniformly ultimately bounded with respect to any ball of radius greater than $\beta_o = \sqrt{\lambda_{\max}(\boldsymbol{P}_m)/\lambda_{\min}(\boldsymbol{P}_m)}R_o$, where $R_o = r_o\varepsilon_o/\lambda_{\min}(\boldsymbol{Q}_m)$ for some $r_o > 0$. That is, if $\boldsymbol{e}(t_0) \in \boldsymbol{\Omega}_{e_0}$, then for a given $b_o > \beta_o$, $\|\boldsymbol{e}(t; t_0, \boldsymbol{e}(t_0))\| \leq b_o$ for $t \geq t_0 + T_{o1} + T_o$, where

$$T_o = \begin{cases} 0 & d_o \leq \tilde{R}_o \\ \frac{\lambda_{\max}(\boldsymbol{P}_m)d_o^2 - \lambda_{\min}(\boldsymbol{P}_m)\tilde{R}_o^2}{2\lambda_{\min}(\boldsymbol{Q}_m)\tilde{R}_o(\tilde{R}_o - R_o)} & d_o > \tilde{R}_o, \end{cases}$$

with

$$d_o = \sqrt{\frac{2c_e}{\lambda_{\min}(\boldsymbol{P}_m)}} \quad \text{and} \quad \tilde{R}_o = \sqrt{\frac{\lambda_{\min}(\boldsymbol{P}_m)}{\lambda_{\max}(\boldsymbol{P}_m)}}b_o.$$

Proof: The proof is similar to that of Theorem 1. ■

The uncertainty ball given in Theorem 2,

$$\left\{\boldsymbol{e} : \|\boldsymbol{e}\| \leq \sqrt{\frac{\lambda_{\max}(\boldsymbol{P}_m) r_o^2 \varepsilon_o^2}{\lambda_{\min}(\boldsymbol{P}_m)\lambda_{\min}^2(\boldsymbol{Q}_m)}}\right\},$$

represents the trade-off between the magnitude of the steady-state tracking error and the peaking of the high-gain tracking error and uncertainty observer.

Remark 3: We assume in Section III that both ψ_i and $\dot{\psi}_i$ are bounded. This assumption is essential in the proof of Theorem 2. In fact, f_i, g_{ij} and d_i are bounded in general and u_{oi}^s is bounded by construction, so it follows from (5) that ψ_i is bounded. However, bounded $\dot{\psi}_i$ implies uniformly continuous ψ_i, which is not always feasible in practice. There is no guarantee that the disturbance d_i is uniformly continuous. Hence, the assumption of bounded $\dot{\psi}_i$ seems to be rather restrictive for practical applications of the proposed control strategy. However, our simulation results in Section V indicate that this is not the case, where the proposed controller can still achieve excellent tracking performance in the presence of the disturbance that is not uniformly continuous.

V. Examples

In this section, we illustrate the performance of our proposed approximation-free output feedback controller on two benchmark examples. In the first example, we apply the proposed tracking control strategy to the output tracking control of a class of single-input single-output (SISO) uncertain systems. In the second example, the proposed controller is tested on a MIMO uncertain system.

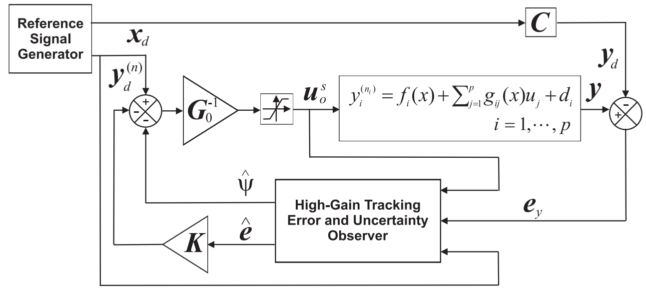

Fig. 4. The closed-loop system driven by the output feedback controller.

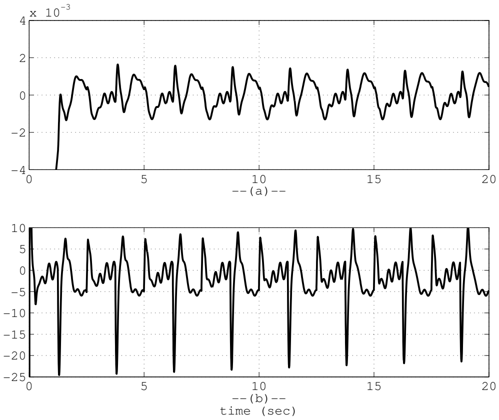

Fig. 5. Output feedback controller performance in Example 1 with $d = 2\sin(20t)$. (a) tracking error e; (b) control input u.

