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PID control of rigid robots actuated by brushless DC motors

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Abstract— **In this note we are concerned with PID control of rigid robots equipped with brushless DC (BLDC) motors when the electric dynamics of these actuators is taken into account. We show that a PID controller suffices to achieve local stability whereas an adaptive PD controller yields stability and global convergence to the desired link positions. We show that the effect of the adaptive part can be rendered arbitrarily small and, hence, virtually the PD controller suffices to achieve the reported global results. We present a theoretical justification for the torque control or current control strategy commonly used in practice to control BLDC motors.**

I. INTRODUCTION

It is widely recognized at present that use of brushless DC (BLDC) motors as actuators in robotics presents a number of advantages with respect to use of brushed DC motors [1],[2],[3],[4]. However, it is also known that control of BLDC motors is more complicated because of the nonlinear and multivariable nature of their model.

Some control schemes have been presented until now for rigid robots actuated by BLDC motors when their electric dynamics is taken into account [1],[2], [3], [4]. However, the mathematical complexity of the BLDC motors model has deviated attention of these works towards the design of complicated nonlinear controllers. It is recognized in [5] pp. 257, 395, 403 that complex control laws increase sensibility to numerical errors and produce input voltage saturation as well as noise amplification in practice. On the other hand, no result has been presented until now for the stability analysis of PID control for robots equipped with BLDC motors when the electric dynamics of these actuators is taken into account, even for regulation tasks. In the present note we are concerned with the analysis and design of this control problem.

Our contribution is presented in two main results. In our first result we show that controller in [6], when linear feedback of electric current is added, ensures local asymptotic stability without requiring the exact knowledge of neither robot not actuator parameters. We stress that this is the first time that a PID controller is shown to achieve stability for robots actuated by BLDC motors.

In our second result we succeed to ensure stability and global convergence to the desired constant link positions when an adaptive PD controller is used. Thanks to adaptation this controller only needs to know exactly the robot's gravitational effects term and the motor's torque constant. We ensure robustness with respect to possible numerical errors and noise amplification introduced when the nonlinear high order terms present in the adaptation law are computed. Further, we also find for the first time, theoretical evidence suggesting that a linear PD controller, implemented by means of the common industrial practice known as torque control [7], [8], suffices to control globally robots equipped with BLDC motors.

This note is organized as follows. In section II we present the dynamic model of rigid robots actuated by BLDC motors. Sections III and IV are devoted to present our main results whereas some conclusions are given in section V.

Finally, some remarks on notation. We use $\lambda_{min}(A(x))$ and $\lambda_{max}(A(x))$ to represent, respectively, the smallest and the largest eigenvalues of the symmetric positive definite matrix $A(x)$, for any $x \in \mathcal{R}^n$. Given an $x \in \mathcal{R}^n$ and a matrix $A(x)$ the norm of x is defined as $\|x\| = \sqrt{x^T x}$ and the spectral norm of $A(x)$ is defined as $\|A\| = \sqrt{\lambda_{max}(A^T A)}$ which implies $\|A\| = \max_i |\lambda_i(A(x))|$, where $|\cdot|$ stands for the absolute value function, if $A(x)$ is a symmetric matrix. Symbol $p = (d/dt)$ denotes the differential operator.

II. DYNAMIC MODEL OF ROBOTS WITH BLDC MOTORS

The dynamic model of an n degrees of freedom rigid robot equipped with a direct-drive BLDC motor at each joint is given as [1], [9]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + F\dot{q} = [K_{T1}I_B + K_{T2}]I_a \quad (1)$$

$$L_a\dot{I}_a + R_aI_a + N_pL_bI_B\dot{q} + K_{T2}\dot{q} = V_a \quad (2)$$

$$L_b\dot{I}_b + R_bI_b - N_pL_aI_A\dot{q} = V_b \quad (3)$$

where:

$$K_{T1} = N_p(L_b - L_a), \quad K_{T2} = \sqrt{\frac{3}{2}}N_pK_B \quad (4)$$

$$V_a = [v_{a1}, v_{a2}, \ldots, v_{an}]^T \in R^n$$
$$V_b = [v_{b1}, v_{b2}, \ldots, v_{bn}]^T \in R^n$$
$$I_a = [i_{a1}, i_{a2}, \ldots, i_{an}]^T \in R^n$$
$$I_b = [i_{b1}, i_{b2}, \ldots, i_{bn}]^T \in R^n$$
$$I_A = diag\{i_{a1}, i_{a2}, \ldots, i_{an}\} \in R^{n\times n}$$
$$I_B = diag\{i_{b1}, i_{b2}, \ldots, i_{bn}\} \in R^{n\times n}$$

Link positions are represented by $q \in \mathcal{R}^n$, $M(q)$ is the $n \times n$ symmetric positive definite inertia matrix, $C(q,\dot{q})\dot{q}$ is the centripetal and Coriolis term, $g(q) = \frac{\partial U(q)}{\partial q}$ is the gravity effects term, where $U(q)$ is a scalar valued function representing the potential energy, and F is a $n \times n$ constant

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diagonal positive definite matrix representing the viscous friction coefficients at each joint. Throughout this note we use $\tilde{q} = q - q_d$ to represent the position error where $q_d \in \mathcal{R}^n$ represents the constant desired link positions. We also assume that robot under study is equipped only with revolute joints.

