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Stabilizing the Psychological Dynamics of People in a Queue

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Abstract— **This work is part of an ongoing investigation aimed at applying the tools of control theory to gain a deeper understanding of crowd psychology and ultimately to control the relevant psychological dynamics. The present treatment investigates an approach to stabilize a queue, that is, a one-dimensional crowd structure. We pursue the problem in which an authoritative figure, termed a control agent, issues signals that propagate through the queue to drive specific components of the queue's state to zero. Simulations confirm that the control strategy performs admirably in ideal conditions; however, in the presence of system noise, performance deteriorates dramatically as the queue length increases.**

I. INTRODUCTION

The study of crowd psychology rose to prominence in late nineteenth-century Europe as a consequence of the masses rising against oppressive political regimes. Seminal contributions were made by a number of intellectuals of the day, of which the most notable was Gustav LeBon [1]. Eventually, interest in the field waned and crowd psychology fell out of favor by the early 1900s. In the mid-twentieth century the field was rekindled by social psychologists eager to treat the subject with greater intellectual scrutiny and scientific rigor. Today, interest in crowd psychology continues within both academic circles and popular culture. The latter point is evidenced by a number of best-selling books (e.g., [2],[3]) that examine the pervasive nature of group behavior in various aspects of society. In this paper we look at one-dimensional crowd structures in which people are arranged contiguously in a collinear fashion to form a queue. The study of queues is interesting in its own right, but also constitutes a gateway for exploring more complex and highly coupled social systems, including two-dimensional crowds.

When a group of people descend upon a venue they often orient themselves in a queue. For example, people typically form queues while waiting to use an ATM, paying for groceries at a busy supermarket, or seeking admission to an amusement park. This prevalence would suggest queues satisfy a cultural need for order and regularity. Indeed, the very structure of a queue provides a history of each individual's arrival status and a definitive order in which awaiting patrons should be served in the interest of fairness and equality. Queues also foster a rich assortment of intriguing social phenomena. Specifically, queues allow ideas, attitudes, and actions to propagate by allowing the social interactions between the comprising members to act as a conduit for information exchange [4],[5],[6]. To illustrate how dynamic social behavior can develop in queues, consider the scenario in which avid fans are patiently waiting in line for admission to a music concert. United by their anticipation for the show, individuals engage one another in conversation. In doing so, they set up a social network in which ideas, attitudes, and the associated actions flow back and forth among members of the queue. Now suppose an individual in line is alerted, perhaps through a cellular-phone call, that reports indicate the headline act is nowhere to be found. Troubled by the development, this individual shares the information with neighboring fans, who in turn pass their views of the subject onto their neighbors. Among the impassioned fans, conflicting viewpoints, regarding whether the concert will proceed or have to be cancelled, evolve and transmute with time. In the event the queue is composed of particularly demonstrative individuals, it is possible the original rumor could incite aggressive and even hostile behavior as it is disseminated. While extreme, this scenario illustrates the notion that queues support the propagation of ideas and attitudes and that in the right setting the resultant behavior can be dramatic.

The remainder of the paper is structured as follows: Section II presents a model of the psychological behavior of people in a queue. Section III discusses the queue topology, our definition of stability, and the control objectives considered. Section IV provides the main theoretical result, a stability theorem for queues. Practical computation of a stabilizing control law is discussed in Section V. Simulation results are provided in Section VI. Finally, Section VII summarizes the prominent ideas of the paper and outlines directions of ongoing and future research.

II. MODELING PSYCHOLOGICAL DYNAMICS IN A QUEUE

Throughout, we refer to people in a crowd or queue as *agents*. This terminology was deemed more direct than other labels, and stresses the ability of people to act and influence their environment. Additionally, the term is used when discussing multi-agent systems, a field in which we believe our formulation of crowd behavior fits naturally.

The dynamics we use to describe the behavior of agents in a queue are based on a model reported in [7]. The aforementioned model was developed by extrapolating the predominant attributes of crowd psychology emphasized by LeBon in [1]. Central to this perspective is the notion of a *psychological crowd*. LeBon stipulates that under appropriate conditions a *trigger event* can cause a collection of independent individuals to form a *psychological crowd*, in which the comprising agents think and behave as a single unified entity. In [7] the authors attempt to capture this paradigm

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and the associated dynamics through a set of nonlinear, discrete-time, state equations interrelating the state of each agent with those of its neighbors. The state of each agent is defined as the composite of four signals, referred to as *prestige*, *action*, *delayed-action*, and *suggestibility*, with each component described below:

