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IF A RETARDED SYSTEM IS LINEARIZABLE AND STABILIZABLE, THEN IT IS INPUT-TO-STATE STABILIZABLE

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Abstract A disturbance adding to the control law is a typical situation in practice because of actuator errors. In this paper, it is proved that a retarded nonlinear system which is linearizable and stabilizable, is also input-to-state stabilizable.

Keywords: Input-to-State Stabilizability, Retarded Nonlinear Systems, Liapunov-Krasovskii Methodology.

INTRODUCTION

In 1989 Sontag showed in the paper [23] that nonlinear systems which are (smooth) feedback stabilizable, are also (smooth) input-to-state stabilizable with respect to disturbances adding to the control input. As well known, those disturbances are very frequent in practice, because of actuator errors. Many contributions concerning the state feedback stabilization and the input-output state feedback linearization of nonlinear retarded systems can be found in the literature (see, for instance, [3,6,8,12,14,15,17,18,26,30]). Liapunov-Krasovskii methodologies for the input-to-state stability of retarded nonlinear systems have been studied in [10,19,28]. As far as the input-to-state stabilizability of stabilizable retarded nonlinear systems is concerned, a contribution is given in [27], where, besides the main results dealing with the relationship between the input-to-state stability and the exponential stability in the unforced case, the input-to-state stabilizability of retarded nonlinear systems which are transformable by a state feedback control law into a linear, delay-free, exponentially stable system is considered, and the formula for the input-to-state stabilizing state feedback control law is provided. In this paper, on the basis of the converse Liapunov-Krasovskii theorem for linear retarded systems (see [4] and References therein), it is proved that a retarded nonlinear system which is linearizable and stabilizable (i.e., there exists a state feedback control law such that the closed loop system, with disturbance equal to zero, is an asymptotically stable linear system with suitable discrete as well as distributed time delays), is also input-to-state stabilizable (as well known, input-to-state stabilizability implies stabilizability).

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Notations

R denotes the set of real numbers, $R^\star$ denotes the extended real line $[-\infty,+\infty]$, R^+ denotes the set of non negative reals $[0,+\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. A function $v: R^+ \to R^m$, m positive integer, is said to be *essentially bounded* if $ess\sup_{t\geq 0}|v(t)| < \infty$. For given times $0 \leq T_1 < T_2$, we indicate with $v_{[T_1,T_2)}: R^+ \to R^m$ the function given by $v_{[T_1,T_2)}(t) = v(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input v is said to be *locally essentially bounded* if, for any $T > 0$, $v_{[0,T)}$ is essentially bounded. For a positive real Δ, $C([-\Delta, 0]; R^n)$ denotes the space of the continuous functions mapping $[-\Delta, 0]$ into R^n, n positive integer. For positive integers m, n, I_m denotes the identity matrix in $R^{m\times m}$, $0_{m,n}$ denotes a matrix of zeros in $R^{m\times n}$. A functional $F : C([-\Delta, 0]; R^n) \to R^{m\times p}$, m, n, p positive integers, is said to be completely continuous if it is continuous and maps closed bounded sets of $C([-\Delta, 0]; R^n)$ into bounded sets of $R^{m\times p}$. Let us here recall that a function $\gamma : R^+ \to R^+$ is: positive definite if it is continuous, zero at zero and $\gamma(s) > 0$ for all $s > 0$; of class $\mathcal{G}$ if it is continuous, zero at zero, and nondecreasing; of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded; of class $\mathcal{L}$ if it monotonically decreases to zero as its argument tends to $+\infty$. A function $\beta : R^+ \times R^+ \to R^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$. With M_a is indicated any functional mapping $C([-\Delta, 0]; R^n)$ into R^+ such that, for some $\mathcal{K}_\infty$ functions $\gamma_a, \bar{\gamma}_a$, the following inequalities hold

$$\gamma_a(|\phi(0)|) \leq M_a(\phi) \leq \bar{\gamma}_a(\|\phi\|_\infty),\ \forall\phi \in C([-\Delta, 0]; R^n) \tag{1}$$

For example, the $\|\cdot\|_{M_2}$ norm, given by (see [2]) $\|\phi\|_{M_2} = \left(|\phi(0)|^2 + \int_{-\Delta}^0 |\phi(\tau)|^2 d\tau\right)^{\frac{1}{2}}$, is a M_a functional. As usual, ISS stands for both input-to-state stable and input-to-state stability.