Model (1)-(4) is obtained after a DQ (Park's) transformation is applied on the original Y-connected 3-phase model of each motor [9], [3], [10]. Thus, V_a and V_b represent, respectively, the DQ transformed phase voltages associated with each motor. I_a and I_b are electric currents defined correspondingly. L_a, L_b, R_a, R_b, N_p, K_B are constant, diagonal, positive definite matrices. We refer to [1], [9] for a complete description of these matrices. Finally, K_{T1} and K_{T2} are diagonal torque constant matrices whereas $\tau = [K_{T1}I_B + K_{T2}]I_a$ is torque applied at robot joints.

On the other hand, as it is by now well known, some important properties of the mechanical part (1) when all joints are revolute, i.e. the class of robots that we consider, are the following.

Property 1. [11], [12] pp. 96. Matrices $M(q)$ and $C(q,\dot{q})$ satisfy $0 < \lambda_{min}(M(q))$, $\lambda_{max}(M(q)) < \beta$, $\forall q \in \mathcal{R}^n$, where β is a finite positive constant scalar, and:

$$\dot{q}^T\left(\frac{1}{2}\dot{M}(q) - C(q,\dot{q})\right)\dot{q} = 0, \quad \forall \dot{q} \in \mathcal{R}^n \tag{5}$$

$$\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q}) \tag{6}$$

Property 2. [11], [13], [12] pp. 102. There exist positive constants k_g, k' and k_c such that for all $w, y, z, q \in \mathcal{R}^n$, we have:

$$\|C(w,y)z\| \leq k_c\|y\|\|z\| \tag{7}$$

$$\left\|\frac{\partial g(q)}{\partial q}\right\| < k_g, \quad \|g(q)\| \leq k' \tag{8}$$

$$\|g(w) - g(y)\| \leq k_g\|w - y\| \tag{9}$$

Property 3. [13] For any constant vector $q_d \in \mathcal{R}^n$, the following function is positive definite and radially unbounded with respect to $\tilde{q} \in \mathcal{R}^n$:

$$U(q_d - \tilde{q}) - U(q_d) - \tilde{q}^T g(q_d) + \frac{k_g}{2}\|\tilde{q}\|^2 \tag{10}$$

Finally, we list some well known properties of the spectral norm. Let $w, y \in \mathcal{R}^n$ be two vectors and let $B(x)$ and $D(x)$ be two $n \times n$ matrices, the former being symmetric and positive definite $\forall x \in \mathcal{R}^n$, then:

$$\pm y^T D(x)w \leq \|y\|\|D(x)\|\|w\| \tag{11}$$

$$\pm y^T B(x)w \leq \|y\|\|B(x)\|\|w\| = \lambda_{max}(B(x))\|y\|\|w\| \tag{12}$$

$$y^T B(x)y \geq \lambda_{min}(B(x))\|y\|^2 \tag{13}$$

$$\|D(x)B(x)\| \leq \|D(x)\|\|B(x)\| \tag{14}$$

III. A LOCAL PID CONTROLLER

In this section we present a PID controller, inspired by [6], which achieves local asymptotic stability. However, instead of the saturation functions introduced in that paper we prefer to use the function introduced in [14], and refined in [13], which, as we show below, has the same useful properties reported in [6]. Define the following scalar potential function:

$$Cos(u) = \begin{cases} 1 - \cos(u), & if\ |u| < \frac{\pi}{2} \\ u - (\pi/2 - 1), & if\ u \geq \frac{\pi}{2} \\ -u - (\pi/2 - 1), & if\ u \leq -\frac{\pi}{2} \end{cases} \tag{15}$$

for $u \in \mathcal{R}$. The first derivative of $Cos(u)$ with respect to u can be expressed as:

$$s(u) = \begin{cases} \sin(u), & if\ |u| < \frac{\pi}{2} \\ 1, & if\ u \geq \frac{\pi}{2} \\ -1, & if\ u \leq -\frac{\pi}{2} \end{cases} \tag{16}$$

Functions $Cos(u)$ and $s(u)$ in (15) and (16) have the following properties:

Property 4. Function $Cos(u)$ is twice continuously differentiable and $Cos(u) > 0,\ \forall u \neq 0$ whereas $Cos(u) = 0$ for $u = 0$.

Property 5. The following properties are adaptations of properties listed in [13]:

$$|u| \geq |s(u)| \geq k_a|u|,\ \forall u \in \mathcal{R}:\ |u| < \xi \tag{17}$$

$$|u| \geq |s(u)| \geq k_a\xi,\ \forall u \in \mathcal{R}:\ |u| \geq \xi \tag{18}$$

$$1 \geq (d/du)s(u) \geq 0 \tag{19}$$

where $\xi = 1$ and $k_a = \sin(\xi) = 0.841$.

Property 6. There is a constant $b > 0$ such that:

$$Cos(u) \geq bs^2(u) > 0,\ \forall u \neq 0 \tag{20}$$

Property 7. There is a constant $k > 0$ such that:

$$u^2 \geq kCos(u) > 0,\ \forall u \neq 0 \tag{21}$$

Property 8.

$$U(q) - U(q_d) - \tilde{q}^T g(q_d) + \frac{1}{4}\tilde{q}^T[2(k_g I + \Lambda)]\tilde{q} > a\|\tilde{q}\|^2 \geq a\|h(\tilde{q})\|^2 \tag{22}$$
$$h(\tilde{q}) = [s(\tilde{q}_1), s(\tilde{q}_2), \ldots, s(\tilde{q}_n)]^T$$

where $a = \frac{1}{2}\lambda_{min}(\Lambda)$ and I, Λ are, respectively, the identity matrix and a diagonal positive definite matrix, both of them $n \times n$ matrices.