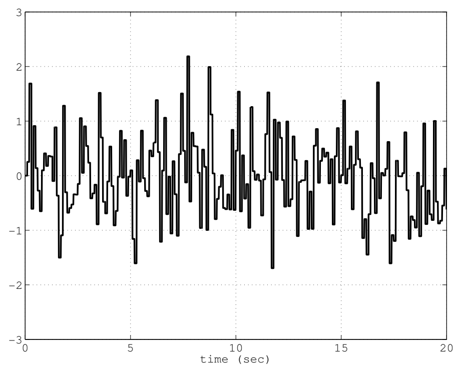

Fig. 6. Disturbance d in Example 1.

Example 1: For the class of MIMO uncertain system (1) with $p = 1$, it becomes a class of SISO feedback linearizable nonlinear systems modeled by

$$y^{(n)} = f(\boldsymbol{x}) + g(\boldsymbol{x})u + d,$$

where g is bounded away from zero. The following nonlinear SISO system model is used for our simulation,

$$\ddot{y} = 4\frac{\sin(4\pi y)}{\pi y}\left(\frac{\sin(\pi\dot{y})}{\pi\dot{y}}\right)^2 + \left(2 + \sin(3\pi(y - 0.5))\right)u + d.$$

The same plant model with zero disturbance was also used in [1], [3], [13] for testing proposed controllers. The reference signal, which was also used in [3], is the output of a low-pass filter with the transfer function $(1 + 0.1s)^{-3}$ driven by a unity amplitude square wave input with frequency of 0.4 Hz and a time average of 0.5. The design parameters for the high-gain tracking error and uncertainty observer are chosen as $\alpha_1^o = 6$, $\alpha_2^o = 12$, $\alpha_3^o = 8$ and $\varepsilon_o = 10^{-3}$. The parameters for the controller are chosen to be $\boldsymbol{k} = [25\ 10]$ and $S = 25$. We select $g_0 = 2$ and the initial conditions to be $y(0) = -0.5$ and $\dot{y}(0) = 2.0$. In Fig. 5, we first show the controller performance with $d = 2\sin(20t)$. Then we select d to be the band-limited white noise shown in Fig. 6. Because the disturbance d is not uniformly continuous, we know that

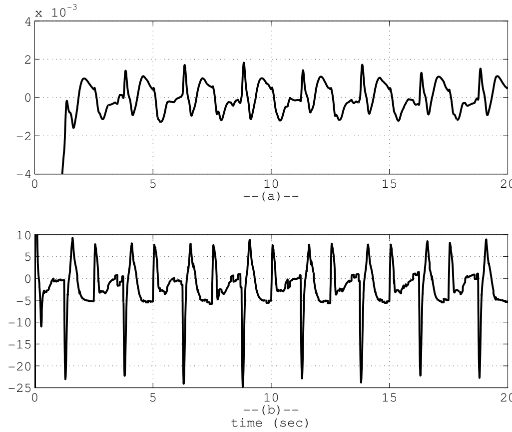

Fig. 7. Output feedback controller performance in Example 1 with the white noise disturbance. (a) tracking error e; (b) control input u.

$\dot{\psi}$ is unbounded. However, the simulation results presented in Fig. 7 show that the proposed tracking control strategy can still achieve the desired performance without satisfying the assumption that $\dot{\psi}$ is bounded. Therefore, the assumption that $\dot{\psi}$ is bounded is not that restrictive for real applications, which we have discussed in Remark 3. Compared with the results in [3], we can see that our proposed controller yields similar performance but with much simpler structure. We do not use any neural network based approximators so that we save a lot of effort to determine the structure of the neural network off-line as it was done in [3].

Example 2: The nonlinear MIMO plant model used for our simulation is the planar articulated two-link manipulator given in [14, p. 394]. Let θ_1 and θ_2 represent the angular positions of joint 1 and 2, respectively, and τ_1 and τ_2 be the applied torques at these two joints. We assume that there exist input disturbances η_1 and η_2 associated with the applied torques τ_1 and τ_2, respectively. Then the dynamics of this two-link rigid robot are given by

$$\begin{aligned} &\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \\ &+ \begin{bmatrix} -h\dot{\theta}_2 & -h\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ h\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 + \eta_1 \\ \tau_2 + \eta_2 \end{bmatrix}, \end{aligned}$$

where $H_{11} = a_1 + 2a_3\cos(\theta_2) + 2a_4\sin(\theta_2)$, $H_{12} = H_{21} = a_2 + a_3\cos(\theta_2) + a_4\sin(\theta_2)$, $H_{22} = a_2$, $h = a_3\sin(\theta_2) - a_4\cos(\theta_2)$, and $a_1 = I_1 + m_1 l_{c1}^2 + I_e + m_e l_{ce}^2 + m_e l_1^2$, $a_2 = I_e + m_e l_{ce}^2$, $a_3 = m_e l_1 l_{ce}\cos(\delta_e)$, $a_4 = m_e l_1 l_{ce}\sin(\delta_e)$. The same plant model was also used in [5], [8], [15] to test the proposed controllers there but without input disturbance, that is, $\boldsymbol{\eta} = \mathbf{0}$. In our simulation, we use the same numerical values as in [5], [8], [14], [15, p. 396], that is, $m_1 = 1.0$, $m_e = 2.0$, $I_1 = 0.12$, $I_e = 0.25$, $l_{c1} = 0.5$, $l_{ce} = 0.6$, $l_1 = 1$, $\delta_e = \pi/6$. The input disturbances η_i, $i = 1, 2$, are the band-limited white noises shown in Fig. 8. The reference signal, which was also used in [8], is defined by $\theta_{d1}(t) = \frac{\pi}{6}\cos(2\pi t)$ and $\theta_{d2}(t) = \frac{\pi}{4}\cos(2\pi t)$. The design parameters for the high-gain tracking error and uncertainty observer are