PRELIMINARIES

In this section, for the reader's convenience, some previously published results which are fundamental for the understanding of the novel results which will be provided in next sections are briefly reported, with some slight modifications for the purposes of this paper. Let us consider the following retarded nonlinear system

$$\begin{aligned} &\dot{x}(t) = f(x_t) + g(x_t)v(t), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{2}$$

where $x(t) \in R^n$, $v(t) \in R^m$ is the input function, measurable and locally essentially bounded, for $t \geq 0$ $x_t : [-\Delta, 0] \to R^n$ is the standard function (see Section 2.1, pp. 38 in [5]) given by $x_t(\tau) = x(t+\tau)$, Δ is the maximum involved delay, f is a locally Lipschitz, completely continuous functional mapping $C([-\Delta, 0]; R^n)$ into R^n, g is a locally Lipschitz, completely continuous functional mapping $C([-\Delta, 0]; R^n)$ into $R^{n\times m}$, $\xi_0 \in C([-\Delta, 0]; R^n)$. It is here supposed that $f(0) = 0$, thus ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t)$ with zero initial conditions. Multiple discrete non-commensurate as well as distributed delays can appear in (2). In the following, the continuity of a functional $V : C([-\Delta, 0]; R^n) \to R^+$ is intended with respect to the supremum norm. Given a locally Lipschitz continuous functional $V : C([-\Delta, 0]; R^n) \to R^+$, the upper right-hand derivative D^+V of the functional V is given by (see [1], Definition 4.2.4, pp. 258)

$$D^+V(\phi, v) = \limsup_{h\to 0^+} \frac{1}{h}\left(V(\phi_h) - V(\phi)\right), \tag{3}$$

where $\phi_h \in C([-\Delta, 0]; R^n)$ is given by

$$\phi_h(\theta) = \begin{cases} \phi(\theta + h), & s \in [-\Delta, -h), \\ \phi(0) + (f(\phi) + g(\phi)v)(\theta + h), & \theta \in [-h, 0] \end{cases} \tag{4}$$

Definition 1: ([23,19]) *The system (2) is said to be input-to-state stable (ISS) if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input v, the solution exists for all $t \geq 0$ and furthermore it satisfies*

$$|x(t)| \leq \beta\left(\|\xi_0\|_\infty, t\right) + \gamma\left(\|v_{[0,t)}\|_\infty\right) \tag{5}$$

Theorem 2: ([19]) *If there exist a locally Lipschitz functional $V : C([-\Delta, 0]; R^n) \to R^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, ρ of class $\mathcal{K}$ such that:*

- *H_1) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(M_a(\phi))$, $\forall\ \phi \in C([-\Delta, 0]; R^n)$;*
- *H_2) $D^+V(\phi, v) \leq -\alpha_3(M_a(\phi))$,*
  $\forall\ \phi \in C([-\Delta, 0]; R^n),\ v \in R^m :\ M_a(\phi) \geq \rho(|v|)$;

then, the system (2) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Remark 3: With respect to the published literature: Theorem 2 makes use of the M_a functional instead of the $\|\cdot\|_a$ norm used in [19], thus weakening the hypotheses. •

Let us consider now the following retarded linear system (see [2,4,5,13,16])

$$\begin{aligned} &\dot{x}(t) = Lx_t, \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{6}$$

where $L : C([-\Delta, 0]; R^n) \to R^n$ is a linear operator which is defined as

$$L\phi = A_0\phi(0) + \sum_{i=1}^{p} A_i\phi(-\Delta_i) + \int_{-\Delta}^{0} A_{0,1}(\theta)\phi(\theta)d\theta, \tag{7}$$