Property 9. The following bound holds for all $\tilde{q} \in \mathcal{R}^n$:

$$\|g(q) - g(q_d)\| \leq \frac{k_{h2}}{k_a}\|h(\tilde{q})\| \tag{23}$$

where k_{h2} is any number satisfying $k_{h2} \geq \frac{2k'}{s(2k'/k_g)}$.

Property 4 is obvious. Property 7 can be proven as follows. Both functions involved in (21) are zero at $u = 0$. Hence, (21) is true for $u \geq 0$ if $(d/du)[u^2] \geq (d/du)[kCos(u)]$, $\forall u \geq 0$. From this condition and the facts that $|u| \geq |s(u)|$ and that both functions in (21) are symmetric with respect to $u = 0$ we find that (21) is true with $k = 2$. Property 6 is proven to hold with $b = 0.5$ proceeding similarly by considering that $(d/du)[Cos(u)] \geq (d/du)[bs^2(u)]$ for $u \geq 0$ if $(d^2/du^2)[Cos(u)] \geq (d^2/du^2)[bs^2(u)]$ for $u \geq 0$. Property 8 is readily obtained using again the fact that

$|u| \geq |s(u)|$ and (10). Property 9 is proven as follows. Using property 5 we obtain:

$$\|h(\tilde{q})\| \geq \begin{cases} k_a\|\tilde{q}\|, & if\ \|\tilde{q}\| < \xi \\ k_a, & if\ \|\tilde{q}\| \geq \xi \end{cases} \quad (24)$$

$$\|h(\tilde{q})\| \leq \begin{cases} \|\tilde{q}\|, & if\ \|\tilde{q}\| < \xi \\ \sqrt{n}, & if\ \|\tilde{q}\| \geq \xi \end{cases} \quad (25)$$

$$s(\|\tilde{q}\|) \leq \frac{1}{k_a}\|h(\tilde{q})\| \quad (26)$$

On the other hand, proceeding as in [12], pp. 105-107, we obtain:

$$\|g(q) - g(q_d)\| \leq k_{h2} s(\|\tilde{q}\|) \quad (27)$$

Finally, using (26) and (27) we obtain (23).

Proposition. 1: Consider the dynamic model (1), (2), (3) together with the following PID controller:

$$\begin{aligned} V_a &= -r_a I_a - (\ \overline{K_P} + \overline{K_I}\ )\ \tilde{q} - \overline{K_D}\ \vartheta \\ &\quad -\overline{K_I}\int_0^t \varepsilon_0 h(\tilde{q}(r))\ dr \quad (28) \\ V_b &= -r_b I_b \quad (29) \\ \vartheta &= diag\left\{\frac{b_i p}{p + a_i}\right\} q \quad (30) \end{aligned}$$

where $A = diag\{a_i\}$, $B = diag\{b_i\}$ are $n \times n$ diagonal positive definite matrices and $h(\cdot)$ is defined in property 8. There always exist a constant scalar $\varepsilon_0 > 0$ and $n \times n$ diagonal positive definite matrices $\overline{K_P}$, $\overline{K_D}$, $\overline{K_I}$, r_a, r_b such that the closed loop system has a unique equilibrium point, where $\tilde{q} = 0$, which is locally asymptotically stable.

Poof. Define $\rho = I_a - R^{-1}\Big(-(\ \overline{K_P} + \overline{K_I}\ )\ \tilde{q} - \overline{K_D}\ \vartheta - \overline{K_I}\int_0^t \varepsilon_0 h(\tilde{q}(r))\ dr\Big)$, where $R = R_a + r_a$. Note that $\dot{\vartheta} = -A\vartheta + B\dot{q}$, is a realization of filter (30). Using these expressions and replacing (28) in (2) we can write:

$$\begin{aligned} L_a\dot{\rho} &= -R\rho - N_p L_b I_B \dot{q} - K_{T2}\dot{q} \\ &+ L_a R^{-1}(\overline{K_P} + \overline{K_I} + \overline{K_D}B)\dot{q} - L_a R^{-1}\overline{K_D}A\vartheta \\ &+ \varepsilon_0 L_a R^{-1}\overline{K_I}h(\tilde{q}) \quad (31) \end{aligned}$$

Now, define $\delta_a = [\delta_{a1}, \delta_{a2}, \ldots, \delta_{an}]^T \in \mathcal{R}^n$ as:

$$\delta_a = -(\ \overline{K_P} + \overline{K_I}\ )\ \tilde{q} - \overline{K_D}\ \vartheta - \overline{K_I}\int_0^t \varepsilon_0 h(\tilde{q}(r))\ dr$$