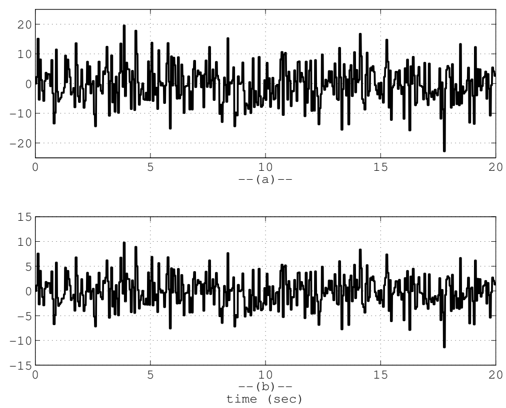

Fig. 8. Input disturbances in Example 2. (a) η_1; (b) η_2.

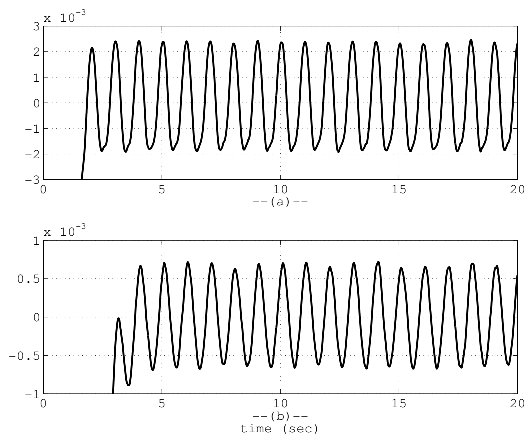

Fig. 9. Tracking errors in Example 2. (a) e_1; (b) e_2.

chosen as $\alpha^o_{i1} = 6$, $\alpha^o_{i2} = 12$, $\alpha^o_{i3} = 8$ for $i = 1, 2$ and $\varepsilon_0 = 10^{-4}$. The parameters for the controller are: $\boldsymbol{k}_1 = [25\ 10]$, $\boldsymbol{k}_2 = [9\ 6]$, $S_1 = 220$ and $S_2 = 100$. We select $g_0 = 1$, that is, $\boldsymbol{G}_0 = \boldsymbol{I}_2$. The manipulator is initially at rest, that is, $\theta_1 = \theta_2 = 0$ and $\dot{\theta}_1 = \dot{\theta}_2 = 0$. In Fig. 9 and Fig. 10, we show the performance of our proposed controller. Compared with the results in [8], our proposed controller has much simpler structure and yields superior performance than the controller in [8]. Moreover, our proposed controller is based on output feedback whereas the proposed controller in [8] requires full state feedback.

VI. Summary

In this paper, novel state feedback and output feedback tracking controllers without either approximation or robustifying components have been proposed for a class of MIMO uncertain systems. The proposed approximation-free tracking control strategies incorporate high-gain observers in the controller implementation. The tracking error of the closed-loop system is guaranteed to be semi-globally uniformly ultimately bounded. The final tracking accuracy is determined by the controller's design parameters.

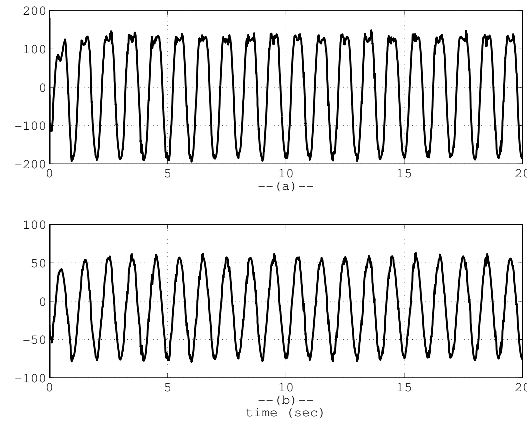

Fig. 10. Control inputs in Example 2. (a) τ_1; (b) τ_2.

The proposed approaches can be directly applied to the output tracking control of the class of uncertain systems (1) with time varying f_i and g_{ij}, that is, $f_i(t, \boldsymbol{x})$ and $g_{ij}(t, \boldsymbol{x})$. All the results hold as long as $f_i(t, \boldsymbol{x})$ and $g_{ij}(t, \boldsymbol{x})$ are bounded with respect to t and continuous or bounded with respect to $\boldsymbol{x}$, and $\boldsymbol{G}(t, \boldsymbol{x})$ is definite.

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