- *Prestige of agent* i: $p_i[k] > 0$ is a measure of agent i's ability to influence the behavior of other agents.
- *Action of agent* i: $a_i[k]$ is a quantification of agent i's behavior as it relates to acceptance of an idea $(a_i[k] > 0)$ or its antitheses $(a_i[k] < 0)$. Action values for which $|a_i[k]| \approx 0$ are indicative of mild acceptance of an idea or its opposite notion, and are associated with a calm and orderly agent. Larger values of $|a_i[k]|$ are indicative of more extreme degrees of acceptance ranging from devoted to frenetic in accordance with $|a_i[k]|$. In the concert example of Section I, positive action may be associated with the belief the headline act will perform, while negative action is consequently associated with the belief the show will be cancelled.
- *Delayed action of agent* i: $b_i[k]$ is the value of $a_i[k]$ one time instant in the past and is introduced so the dynamics may be expressed in state form.
- *Suggestibility of agent* i: $s_i[k] > 0$ is a measure of agent i's affinity to incorporate the behavior of neighboring agents into their own behavior, by mimicking the actions and conduct of others.

The psychological dynamics relating agent states in [7][1] are given, in the case of agent i, by:

$$p_i[k+1] = c_p p_i[k] + \mu_{pa,i}|a_i[k]| \tag{1}$$

$$a_i[k+1] = c_a a_i[k] + \mu_{apa,i} s_i^2[k] \sum_{j\in\mathcal{N}_i} p_j[k] a_j[k] \tag{2}$$

$$b_i[k+1] = a_i[k] \tag{3}$$

$$s_i[k+1] = \mu_{s,i} S \alpha^{c_s \beta_i[k]} \tag{4}$$

$$\beta_i[k] := \mu_{sa,i}\left(a_i[k] - b_i[k]\right)^2 + \mu_{sap,i} \sum_{j\in\mathcal{N}_i} p_j[k]|a_j[k]| + \mu_{ssp,i} \sum_{j\in\mathcal{N}_i} p_j[k]\left(s_j[k] - s_i[k]\right). \tag{5}$$

The constants and constructs used above warrant a brief explanation. Borrowing from the multi-agent terminology, $\mathcal{N}_i$ is referred to as the neighbor set of agent i. In our formulation, $\mathcal{N}_i$ represents the set of agent indices for which agent i has direct interaction with the associated agents through conversation, gesturing, or other social exchanges. In (4), $\alpha > 1$ is a constant and S is a nominal suggestibility value. The μ parameters are agent-specific positive gains used to scale the contributions of various social effects in (1)–(5). The constants c_p, c_a, and c_s reside in the interval $(0,1)$ and capture the tendency of $p_i[k]$ and $a_i[k]$ to decay towards zero and $s_i[k]$ to approach $\mu_{s,i}S$ in the absence of adequate social excitation from neighboring agents. A detailed account chronicling the development of (1)–(5) can be found in [7].

[1]The suggestibility equation in (4) has been modified slightly from its original form in [7]. Specifically, the new equation is strictly positive, which we believe to be more consistent with LeBon's view of suggestibility.

III. Problem Formulation

The problem we pursue considers a queue of n agents that have entered the psychological crowd mindset and are thus subject to (1)–(5). The overall state is taken as the ordered collection of all agent states and is denoted by

$$x[k] = [p_1[k], \ldots, p_n[k], a_1[k], \ldots, a_n[k], b_1[k], \ldots, b_n[k], s_1[k], \ldots, s_n[k]]^T. \tag{6}$$

The nonlinear interplay in (1)–(5) suggests the dynamics are highly unstable, an inclination verified through linearization arguments and simulations in [7]. To maintain orderly behavior among agents we position a *control agent* at the left end of the queue, thereby increasing the queue length by one. We view the control agent as an authoritative figure (such as a security officer) with keen judgement, a disposition towards preserving decorum, and possessing the self-composure and retrospective psyche needed to resist being caught up in the crowd mentality. Given these traits, the control agent is modeled as being able to sense $x[k]$ and use this knowledge alongside an understanding of the social dynamics in (1)–(5) to consciously influence the behavior of agents in the queue. These attributes are captured by assigning the control agent state according to (7)–(9) below:

$$p[k] = \hat{p} \tag{7}$$

$$a[k] = u[k] = u(x[k]) \tag{8}$$

$$s[k] = 0. \tag{9}$$

These three equations constitute the controller. Having the prestige equal the constant $\hat{p} > 0$ in (7) attests to the control agent maintaining a constant level of influence. The functional dependence of the control signal $u[k]$ on $x[k]$ in (8) indicates the action of the control agent is predicated on the behavior of agents in the queue. Finally, using $s[k] = 0$ in (9) signifies the control agent is impervious to suggestion and thus acts as an individual rather than a member of the psychological crowd. From a systems perspective, the control agent introduces the familiar concept of feedback. However, in this case, the role of both sensor and actuator is performed by a person, specifically the control agent, as opposed to one or more mechanical devices, as is often the case.