Replacing (29) in (3) and using the definition of ρ we can write:

$$L_b\dot{I}_b = -\bar{R}I_b + N_p L_a \dot{Q}\rho + N_p L_a \dot{Q} R^{-1}\delta_a \quad (32)$$

where we have defined $\dot{Q} = diag\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\} \in \mathcal{R}^{n\times n}$ and $\bar{R} = R_b + r_b$. Note that we can write:

$$K_{T2}R^{-1}\delta_a = -K_P\ \tilde{q} - K_D\ \vartheta - K_I\ z + g(q_d) \quad (33)$$

by defining:

$$\begin{aligned} K_P &= K_{T2}R^{-1}\overline{K_P} \quad (34) \\ K_D &= K_{T2}R^{-1}\overline{K_D} \quad (35) \\ K_I &= K_{T2}R^{-1}\overline{K_I} \quad (36) \\ z &= \tilde{q} + \int_0^t \varepsilon_0 h(\tilde{q}(r))dr + (K_I)^{-1}g(q_d) \quad (37) \end{aligned}$$

Thus, replacing (33) in (32):

$$L_b\dot{I}_b = -\bar{R}I_b + N_p L_a \dot{Q}\rho + N_p L_a \dot{Q} K_{T2}^{-1}\delta_a^* \quad (38)$$

where $\delta_a^* = [-K_P\tilde{q} - K_D\vartheta - K_I z + g(q_d)]$. On the other hand, (1) can be written as:

$$\begin{aligned} &M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + F\dot{q} \\ &= [K_{T1}I_B + K_{T2}]\rho + K_{T1}I_B R^{-1}\delta_a + K_{T2}R^{-1}\delta_a \quad (39) \end{aligned}$$

Using (33) we can write (39) as:

$$\begin{aligned} &M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - g(q_d) + F\dot{q} \\ &= [K_{T1}I_B + K_{T2}]\rho - K_P\tilde{q} - K_D\vartheta - K_I\ z \\ &\quad + K_{T1}I_B K_{T2}^{-1}\delta_a^* \quad (40) \end{aligned}$$

Thus, the closed loop dynamics is given by (40), (31), (38) together with:

$$\dot{z} = \dot{q} + \varepsilon_0 h(\tilde{q}), \quad \dot{\vartheta} = -A\vartheta + B\dot{q} \quad (41)$$

Note that $(\tilde{q}, \dot{q}, z, \vartheta, \rho, I_b) = (0,0,0,0,0,0)$ is the unique equilibrium point of the closed loop dynamics (40), (31), (38), (41). Now, we proceed to study the stability of this equilibrium point. In [6] it was proven, by means of properties 6, 7 and 8, that the following function is positive definite and radially unbounded:

$$\begin{aligned} V_1(\tilde{q}, \dot{q}, z, \vartheta) &= \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P\tilde{q} \quad (42) \\ &+ \varepsilon_0 h^T(\tilde{q})M(q)\dot{q} + \sum_{i=1}^n \varepsilon_0 f_i Cos(\tilde{q}_i) + \frac{1}{2}z^T K_I z \\ &+ U(q) - U(q_d) - \tilde{q}^T g(q_d) + \frac{1}{2}\vartheta^T K_D B^{-1}\vartheta \end{aligned}$$

where f_i stands for the diagonal entries of matrix F, if:

$$\begin{aligned} \lambda_{min}(K_P) &\geq \frac{4\varepsilon_0^2}{b\ k}\lambda_{max}(M(q)) \quad (43) \\ \lambda_{min}(K_P) &> 2(k_g + \lambda_{min}(\Lambda)) \quad (44) \end{aligned}$$

for b, k and Λ defined in (20)-(22). Thus, the following scalar function qualifies as a Lyapunov function candidate:

$$\begin{aligned} W(\tilde{q}, \dot{q}, z, \vartheta, \rho, I_b) &= V_1(\tilde{q}, \dot{q}, z, \vartheta) + V_2(\rho) + V_3(I_b) \\ V_2(\rho) = \frac{1}{2}\rho^T L_a\rho, &\qquad V_3(I_b) = \frac{1}{2}I_b^T L_b I_b \quad (45) \end{aligned}$$

Using (4) and the diagonal nature of all the involved matrices, we have $\dot{q}^T K_{T1}I_B\rho - \rho^T N_p L_b I_B\dot{q} + I_b^T N_p L_a\dot{Q}\rho = 0$ and $\dot{q}^T K_{T1}I_B K_{T2}^{-1}\delta_a^* + I_b^T N_p L_a \dot{Q} K_{T2}^{-1}\delta_a^* = I_b^T N_p L_b \dot{Q} K_{T2}^{-1}\delta_a^*$. These facts as well as (5), (6) and $g(q) = \frac{\partial U(q)}{\partial q}$ allow to find the following time derivative along the trajectories of dynamics (40), (31), (38), (41):

$$
\begin{aligned}
\dot{W} = & -\dot{q}^T\left[F-\varepsilon_0\frac{\partial h(\tilde{q})}{\partial\tilde{q}}M(q)\right]\dot{q}\\
& +\varepsilon_0 h^T(\tilde{q})C^T(q,\dot{q})\dot{q}+\varepsilon_0 h^T(\tilde{q})(g(q_d)-g(q))\\
& +\varepsilon_0 h^T(\tilde{q})K_{T2}\rho-\varepsilon_0 h^T(\tilde{q})K_P\tilde{q} \qquad (46)\\
& -\varepsilon_0 h^T(\tilde{q})K_D\vartheta-I_b^T\bar{R}I_b-\vartheta^T K_D B^{-1}A\vartheta\\
& -\rho^T R\rho+\rho^T L_a K_{T2}^{-1}[(K_P+K_I+K_D B)\dot{q}\\
& -K_D A\vartheta+\varepsilon_0 K_I h(\tilde{q})]+G,\\
G = & \dot{q}^T N_p L_b I_B K_{T2}^{-1}\delta_a^*+\varepsilon_0 h^T(\tilde{q})K_{T1}I_B\rho\\
& +\varepsilon_0 h^T(\tilde{q})K_{T1}I_B K_{T2}^{-1}\delta_a^*
\end{aligned}
$$