$A_j \in R^{n\times n}, j = 0, 1, \ldots, p$, p is a positive integer, $A_{0,1}$ is a $n \times n$ matrix of piecewise continuous functions which are defined in $[-\Delta, 0]$ and take values in R, $0 < \Delta_1 < \Delta_2 < \cdots < \Delta_p = \Delta$ are arbitrary (non-commensurate) time delays, $\xi_0 \in C([-\Delta, 0]; R^n)$. The following converse Liapunov-Krasovskii Theorem (Proposition 7.4, pp. 240 in [4]) holds.

Theorem 4: *If the system (6) is asymptotically stable, then there exists a Liapunov-Krasovskii functional*

$$V(\phi) = \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-\Delta}^{0} Q(\xi)\phi(\xi)d\xi + \int_{-\Delta}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi + \int_{-\Delta}^{0}\int_{-\Delta}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta d\xi, \tag{8}$$

where $P = P^T \in R^{n\times n}$ and the matrix functions $Q(\xi) \in R^{n\times n}$, $S(\xi) = S^T(\xi) \in R^{n\times n}$, $R(\xi, \eta) = R^T(\eta, \xi) \in R^{n\times n}$, such that, for all $\phi \in C([-\Delta, 0]; R^n)$,

$$V(\phi) \geq s|\phi(0)|^2 \tag{9}$$

and

$$D^+V(\phi) \leq -s|\phi(0)|^2 \tag{10}$$

are satisfied for some real $s > 0$. Furthermore, the matrix functions $Q(\xi), S(\xi)$ and $R(\xi, \eta)$ are continuous everywhere except at pointwise delays $\xi, \eta = \Delta_i, i = 1, 2, \ldots, p-1$.

MAIN RESULTS

Let us consider the following retarded nonlinear system, corresponding to (2) when the input v is given as the sum of the control input and of the disturbance:

$$\begin{aligned} &\dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{11}$$

where $u(t) \in R^m$ is the control input, $d(t) \in R^m$ is the disturbance, measurable and locally essentially bounded.

Theorem 5: *Let there exist a locally Lipschitz, completely continuous functional $k : C([-\Delta, 0]; R^n) \to R^m$, such that the functional $L : C([-\Delta, 0]; R^n) \to R^n$, defined as*

$$L\phi = f(\phi) + g(\phi)k(\phi) \tag{12}$$

is linear (see (6),(7)), and the system

$$\dot{x}(t) = Lx_t \tag{13}$$

is asymptotically stable. Let the functional

$$\begin{aligned} V_{0,L}(\phi) =& \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-\Delta}^{0} Q(\xi)\phi(\xi)d\xi + \\ & \int_{-\Delta}^{0} \phi^T(\xi)S(\xi)\phi(\xi)d\xi + \\ & \int_{-\Delta}^{0}\int_{-\Delta}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta d\xi, \end{aligned} \tag{14}$$

be the Liapunov-Krasovskii functional by which the asymptotic stability of system (13) can be proved, as in Theorem 4. Let $D_a^+ V_{0,L} : C([-\Delta, 0]; R^n) \to R$ be the continuous derivative of the functional $V_{0,L}$ with respect to the linear unforced system (13), according to (3) ($v = 0$). Let $p : C([-\Delta, 0]; R^n) \to R^m$ be defined as

$$\begin{aligned} p(\phi) = & \big(D_a^+ V_{0,L}(\phi) + q_2 s|\phi(0)|^2 - q_1 s|\phi(-\Delta)|^2 - \\ & - \frac{q_2 - q_1}{\Delta} s \int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau\Big) \cdot \\ & \left(g^T(\phi)P\phi(0) + g^T(\phi)\int_{-\Delta}^{0} Q(\xi)\phi(\xi)d\xi\right), \end{aligned} \tag{15}$$

where s is the positive real in (9) (10) associated with functional (14) and system (13), and q_1, q_2 are any positive reals satisfying $0 < q_1 < q_2 < 1$.