As a means to concisely refer to specific members of the queue, as well as to facilitate succinct representation of the queue itself, we denote the i^{th} agent as O_i and the lone control agent by X_0. The queue topology under consideration may then be expressed symbolically as

$$X_0 O_1 \cdots O_n. \tag{10}$$

We restrict inclusion in the neighbor set of each member of the queue to the indices of those agents situated immediately adjacent to the member in question. Using the notational scheme in (10), $\mathcal{N}_i$ may therefore be expressed as

$$\mathcal{N}_i = \begin{cases} \{1\} & \text{for} \quad i = 0 \\ \{i-1, i+1\} & \text{for} \quad i = 1, \ldots, n-1 \\ \{n-1\} & \text{for} \quad i = n. \end{cases} \tag{11}$$

From a control perspective, a natural objective is to stabilize the queue dynamics and prevent state signals from growing without bound. Along these lines, we introduce the following definition of stability:

Definition 1 *The queue $X_0O_1\cdots O_n$ is defined to be $Q(\lambda)$–stabilizable for finite λ if there exists a causal control law, (7)–(9), capable of driving, from any initial state $x[0]$, the action state of all agents to zero in λ time instants and subsequently holding all action states at zero. If such a control law is implemented, the queue is said to be $Q(\lambda)$–stable and the controller is $Q(\lambda)$–stabilizing.* $\diamond$

$Q(\lambda)$–stability involves regulating all action states in finite time, which implies from (1), (4), and (5) that $p_i[k]$ will tend towards zero asymptotically and $s_i[k]$ will approach $\mu_{s,i}S$ for all $i = 1\ldots n$. Therefore this notion of stability is consistent with the idea of ensuring state signals remain bounded and all agents act in a calm and orderly manner.

Having introduced the necessary notation and terminology, the control objective of this paper may be stated succinctly as developing a control law, (7)–(9), that $Q(\lambda)$–stabilizes the queue in (10). Since $u[k]$ is a function of $x[k]$ the control law has a state-feedback structure which, given the complex nature of (1)–(5), will likely be nonlinear in form. Before considering stability results in greater depth, we note that the topology in (10) and expression for $\mathcal{N}_i$ in (11) indicate X_0 has direct influence on only O_1. The effect of X_0 on the remaining agents is indirect and a consequence of $\hat{p}$ and $u[k]$ propagating through the highly coupled network linking the state of each agent with those of its neighbors. Therefore, designing the control law in (8) to stabilize the queue must be done prudently and with consideration for the highly entwined nature of (1)–(5).

IV. A Stability Theorem for Queues

This section discusses the existence of a control law that $Q(\lambda)$–stabilizes the queue in (10) with $\lambda = n$. This result is developed progressively through three propositions. Proposition 1 establishes that the action $a_i[k+i]$ has rich structure with respect to $u[k]$. Proposition 2 extends this result by certifying that $a_i[k+i]$ may be driven to zero by appropriate selection of $u[k]$. Proposition 3 investigates the implications of holding the action of the rightmost agent in the queue at zero indefinitely. Finally, Propositions 2 and 3 are leveraged in Theorem 1 to yield the main stability result. Central to the arguments that follow is an appreciation for the temporal structure with which $u[k]$ propagates through the queue. To gain insight into this paradigm it is useful to first consider a queue for which enumeration of the agent states is algebraically tractable. We consider the simple case in which $n = 2$ below.

A. The Queue $X_0O_1O_2$

Consider the simple queue $X_0O_1O_2$ and the goal of driving the action state of the last agent to zero at time 2. Given the queue structure, X_0 cannot affect the state of O_2 directly, but only indirectly by issuing control signals to O_1 that subsequently influence O_2 through the interaction between the two agents. Due to the small size of the queue being considered, it is possible to enumerate the relevant state signals and study the algebraic forms that emerge. To this end, the state of O_1 at time 1 may be calculated using (1)–(5) and is given by:

$$\begin{aligned}
p_1[1] &= c_p p_1[0] + \mu_{pa,1}|a_1[0]| && (12)\\
a_1[1] &= c_a a_1[0] + \mu_{apa,1} s_1^2[0]\left(\hat{p}u[0] + p_2[0]a_2[0]\right) && (13)\\
s_1[1] &= \mu_{s,1} S \alpha^{c_s \beta_1[0]} && (14)\\
\beta_1[0] &= \mu_{sa,1}\left(a_1[0]-b_1[0]\right)^2 - \mu_{ssp,1}\hat{p}s_1[0] + \\
&\quad \mu_{ssp,1}p_2[0]\left(s_2[0]-s_1[0]\right) + \\
&\quad \mu_{sap,1}\left(\hat{p}|u[0]| + p_2[0]|a_2[0]|\right). && (15)
\end{aligned}$$