According to (19), $\frac{\partial h(\tilde{q})}{\partial\tilde{q}}$ is a diagonal matrix whose entries are nonnegative and smaller than or equal to 1. On the other hand, using property 9 and taking advantage of the facts that K_P is a diagonal matrix and that $|u|\geq|s(u)|$ we can write $\varepsilon_0 h^T(\tilde{q})(g(q)-g(q_d))+\varepsilon_0 h^T(\tilde{q})K_P\tilde{q}\geq\varepsilon_0[-\frac{k_{h2}}{k_a}+\lambda_{min}(K_P)]\|h(\tilde{q})\|^2$. As proposed in [6], the following condition is important for our purposes:

$$
-\frac{k_{h2}}{k_a}+\lambda_{min}(K_P)\geq a+\frac{1}{2}\lambda_{max}(K_D) \qquad (47)
$$

Also note that, according to (25), we can bound $\|h(x)\|\leq\sqrt{n},\ \forall x\in\mathcal{R}^n$. Thus, from (7) we obtain that $\varepsilon_0 h(\tilde{q})^T C^T(q,\dot{q})\dot{q}\leq\varepsilon_0\|h(\tilde{q})\|\|C^T(q,\dot{q})\dot{q}\|\leq\varepsilon_0\sqrt{n}k_c\|\dot{q}\|^2$. Finally, from $(\|h(\tilde{q})\|-\|\vartheta\|)^2\geq 0$ we obtain $\|h(\tilde{q})\|^2+\|\vartheta\|^2\geq 2\|h(\tilde{q})\|\ \|\vartheta\|$. Inspired by [6] we can use these facts as well as (11)-(14) to obtain:

$$
\dot{W}\leq -\begin{bmatrix}\|\dot{q}\|\\ \|h(\tilde{q})\|\\ \|\vartheta\|\\ \|\rho\|\\ \|I_b\|\end{bmatrix}^T P\begin{bmatrix}\|\dot{q}\|\\ \|h(\tilde{q})\|\\ \|\vartheta\|\\ \|\rho\|\\ \|I_b\|\end{bmatrix} \qquad (48)
$$

where entries of matrix P are:

$$
\begin{aligned}
P_{11} &= \lambda_{min}(F)-\varepsilon_0[\lambda_{max}(M(q))+\sqrt{n}k_c]\\
P_{22} &= a\varepsilon_0\\
P_{33} &= \lambda_{min}(K_D B^{-1}A)-\frac{\varepsilon_0}{2}\lambda_{max}(K_D)\\
P_{44} &= \lambda_{min}(R),\ P_{55}=\lambda_{min}(\bar{R})\\
P_{12} &= P_{21}=P_{13}=P_{31}=P_{45}=P_{54}=0\\
P_{14} &= P_{41}=-\frac{1}{2}\lambda_{max}(L_a K_{T2}^{-1}[K_P+K_I+K_D B])\\
P_{15} &= P_{51}=-\frac{1}{2}\lambda_{max}(N_p L_b K_{T2}^{-1})\|g(q_d)\|\\
& -\frac{1}{2}\lambda_{max}(N_p L_b K_{T2}^{-1}K_I)\|z\|\\
& -\frac{1}{2}\lambda_{max}(N_p L_b K_{T2}^{-1}K_P)\|\tilde{q}\|\\
P_{23} &= P_{32}=-\frac{1}{2}\varepsilon_0\lambda_{max}(K_{T2}^{-1}K_D)\\
& \times\max_i|\lambda_i(K_{T1})|\ \|I_b\|\\
P_{24} &= P_{42}=-\frac{\varepsilon_0}{2}\lambda_{max}(K_{T2})\\
& -\frac{\varepsilon_0}{2}\lambda_{max}(L_a K_{T2}^{-1}K_I)\\
& -\frac{\varepsilon_0}{2}\max_i|\lambda_i(K_{T1})|\ \|I_b\|\\
P_{25} &= P_{52}=\varepsilon_0\max_i|\lambda_i(K_{T1})|\times\\
& \Big[-\frac{1}{2}\lambda_{max}(K_{T2}^{-1})\|g(q_d)\|\\
& -\frac{1}{2}\lambda_{max}(K_{T2}^{-1}K_I)\|z\|\\
& -\frac{1}{2}\lambda_{max}(K_{T2}^{-1}K_P)\|\tilde{q}\|\Big]\\
P_{34} &= P_{43}=-\frac{1}{2}\lambda_{max}(L_a K_{T2}^{-1}K_D A)\\
P_{35} &= P_{53}=-\frac{1}{2}\lambda_{max}(N_p L_b K_{T2}^{-1}K_D)\|\dot{q}\|
\end{aligned}
$$