Then the following feedback control law

$$u(t) = k(x_t) + p(x_t) \tag{16}$$

is such that the closed loop system (11), (16), described by the following nonlinear equations

$$\dot{x}(t) = Lx_t + g(x_t)p(x_t) + g(x_t)d(t), \tag{17}$$

is input-to-state stable with respect to the disturbance $d(t)$.

Proof.

For any given $\phi \in C([-\Delta, 0]; R^n)$, $h \in [0, \Delta]$, let us define $\phi_h^a \in C([-\Delta, 0]; R^n)$ and $\delta_h^g \in C([-\Delta, 0]; R^{n\times m})$ as follows

$$\begin{aligned} \phi_h^a(\theta) &= \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h), \\ \phi(0) + (\theta + h)L\phi, & \theta \in [-h, 0], \end{cases} \\ \delta_h^g(\theta) &= \begin{cases} 0_{n\times m}, & \theta \in [-\Delta, -h), \\ (\theta + h)g(\phi), & \theta \in [-h, 0]. \end{cases} \end{aligned} \tag{18}$$

Let $V_1 : C([-\Delta, 0]; R^n) \to R^+$ be defined as $V_1(\phi) = \int_{-\Delta}^{0} q(\tau)s|\phi(\tau)|^2 d\tau$, with $q : [-\Delta, 0] \to R^+$ given by

$$q(\tau) = -\frac{\tau}{\Delta}q_1 + \frac{\tau + \Delta}{\Delta}q_2, \qquad \tau \in [-\Delta, 0] \tag{19}$$

Let $V_0(\phi) = V_{0,L}(\phi) + V_1(\phi)$. The following equality holds

$$\begin{aligned} \limsup_{h\to 0^+} \frac{V_1(\phi_h^a) - V_1(\phi)}{h} = q_2 s|\phi(0)|^2 - q_1 s|\phi(-\Delta)|^2 - \\ - \frac{q_2 - q_1}{\Delta} s \int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau, \end{aligned} \tag{20}$$

Now, let us prove that, for any $M \in R^m$, the following equality holds

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = \\ &2\phi^T(0)Pg(\phi)M + 2\left(\int_{-\Delta}^{0} Q(\xi)\phi(\xi)d\xi\right)^T g(\phi)M \end{aligned} \tag{21}$$

The following equality holds

$$\limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g M) - V_0(\phi_h^a)}{h} = l_1 + l_2 + l_3 + l_4 + l_5, \tag{22}$$

where

$$\begin{aligned} l_1 = & \lim_{h\to 0^+} \\ & \frac{(\phi(0) + hL\phi + hg(\phi)M)^T P (\phi(0) + hL\phi + hg(\phi)M)}{h} \\ & - \frac{(\phi(0) + hL\phi)^T P (\phi(0) + hL\phi)}{h}, \end{aligned} \tag{23}$$

$$\begin{aligned} l_2 = & \lim_{h\to 0^+} \frac{1}{h} 2 (\phi(0) + hL\phi + hg(\phi)M)^T \cdot \\ & \cdot \int_{-\Delta}^{0} Q(\xi)(\phi_h^a(\xi) + \delta_h^g(\xi)M)d\xi \\ & - \frac{2 (\phi(0) + hL\phi)^T \int_{-\Delta}^{0} Q(\xi)\phi_h^a(\xi)d\xi}{h}, \end{aligned} \tag{24}$$

$$l_3 = \lim_{h\to 0^+}$$
$$\int_{-\Delta}^{0} \left(\phi_h^a(\xi) + \delta_h^g(\xi)M\right)^T S(\xi) \left(\phi_h^a(\xi) + \delta_h^g(\xi)M\right) d\xi -$$
$$- \frac{\int_{-\Delta}^{0} \left(\phi_h^a(\xi)\right)^T S(\xi) \left(\phi_h^a(\xi)\right) d\xi}{h}, \tag{25}$$