Similarly, the state of O_2 at time 1 is given by:

$$\begin{aligned}
p_2[1] &= c_p p_2[0] + \mu_{pa,2}|a_2[0]| && (16)\\
a_2[1] &= c_a a_2[0] + \mu_{apa,2} s_2^2[0] p_1[0] a_1[0] && (17)\\
s_2[1] &= \mu_{s,2} S \alpha^{c_s \beta_2[0]} && (18)\\
\beta_2[0] &= \mu_{sa,2}\left(a_2[0]-b_2[0]\right)^2 + \\
&\quad \mu_{ssp,2}p_1[0]\left(s_1[0]-s_2[0]\right) + \\
&\quad \mu_{sap,2}p_1[0]|a_1[0]|. && (19)
\end{aligned}$$

At time 1 the term $u[0]$ has no effect on O_2, as illustrated by the absence of $u[0]$ in (16)–(19). Rather, the implications of $u[0]$ on O_2 are experienced at time 2, at which point

$$a_2[2] = c_a a_2[1] + \mu_{apa,2} s_2^2[1] p_1[1] a_1[1]. \quad (20)$$

Substituting (12), (13), (17), and (18) in (20) permits $a_2[2]$ to be written entirely in terms of system parameters and agent states at time 0 through the expression

$$\begin{aligned}
a_2[2] &= c_a^2 a_2[0] + c_a \mu_{apa,2} s_2^2[0] p_1[0] a_1[0] + \\
&\quad \mu_{apa,2}\mu_{s,2}^2 S^2 \alpha^{2c_s\beta_2[0]} \times \\
&\quad \left(c_p p_1[0] + \mu_{pa,1}|a_1[0]|\right) \times \\
&\quad \left(c_a a_1[0] + \mu_{apa,1} s_1^2[0]\left(\hat{p}u[0] + p_2[0]a_2[0]\right)\right).
\end{aligned}$$

The coefficient of $u[0]$ in the above relation, namely $\mu_{apa,2}\mu_{s,2}^2 S^2 \alpha^{2c_s\beta_2[0]}\left(c_p p_1[0] + \mu_{pa,1}|a_1[0]|\right)\mu_{apa,1}s_1^2[0]\hat{p}$, consists of system parameters and positive functions of the agent states at time 0. This implies $a_2[2]$ is a positively sloped linear function of $u[0]$ and may be driven to any real value by appropriate selection of $u[0]$. In summary, by explicitly propagating $u[0]$ through the queue dynamics, an equation suitable for determining the causal control law needed to drive $a_2[2]$ to zero may be developed. However, in developing this result, for a relatively small queue, considerable algebraic complexity has emerged. Applying an enumerative approach of this form to gauge if similar results apply to larger queues quickly becomes infeasible given the highly coupled and nonlinear nature of the dynamics.

B. The Queue $X_0O_1\cdots O_n$

To extend the results of Section IV-A to larger queues it is necessary to make the most of the structure present in (1)–(5). To this end, the following proposition exploits the fact that the terms inside the summation in (2) are linear with respect to the action state of neighboring agents:

Proposition 1 *Consider the queue $X_0 O_1 \cdots O_n$. For $i = 1 \ldots n$ and $k \geq 0$ the action state of O_i at time $k+i$ may be expressed in the form*

$$a_i[k+i] = a_i^{\overline{u}}[k+i] + a_i^u[k+i]u[k] \tag{21}$$

where $a_i^{\overline{u}}[k+i]$ and $a_i^u[k+i]$ are both functions of only $x[k]$ and in particular are independent of $u[k]$. Furthermore, the term $a_i^u[k+i]$ is nonzero. $\diamond$

Proof: From (2), the action state of agent O_1 at time $k+1$ is given by $c_a a_1[k] + \mu_{apa,1} s_1^2[k] \left(\hat{p}u[k] + p_2[k]a_2[k]\right)$. Defining $\mu_{apa,1} s_1^2[k]\hat{p}$ and $c_a a_1[k] + \mu_{apa,1} s_1^2[k] p_2[k] a_2[k]$ as $a_1^u[k+1]$ and $a_1^{\overline{u}}[k+1]$ respectively and noting that $\hat{p}$ and $s_1[k]$ are positive, (21) is immediately confirmed for $i = 1$. Proceeding along inductive lines, for some $m \in [1, n-1]$, assume the proposition statement is true for $i = m$. It is noted that this condition has been verified for $m = 1$ above. For $m \in [1, n-2]$ it follows from the action-state update equation of O_{m+1} and the induction assumption that

$$\begin{aligned} a_{m+1}[k+m+1] = c_a a_{m+1}[k+m] + \mu_{apa,m+1} s_{m+1}^2[k+m] \times \\ \left(p_m[k+m]\left(a_m^{\overline{u}}[k+m] + a_m^u[k+m]u[k]\right) + \right. \\ \left. p_{m+2}[k+m] a_{m+2}[k+m]\right). \end{aligned}$$