Matrix P is positive definite if and only if:

$$
\begin{aligned}
P_{11} &> 0,\quad P_{22}>0 && (49)\\
\delta_3 &= P_{33}P_{22}P_{11}-P_{23}P_{11}P_{32}>0 && (50)\\
\delta_4 &= P_{44}\delta_3-P_{14}[P_{41}(P_{22}P_{33}-P_{23}P_{32})]\\
& +P_{24}[P_{11}(P_{32}P_{43}-P_{42}P_{33})]\\
& -P_{34}[P_{11}(P_{22}P_{43}-P_{42}P_{23})]>0 && (51)\\
\delta_4 P_{55} &+ P_{15}\bar{P}_{15}-P_{25}\bar{P}_{25}+P_{35}\bar{P}_{35}>0 && (52)
\end{aligned}
$$

where $\bar{P}_{15}$, $\bar{P}_{25}$, $\bar{P}_{35}$ are factors which can be readily obtained from matrix P. Conditions in (49) are always satisfied by choosing a small $\varepsilon_0>0$ and a positive a. Note that $P_{33}>0$ is ensured by selecting suitable positive definite matrices A, B, K_D and a small $\varepsilon_0>0$. On the other hand, from property 8 and (44) we realize that a large a is obtained by means of a large positive definite K_P. Hence, this selection of gains suffices to satisfy (50) because any of P_{23}, P_{11}, P_{32} do not grow when K_P is enlarged. However, it is important to see that this is possible only for small values of $\|I_b\|$. Condition (51) is always satisfied by choosing a large matrix R, i.e. by means of a large r_a, because only the product $P_{44}\delta_3$ grows as R is enlarged. Note, however, that this also requires $\|I_b\|$ to be small. Finally, expressions for the second, third and fourth terms in (52) are cumbersome to be written in this note. However, the reader can compute them from matrix P in a rather easy manner to verify that any of them do not grow as $\bar{R}$ grows. Hence, it is always possible to choose a large $\bar{R}$, i.e. a large r_b, to ensure that product $\delta_4 P_{55}$ dominates all the other terms to render (52) true. Also note that this requires $\|I_b\|$ $\|\dot{q}\|$, $\|\tilde{q}\|$ and $\|z\|$ to be small. Hence, we conclude that $\dot{W}$, given in (48), can always be rendered locally negative semidefinite by choosing a small $\varepsilon_0>0$ and suitable positive definite matrices K_P, K_D, K_I, B, A, r_a and r_b. This, together with the positive definiteness of W ensure, by means of the LaSalle invariance principle, local asymptotic stability of $(\tilde{q},\dot{q},z,\vartheta,\rho,I_b)=(0,0,0,0,0,0)$. This completes the proof of proposition 1.

Remark. 1: In spite of its locality, result in proposition 1 is important because it shows that a PID controller plus linear feedback of electric current suffices to locally regulate position in rigid robots equipped with BLDC motors as actuators. This is the first time that such a result is presented in the case when the electric dynamics of such actuators is taken into account.

IV. An adaptive PD controller

In the following proposition we present an adaptive PD controller which ensures global convergence to the desired link positions.

Proposition. 2: Consider the dynamic model (1), (2), (3) together with the control law:

$$\begin{aligned} V_a &= -r_a I_a - \overline{K_P}\,\tilde{q} - \overline{K_D}\,\vartheta \\ &\quad + RK_{T2}^{-1}g(q_d) \qquad (53) \\ V_b &= -\dot{Q}\bar{\Delta}_a\widehat{\theta}_1 - \varepsilon\tilde{Q}I_A\widehat{\theta}_2 \qquad (54) \\ \frac{d}{dt}\begin{bmatrix}\widehat{\theta}_1\\ \widehat{\theta}_2\end{bmatrix} &= \Gamma\begin{bmatrix} I_B\,\dot{Q}\,\bar{\delta}_a \\ \varepsilon I_B\tilde{Q}\,I_a\end{bmatrix} \qquad (55) \\ \varepsilon &= \frac{\varepsilon_0}{1+\|\tilde{q}\|}, \quad R = R_a + r_a \\ \vartheta &= diag\left\{\frac{b_i p}{p+a_i}\right\}q \qquad (56) \\ \tilde{Q} &= diag\{\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n\} \in \mathcal{R}^{n\times n} \\ \bar{\Delta}_a &= diag\{\bar{\delta}_{a1}, \bar{\delta}_{a2}, \ldots, \bar{\delta}_{an}\} \in \mathcal{R}^{n\times n} \qquad (57) \end{aligned}$$

where $\bar{\delta}_a = -\overline{K_P}\;\tilde{q} - \overline{K_D}\;\vartheta + RK_{T2}^{-1}g(q_d) = [\bar{\delta}_{a1}, \bar{\delta}_{a2}, \ldots, \bar{\delta}_{an}]^T \in \mathcal{R}^n$, $A = diag\{a_i\}$ and $B = diag\{b_i\}$ are $n\times n$ diagonal positive definite matrices, Γ is an arbitrary $2n\times 2n$ diagonal positive definite matrix, $\widehat{\theta}_1$, $\widehat{\theta}_2$ are the estimates of parameters defined as:

$$\begin{aligned} \theta_1^* &= \left[\frac{N_{p1}L_{b1}}{R_1}, \ldots, \frac{N_{pn}L_{bn}}{R_n}\right]^T \in \mathcal{R}^n \qquad (58) \\ \theta_2^* &= [K_{T1_1}, \ldots, K_{T1_n}]^T \in \mathcal{R}^n \end{aligned}$$

where subindex indicates a diagonal entry of the corresponding matrix. There always exist diagonal positive definite matrices $\overline{K_P}$, $\overline{K_D}$, r_a and a constant scalar $\varepsilon_0 > 0$ such that the closed loop system has an equilibrium point where $\tilde{q} = 0$ which is stable and global convergence $\lim_{t\to\infty} q(t) = q_d$ is ensured.