$$l_4 = \lim_{h\to 0^+} \frac{1}{h} \int_{-\Delta}^{0} \int_{-\Delta}^{0}$$
$$\left(\phi_h^a(\xi) + \delta_h^g(\xi)M\right)^T R(\xi,\eta) \left(\phi_h^a(\eta) + \delta_h^g(\eta)M\right) d\eta d\xi$$
$$- \frac{1}{h} \int_{-\Delta}^{0} \int_{-\Delta}^{0} \phi_h^a(\xi)^T R(\xi,\eta) \phi_h^a(\xi) d\eta d\xi \tag{26}$$

$$\begin{aligned} l_5 = \lim_{h\to 0^+} \frac{1}{h} \int_{-\Delta}^{0} q(\xi) s \cdot \\ \cdot \left(\phi_h^a(\xi) + \delta_h^g(\xi)M\right)^T \left(\phi_h^a(\xi) + \delta_h^g(\xi)M\right) d\xi - \\ \frac{1}{h} \int_{-\Delta}^{0} q(\xi) s \left(\phi_h^a(\xi)\right)^T \left(\phi_h^a(\xi)\right) d\xi, \end{aligned} \tag{27}$$

As far as l_1 is concerned, the following equalities hold

$$\begin{aligned} l_1 = \lim_{h\to 0^+} \frac{1}{h} 2(\phi(0) + hL\phi)^T P h g(\phi) M + \\ \frac{1}{h} h^2 M^T g^T(\phi) P g(\phi) M = 2\phi^T(0) P g(\phi) M \end{aligned} \tag{28}$$

As far as l_2 is concerned, the following equalities hold

$$\begin{aligned} l_2 = \\ \lim_{h\to 0^+} \frac{1}{h} 2(\phi(0) + hL\phi)^T \int_{-h}^{0} Q(\xi)(\xi + h) g(\phi) M d\xi + \\ \frac{1}{h} 2h M^T g^T(\phi) \int_{-\Delta}^{0} Q(\xi)(\phi_h^a(\xi) + \delta_h^g(\xi)M) d\xi = \\ \lim_{h\to 0^+} \frac{1}{h} 2(\phi(0) + hL\phi)^T \int_{-h}^{0} Q(\xi) \xi g(\phi) M d\xi + \\ + \frac{1}{h} 2(\phi(0) + hL\phi)^T \int_{-h}^{0} Q(\xi) h g(\phi) M d\xi + \\ + \frac{2h M^T g^T(\phi) \int_{-\Delta}^{0} Q(\xi)(\phi_h^a(\xi) + \delta_h^g(\phi)M) d\xi}{h} \end{aligned} \tag{29}$$

Taking into account the continuity at 0 of the matrix of functions $Q(\xi)$ and the uniform convergence of $\phi_h^a + \delta_h^g(\phi)M$ to ϕ, the following equalities hold

$$\begin{aligned} l_2 = \lim_{h\to 0^+} 2M^T g^T(\phi) \int_{-\Delta}^{0} Q(\xi)(\phi_h^a(\xi) + \delta_h^g(\xi)M) d\xi) = \\ 2 \left(\int_{-\Delta}^{0} Q(\xi)\phi(\xi) d\xi \right)^T g(\phi) M \end{aligned} \tag{30}$$

As far as l_3 is concerned, the following equalities hold

$$\begin{aligned} l_3 = \lim_{h\to 0^+} \frac{1}{h} \\ 2 \int_{-h}^{0} (\phi(0) + (\xi + h)L\phi)^T S(\xi)(\xi + h) g(\phi) M d\xi + \\ \frac{1}{h} \int_{-h}^{0} (\xi + h)^2 M^T g^T(\phi) S(\xi) g(\phi) M d\xi = \\ \lim_{h\to 0^+} \frac{2 \int_{-h}^{0} (\phi(0) + \xi L\phi)^T S(\xi) g(\phi) M \xi d\xi}{h} \\ + 2 \int_{-h}^{0} (\phi(0) + \xi L\phi)^T S(\xi) g(\phi) M d\xi + \\ 2 \int_{-h}^{0} (L\phi)^T S(\xi) g(\phi) M \xi d\xi \\ + 2h \int_{-h}^{0} (L\phi)^T S(\xi) g(\phi) M d\xi + \\ \frac{\int_{-h}^{0} \xi^2 M^T g^T(\phi) S(\xi) g(\phi) M d\xi}{h} + \\ \int_{-h}^{0} 2\xi M^T g^T(\phi) S(\xi) g(\phi) M d\xi + \\ h \int_{-h}^{0} M^T g^T(\phi) S(\xi) g(\phi) M d\xi \end{aligned} \tag{31}$$