The above expression has the form in (21) for $i = m+1$ with:

$$a_{m+1}^u[k+m+1] := \mu_{apa,m+1} s_{m+1}^2[k+m] p_m[k+m] a_m^u[k+m] \tag{22}$$

$$\begin{aligned} a_{m+1}^{\overline{u}}[k+m+1] := c_a a_{m+1}[k+m] + \mu_{apa,m+1} s_{m+1}^2[k+m] \times \\ \left(p_m[k+m] a_m^{\bar{u}}[k+m] + p_{m+2}[k+m] a_{m+2}[k+m]\right). \end{aligned} \tag{23}$$

Repeating this procedure for $m = n-1$ yields an expression for $a_{m+1}^u[k+m+1]$ identical to that in (22). The expression for $a_{m+1}^{\bar{u}}[k+m+1]$ is equal to the relation in (23) with the term $\mu_{apa,m+1} s_{m+1}^2[k+m] p_{m+2}[k+m] a_{m+2}[k+m]$ removed.

Since $u[k]$ propagates through the queue, in that it influences agent states, at a rate no faster than one agent per discrete time interval, all prestige, action, and suggestibility states of agents O_{m+1} and, for $m < n-1$, O_{m+2} are independent of $u[k]$ at time $k+m$. Furthermore, from (1), the prestige of O_m is a function of agent m's prestige and action at the previous instant and therefore $p_m[k+m]$ is also independent of $u[k]$. Hence, for $m \in [1, n-1]$, a_{m+1}, a_{m+2}, s_{m+1}, p_m, and p_{m+2} in the relations for $a_{m+1}^u[k+m+1]$ and $a_{m+1}^{\overline{u}}[k+m+1]$ are independent of $u[k]$. By the induction assumption, the terms $a_m^u[k+m]$ and $a_m^{\overline{u}}[k+m]$ are independent of $u[k]$ and it follows that $a_{m+1}^u[k+m+1]$ and $a_{m+1}^{\overline{u}}[k+m+1]$ are independent of $u[k]$.

Next, by the induction assumption, $a_m^u[k+m]$ and $a_m^{\overline{u}}[k+m]$ are exclusive functions of $x[k]$. Other signals appearing in the relations for $a_{m+1}^u[k+m+1]$ and $a_{m+1}^{\overline{u}}[k+m+1]$ can be expressed in terms of $x[k]$ by propagating elements of the state at time k through the queue dynamics and it follows that $a_{m+1}^u[k+m+1]$ and $a_{m+1}^{\overline{u}}[k+m+1]$ can be viewed as exclusive functions of $x[k]$.

Finally, again by the induction assumption, $a_m^u[k+m]$ is nonzero. Since $s_{m+1}[k+m]$ and $p_m[k+m]$ are positive it follows that $a_{m+1}^u[k+m+1]$ in (22) is also nonzero.

This series of results confirm the proposition statement is true for $i = m+1$. It follows from induction that for $i = 1 \ldots n$ and $k \geq 0$ the action state of O_i may be expressed as $a_i^{\overline{u}}[k+i] + a_i^u[k+i]u[k]$ where $a_i^{\overline{u}}[k+i]$ and $a_i^u[k+i]$ are both functions of only $x[k]$; moreover, $a_i^u[k+m]$ is nonzero. ■

Proposition 1 can be used to conclude that $a_i[k+i]$ can be driven to zero by appropriate selection of $u[k]$:

Proposition 2 *Consider the queue $X_0 O_1 \cdots O_n$. For $i = 1 \ldots n$ and $k \geq 0$ there exists a causal, state-feedback control law, (7)–(9), capable of driving $a_i[k+i]$ to zero.* $\diamond$

Proof: It follows from (21) in Proposition 1 that for $i = 1 \ldots n$ and $k \geq 0$, $a_i[k+i]$ may be written in the form

$$a_i[k+i] = a_i^{\overline{u}}[k+i] + a_i^u[k+i]u[k] \tag{24}$$

where $a_i^{\overline{u}}[k+i]$ and $a_i^u[k+i]$ are both independent of $u[k]$ and $a_i^u[k+i]$ is nonzero. Since (24) is linear in $u[k]$, the action state of O_i at time $k+i$ may be driven to zero by selecting

$$u[k] = -\frac{a_i^{\overline{u}}[k+i]}{a_i^u[k+i]}. \tag{25}$$

From Proposition 1, $a_i^{\overline{u}}[k+i]$ and $a_i^u[k+i]$ are functions of only $x[k]$; hence, $u[k]$ in (25) depends only on $x[k]$ and (7)–(9) may be regarded as a causal, state-feedback control law. ■