Proof of this proposition follows proceeding as in proof of proposition 1 by using $\varrho = I_a - R^{-1}\bar{\delta}_a$, instead of ρ, $\tilde{\theta} = \widehat{\theta} - \theta^* = [(\widehat{\theta}_1 - \theta_1^*)^T, (\widehat{\theta}_2 - \theta_2^*)^T]^T$ and using the positive definite and radially unbounded Lyapunov function:

$$\begin{aligned} \nu(\tilde{q}, \dot{q}, \varrho, I_b, \vartheta, \tilde{\theta}) &= \nu_1(\tilde{q}, \dot{q}) + \nu_2(\vartheta, \tilde{\theta}) + \nu_3(\varrho, I_b) \\ \nu_1(\tilde{q}, \dot{q}) &= \frac{1}{2}\dot{q}^T M(q)\dot{q} - U(q_d) - \tilde{q}^T g(q_d) \\ +(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}) &\quad \tilde{q}^T K_P \tilde{q} + \varepsilon\tilde{q}^T M(q)\dot{q} + U(q) \\ \nu_2(\vartheta, \tilde{\theta}) &= \frac{1}{2}\vartheta^T K_D B^{-1}\vartheta + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \\ \nu_3(\varrho, I_b) &= \frac{1}{2}\varrho^T L_a \varrho + \frac{1}{2}I_b^T L_b I_b \end{aligned}$$

where K_P and K_D are defined as in (34), (35) and conditions (7), (8), (9) given in [15] are satisfied, to find that:

$$\dot{\nu} \le -\begin{bmatrix}\|\dot{q}\|\\ \|\vartheta\|\\ \|\varrho\|\\ \|I_b\|\end{bmatrix}^T E \begin{bmatrix}\|\dot{q}\|\\ \|\vartheta\|\\ \|\varrho\|\\ \|I_b\|\end{bmatrix} - \varepsilon\begin{bmatrix}\|\vartheta\|\\ \|\tilde{q}\|\\ \|\varrho\|\end{bmatrix}^T \bar{A}\begin{bmatrix}\|\vartheta\|\\ \|\tilde{q}\|\\ \|\varrho\|\end{bmatrix}$$

where:

$$\begin{aligned} E_{11} &= \lambda_{min}(F) - \varepsilon_0(k_c + 2\lambda_{max}(M(q))) \\ E_{22} &= \frac{1}{2}\lambda_{min}(K_D B^{-1}A), \quad E_{31} = E_{13} \\ E_{33} &= \frac{1}{2}\lambda_{min}(R), \quad E_{12} = E_{21} = E_{14} = E_{41} = 0 \\ E_{44} &= \lambda_{min}(R_b), \quad E_{42} = E_{24} = E_{34} = E_{43} = 0 \\ E_{13} &= -\frac{1}{2}[\lambda_{max}(L_a K_{T2}^{-1}K_P) + \lambda_{max}(L_a K_{T2}^{-1}K_D B)] \\ E_{32} &= E_{23} = -\frac{1}{2}[\lambda_{max}(L_a K_{T2}^{-1}K_D A)] \\ \bar{A}_{11} &= \frac{1}{2\varepsilon_0}\lambda_{min}(K_D B^{-1}A), \quad \bar{A}_{13} = \bar{A}_{31} = 0 \\ \bar{A}_{22} &= \lambda_{min}(K_P) - k_g, \quad \bar{A}_{33} = \frac{1}{2\varepsilon_0}\lambda_{min}(R) \\ \bar{A}_{23} &= \bar{A}_{32} = -\frac{1}{2}\lambda_{max}(K_{T2}), \;\; \bar{A}_{12} = -\frac{1}{2}\lambda_{max}(K_D) \end{aligned}$$

and $\bar{A}_{21} = \bar{A}_{12}$. Matrices E and $\bar{A}$ are positive definite if:

$$\begin{aligned} E_{11} > 0,\; E_{22} > 0,\; E_{33} > 0,\; E_{44} &> 0 \qquad (59) \\ E_{11}E_{22}E_{33} - E_{13}E_{22}E_{31} - E_{23}E_{11}E_{32} &> 0 \\ \bar{A}_{11} > 0,\; \sigma = \bar{A}_{11}\bar{A}_{22} - \bar{A}_{21}\bar{A}_{12} &> 0 \\ \sigma\bar{A}_{33} - \bar{A}_{23}\bar{A}_{11}\bar{A}_{32} &> 0 \end{aligned}$$