The following proposition investigates the implications of holding the action state of the rightmost agent in the queue, namely O_n, at zero indefinitely. Specifically, holding $a_n[k]$ at zero for $k \geq \widetilde{k} \geq 0$ is shown to imply $a_{n-1}[k], \ldots, a_1[k]$ are also zero for $k \geq \widetilde{k}$:

Proposition 3 *Consider the queue $X_0 O_1 \cdots O_n$. Let $\widetilde{k} \geq 0$ denote an instant in time. If the action state of O_n is sustained at zero for all $k \geq \widetilde{k}$ then the action state of each agent in the queue is identically zero for all $k \geq \widetilde{k}$.* $\diamond$

Proof: As indicated in the proposition statement, consider the scenario in which the action state of the rightmost agent in the queue is zero for $k \geq \widetilde{k}$. The implications on other agents in the queue may be inferred through an inductive argument. To this end, for some $m \in [2, n]$, assume $a_i[k] = 0$ for all $i \geq m$ and $k \geq \widetilde{k}$. It is noted that, given the context being considered, this condition is true for $m = n$. The action-state-update equation of O_m gives

$$a_m[k+1] = c_a a_m[k] + \mu_{apa,m} s_m^2[k] \sum_{j \in \mathcal{N}_m} p_j[k] a_j[k].$$

From the induction assumption, $a_m[k] = 0$ for $k \geq \widetilde{k}$, and

$$0 = \mu_{apa,m} s_m^2[k] \sum_{j \in \mathcal{N}_m} p_j[k] a_j[k], \; k \geq \widetilde{k}. \tag{26}$$

In the event $m = n$ the neighbor set $\mathcal{N}_m$ is simply $\{m-1\}$ and (26) reduces to

$$0 = \mu_{apa,m} s_m^2[k] p_{m-1}[k] a_{m-1}[k], \; k \geq \widetilde{k}. \tag{27}$$

Alternatively, if $m \neq n$, $\mathcal{N}_m = \{m-1, m+1\}$; however, it follows from the induction assumption that $a_{m+1}[k] = 0$ for

$k \geq \widetilde{k}$, and (26) once again reduces to (27). Since $s_m[k]$ and $p_{m-1}[k]$ are positive, equality in (27) necessitates $a_{m-1}[k]=0$ for $k \geq \widetilde{k}$. This result establishes that $a_i[k]=0$ for $i \geq m-1$ and $k \geq \widetilde{k}$. It follows from induction that $a_i[k]=0$ for $i = 1\dots n$ and $k \geq \widetilde{k}$, implying that if $a_n[k]$ can be held at zero for $k \geq \widetilde{k}$ the action states $a_{n-1}[k],\dots,a_1[k]$ are also zero for $k \geq \widetilde{k}$. ■

Propositions 2 and 3 can be combined to yield a stability result for queues:

Theorem 1 *There exists a control law, (7)–(9), that $Q(n)$–stabilizes the queue $X_0O_1\cdots O_n$.* ⋄

Proof: To provide $Q(n)$–stability, $u[k]$ must ensure the action state of the rightmost agent in the queue is zero at time n and is subsequently held at zero for all future times. From Proposition 2 this condition may be realized using the causal, state-feedback controller, (7)–(9), with (8) given by (25) for $i=n$ and $k \geq 0$. That is, $a_n[k]$ may be driven and held at zero for $k \geq n$ by selecting $u[k]$ according to

$$u[k] = -\frac{a_n^{\overline{u}}[k+n]}{a_n^{u}[k+n]},\ k \geq 0. \tag{28}$$

By selecting $u[k]$ to hold $a_n[k]$ at zero for $k \geq n$, as in (28), it follows from Proposition 3 that $a_i[k]=0$ for $i=1\dots n$ and $k \geq n$. Therefore, the queue $X_0O_1\cdots O_n$ is $Q(n)$–stabilized using (7)–(9) with (8) given by (28). ■

V. Computation of a Stabilizing Control Law

While Theorem 1 guarantees the existence of a stabilizing control law, it does not explicitly address a practical approach to calculate $u[k]$. Theorem 1 establishes that the queue $X_0O_1\cdots O_n$ may be $Q(n)$–stabilized using the result in (28). However, evaluation of (28) directly amounts to an enumerative approach that, as discussed in Section IV-A, is unwieldy for all but the smallest of queues. Alternatively, here we consider a recursive method to compute $u[k]$ that is well-suited to numeric evaluation. Development of this approach requires no more than an added degree of bookkeeping throughout Propositions 1 and 2 to keep track of the relationships that emerge in relating $a_i^u[k+i]$ and $a_i^{\overline{u}}[k+i]$ to signals at time $k+i-1$ for $i=1\dots n$. The recursive equations that result from such an analysis are given below in (29) and (30) for the case in which $n \geq 2$:

$$a_i^u[k+i] = \begin{cases} \mu_{apa,1}s_1^2[k]\hat{p} & ,\ i=1 \\ \mu_{apa,i}s_i^2[k+i-1]p_{i-1}[k+i-1]a_{i-1}^u[k+i-1], & i=2\dots n \end{cases} \tag{29}$$

$$a_i^{\overline{u}}[k+i] = \begin{cases} c_a a_1[k] + \mu_{apa,1}s_1^2[k]p_2[k]a_2[k] & ,\ i=1 \\ c_a a_i[k+i-1] + \mu_{apa,i}s_i^2[k+i-1]\,(p_{i-1}[k+i-1]\times \\ a_{i-1}^{\overline{u}}[k+i-1] + p_{i+1}[k+i-1]a_{i+1}[k+i-1]) & ,\ 2 \leq i \leq n-1 \\ c_a a_i[k+i-1]+ \\ \mu_{apa,n}s_n^2[k+n-1]p_{n-1}[k+n-1]a_{n-1}^{\overline{u}}[k+n-1] & ,\ i=n. \end{cases} \tag{30}$$

Evaluation of (29) and (30) requires knowledge of specific agent states over the time interval $[k,k+n-1]$. These requisite signals may be determined using a computer routine that iteratively computes the state, using (1)–(5), over the appropriate time interval. Having garnered all of the necessary state values, it is straightforward to calculate $u[k]$ using (28), (29), and (30). It is worth reinforcing that, given an understanding of the social dynamics, all signals defined at time instants on the interval $[k,k+n-1]$ may be expressed, using (1)–(5), as functions of the the components of $x[k]$. Therefore, the control law is indeed causal as implied by the existence of a $Q(n)$–stabilizing control law in Theorem 1.

VI. Simulation Results

This section substantiates the stability result in Theorem 1 through a number of simulations. We also consider the performance of the control law in cases where the control agent is unable to definitively sense $x[k]$ or issue $u[k]$ with absolute precision. In all simulations, agents are assumed to have the same set of μ parameters with each parameter value randomly selected within a predefined range as described in [7]. The μ parameters used for simulation are given by:

$$\mu_{pa,i}=0.02295,\ \mu_{apa,i}=0.01000,\ \mu_{sa,i}=0.04346,$$
$$\mu_{ssp,i}=0.04064,\ \mu_{sap,i}=0.04796,\ \mu_{s,i}=1.07062$$

for $i=1\dots n$. Remaining parameter values used include:

$$c_p=0.95,\ c_a=0.55,\ c_s=0.75,\ \alpha=1.2,\ S=20.$$

Initial agent states were chosen randomly in the intervals

$$p_i[0]\in(0,0.2),\ a_i[0],\ b_i[0]\in(-0.2,0.2),\ s_i[0]\in(20,25)$$

for $i=1\dots n$. Finally, the control agent's prestige value was set at $\hat{p}=0.5$. The functionality of the $Q(n)$–stabilizing control law in (7)–(9) with (8) given by (28) for the cases in which $n=3$ and $n=6$ is illustrated in Figures 1 and 2 respectively. Noteworthy among the plots is that many of the elements of the control signal $u[k]$ used to stabilize the queue $X_0O_1O_2O_3O_4O_5O_6$ are considerably larger than their counterparts in the control signal used to stabilize the queue $X_0O_1O_2O_3$. This observation is explored in the second column of Table I, which lists the first element of the control signal needed to stabilize (10) for various values of n. These entries suggest a trend in which $|u[0]|$ increases as n is increased. Intuitively, given the unstable dynamics and limited rate at which $u[k]$ propagates through the queue, the value of $|u[0]|$ rises as n increases since $u[0]$ must offset states at time n that have had more time to swell in magnitude.

The last two columns of Table I provide two measures by which to assess the viability of the control scheme in non-ideal conditions. To investigate the effects of a human control agent being unable to sense $x[k]$ with complete accuracy, column 4 of Table I provides a measure of the action of the rightmost agent in the queue resulting from a sensed $x[0]$ in which each element of $x[0]$ is randomly skewed by $\pm1\%$ from its true value. To investigate this scenario in more detail, Figure 3 traces the progression of the relevant state signals of agents in the queue $X_0O_1O_2O_3O_4O_5O_6$ for the case in which the sensed value of $x[k]$ is randomly skewed in the manner described above. Finally, to study the ramifications of a human actuator being unable to issue signals with absolute precision, column 5 of Table I lists the value of $a_n[n]$ when $u[0]$ is randomly altered by ± 1 percent from its true value. For the simulation values used, Table I suggests an increase in n tends to cause an increase in the magnitude of $u[0]$ and, additionally, the control law becomes dramatically more sensitive to sensor and actuator noise.