According to (34), (35) given any R it is always possible to adjust $\overline{K_P}$ and $\overline{K_D}$ to maintain the desired values of both K_P and K_D. Hence, only E_{33} and $\bar{A}_{33}$ grow as R grows. Thus, all of the previous inequalities can always be satisfied by using suitable matrices $\overline{K_P}$, $\overline{K_D}$, B, A, a small $\varepsilon_0 > 0$ and a large R, i.e. a large r_a. Thus, $\dot{\nu}$ can always be rendered globally negative semidefinite. This, together with the global positive definiteness and radial unboundedness of ν ensure stability of $(\tilde{q}, \dot{q}, \vartheta, \varrho, I_b, \tilde{\theta}) = (0, 0, 0, 0, 0, 0)$, i.e. the whole state is bounded. Convergence $q(t) \to q_d$ as $t \to \infty$ follows using standard adaptive control arguments. From expression for $\dot{\nu}$ we can show that $\tilde{q}$ is square integrable. Recall that $\dot{q}$, the time derivative of $\tilde{q}$, is also bounded. Note that these properties hold globally. Thus, global convergence $q(t) \to q_d$ as $t \to \infty$ is ensured. This completes the proof of proposition 2. Conditions ensuring result in proposition 2 are given by (7), (8), (9) in [15] and conditions (59) in the present note. It is important to say that this result as well as result in proposition 1 are possible thanks to the realistic assumption on viscous friction at robot joints: even if a small viscous friction, F, is present it is enough to choose a small $\varepsilon_0 > 0$. Finally, we stress that proposition 2 is valid for any values of L_a and L_b, i.e. contrary to the common assumption we do not require inductance to be small.

Remark. 2: It is important to say that the adaptive part of the controller, i.e. V_b in (54), has no effect on the final value of $\tilde{q}$. This can be seen from the fact that $(\tilde{q}, \dot{q}, \vartheta, \varrho, I_b, \tilde{\theta}) = (0, 0, 0, 0, 0, \theta_0)$ for any constant $\theta_0 \in \mathcal{R}^{2n}$ qualifies as an equilibrium point of the closed loop dynamics. As a matter of fact $L_b \dot{I}_b = -R_b I_b + N_p L_a \dot{Q} \rho + N_p L_a \dot{Q} K_{T2}^{-1} \bar{\delta}_a^* - \dot{Q} \bar{\Delta}_a \tilde{\theta}_1 - \varepsilon \tilde{Q} I_A \tilde{\theta}_2 - \dot{Q} \bar{\Delta}_a \theta_1^* - \varepsilon \tilde{Q} I_A \theta_2^*$, where $\bar{\delta}_a^* = K_{T2} R^{-1} \bar{\delta}_a$, is the only closed loop equation which is affected by the estimation error $\tilde{\theta}$. On the other hand, note that the adaptive gain matrix Γ is any arbitrary positive definite diagonal matrix. Also note that the global character of controller in proposition 2 allows us to choose any finite initial values for the estimated parameters. Thus, we can always choose $\widehat{\theta_1}(0) = 0$, $\widehat{\theta_2}(0) = 0$ and Γ as a diagonal matrix whose diagonal entries are arbitrarily close to zero. This ensures that V_b, in (54), can always be kept as close to zero as desired to render negligible its effect. This implies robustness with respect to numerical errors and noise amplification as well as avoidance of undesired input voltage saturations which, as pointed out in [5] pp. 257, 395, 403, can be produced when computing the complex high order terms appearing in the adaptive part of the controller, i.e. V_b in (54). Moreover, this also means that virtually only the PD controller plus linear current feedback, given in (53), is applied. This fact, together with the following remark, means that theoretical evidence has been found, for the first time, suggesting that a linear PD implemented by means of the common practice known as torque control [8], [7], suffices to control globally robots equipped with BLDC motors.

Remark. 3: In industrial practice it is common to consider that the torque applied by BLDC motors to robot joints is proportional to current. Further, the drives for those motors include some current controllers ensuring the generation of the desired torque. This is known as torque control or current control [8]. In the following we recall the procedure presented in [7] to implement this strategy for controlling BLDC motors under the assumption that $L_a = L_b$. In such a case torque applied by motors to robot joints is given as $\tau = K_{T2} I_a$ and torque control can be written as:

$$V_a = K_d(I_a^* - I_a) \tag{60}$$

where K_d is a diagonal positive definite matrix and I_a^* represents the value of the electric current I_a necessary to generate the desired torque τ^*, i.e.:

$$I_a^* = K_{T2}^{-1} \tau^* \tag{61}$$

Additionally, $V_b = 0$ is assumed. Suppose that a PD control law is used as the desired torque:

$$\tau^* = -\kappa_p \, \tilde{q} - \kappa_d \, \vartheta + g(q_d) \tag{62}$$

We stress that, in practice [7], it is always chosen $r_a \gg R_a$ and, hence $R \approx r_a$. Thus, V_a given in (53) is retrieved from (60), (61), (62) by setting $r_a = K_d$, $\overline{K_P} = K_d K_{T2}^{-1} \kappa_p$, $\overline{K_D} = K_d K_{T2}^{-1} \kappa_d$ and $K_d K_{T2}^{-1} g(q_d) \approx R K_{T2}^{-1} g(q_d)$. This relaxes the requirement on the exact knowledge of R_a. Aside from these facts, it is important to stress that our result is valid even if $L_a \neq L_b$.

V. Conclusions

We have presented a stability analysis for rigid robots actuated by BLDC motors when the electric dynamics of these actuators is taken into account. We have found, for the first time, theoretical evidence indicating that a linear PD controller suffices to globally regulate position under these conditions. Contrary to the common assumption, our controller does not require inductance to be small. We have presented for the first time a formal stability analysis for torque control. The proposed Lyapunov functions are partitioned into one Lyapunov function for each closed loop dynamical equation. Results in proposition 2 have been recently extended by the authors to design an adaptive PID controller achieving global convergence to the desired positions. This means that the exact knowledge of any robot or actuator parameter is not required. Such a result has been submitted for publication some where else.

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