VII. Conclusions and Future Directions

In this paper we proposed a causal, state-feedback control law that stabilizes the highly coupled social dynamics present in queues. Simulations attest to the functionality of the approach in ideal conditions but indicate that, in the presence of system noise, performance may suffer dramatically as the queue length increases. There are a number of avenues in which to build upon the results discussed in this paper. First, from a pragmatic standpoint it would be worthwhile to consider control schemes that do not require complete knowledge of the state $x[k]$. Second, it may also prove lucrative to consider queue stability from a Lypaunov perspective. Third, in light of the limitations incurred using a single control agent to stabilize large queues, it is natural to pursue schemes involving multiple control agents in hopes of addressing these shortcomings. Presently, we are investigating cooperative information-sharing protocols between multiple control agents, interspersed throughout a queue, with the goal of improving performance and functionality.

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No. of agents n	$u[0]$	Ideal value of $a_n[n]$	$a_n[n]$ for $\pm 1\%$ error in sensed $x[0]$	$a_n[n]$ for $\pm 1\%$ error in $u[0]$
2	-0.112	0.0	0.000	-0.001
3	-0.285	0.0	-0.003	0.002
4	-0.099	0.0	-0.010	0.000
5	1.046	0.0	0.012	-0.002
6	2.563	0.0	0.021	0.002
7	-9.290	0.0	0.012	0.006
8	-46.565	0.0	-0.130	0.027
9	-150.320	0.0	-0.293	-0.173
10	-256.583	0.0	-4.968	1.004
11	-302.837	0.0	-54.765	7.291

TABLE I

Data illustrating that, as the queue length increases, $u[0]$ tends to grow in magnitude and the control law becomes dramatically more sensitive to noise.

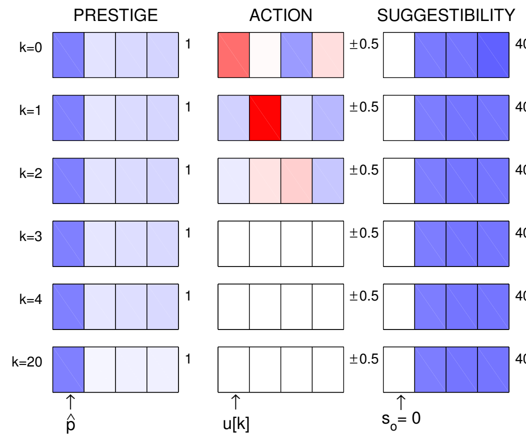

Fig. 1. *Progression of prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3$ subject to the $Q(3)$–stabilizing control law. Various shades of blue and red are used to denote positive and negative values respectively. White denotes a value of zero. The number at the top right of each plot is a color saturation value, e.g., the action plot at $k = 0$ has maximum intensity for any member of the queue possessing an action value with magnitude ≥ 0.5. The components of the control agent's state, including the elements of the stabilizing control signal $u[k]$, are displayed in the leftmost column of each plot.*

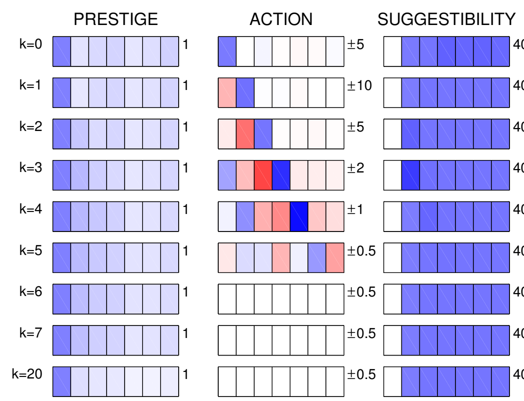

Fig. 2. *Progression of the prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3O_4O_5O_6$ subject to the $Q(6)$–stabilizing control law. The color saturation value is varied with time to provide greater contrast among the colors used to denote state values in each plot.*

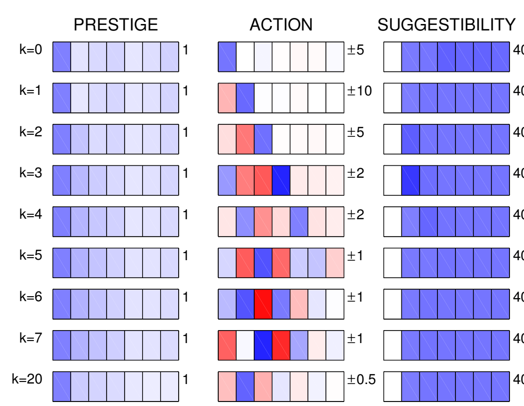

Fig. 3. *Progression of the prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3O_4O_5O_6$ subject to the stabilizing control law (7)–(9) with (8) given by (28) for a random $\pm 1\%$ error in the sensed value of $x[k]$ used to construct $u[k]$.*