The terms which are not divided by h go to zero, since the interval of integration goes to zero (recall that $S(\xi)$ is continuous in $(-\Delta_1, 0]$, see Theorem 4). The terms

$$\begin{aligned} \frac{2 \int_{-h}^{0} (\phi(0) + \xi L\phi)^T S(\xi) g(\phi) M \xi d\xi}{h}, \\ \frac{\int_{-h}^{0} \xi^2 M^T g^T(\phi) S(\xi) g(\phi) M d\xi}{h} \end{aligned} \tag{32}$$

go to zero since the functions inside the integrals are zero at zero. Therefore, $l_3 = 0$. As far as l_4 is concerned, the following equality holds

$$\begin{aligned} l_4 = \lim_{h\to 0^+} \frac{1}{h} \Big(\int_{-\Delta}^{0} \phi_h^a(\xi)^T \int_{-h}^{0} R(\xi,\eta) \delta_h^g(\eta) M d\eta d\xi + \\ \int_{-h}^{0} \left(\delta_h^g(\xi)M\right)^T \int_{-\Delta}^{0} R(\xi,\eta) \phi_h^a(\eta) d\eta d\xi \\ + \int_{-h}^{0} \int_{-h}^{0} \left(\delta_h^g(\xi)M\right)^T R(\xi,\eta) \left(\delta_h^g(\eta)M\right) d\eta d\xi \Big) \end{aligned} \tag{33}$$

Let $\bar{R} = \sup_{(\xi,\eta)\in[-\Delta,0]^2} |R(\xi,\eta)|$. Then, from the inequalities

$$
\begin{aligned}
&\left|\int_{-\Delta}^{0} \phi_h^a(\xi)^T \int_{-h}^{0} R(\xi,\eta)\delta_h^g(\eta) M d\eta d\xi\right| \leq \\
&\qquad h^2\bar{R}|g(\phi)||M| \int_{-\Delta}^{0} |\phi_h^a(\xi)| d\xi, \\
&\left|\int_{-h}^{0} (\delta_h^g(\xi)M)^T \int_{-\Delta}^{0} R(\xi,\eta)\phi_h^a(\eta) d\eta d\xi\right| \leq \\
&\qquad h^2\bar{R}|g(\phi)||M| \int_{-\Delta}^{0} |\phi_h^a(\eta)| d\eta, \\
&\left|\int_{-h}^{0}\int_{-h}^{0} (\delta_h^g(\xi)M)^T R(\xi,\eta)\left(\delta_h^g(\eta)M\right) d\eta d\xi\right| \leq \\
&\qquad h^4\bar{R}|g(\phi)|^2|M|^2,
\end{aligned}
\tag{34}
$$

taking into account that ϕ_h^a converges uniformly to ϕ as h goes to zero, it follows that $l_4 = 0$. As far as l_5 is concerned, by the same reasoning used for l_3 it follows that $l_5 = 0$.

Therefore, since the term $l_1 + l_2 + l_3 + l_4 + l_5$ in the right hand side of (22) is equal to the right-hand side of (21), the equality (21) is proved. Let

$$
r(\phi) = \left(2\phi^T(0)Pg(\phi) + 2\left(\int_{-\Delta}^{0} Q(\xi)\phi(\xi)d\xi\right)^T g(\phi)\right)^T
\tag{35}
$$

From (20) it follows that the functional $p(\phi)$ in (*15*) is equal to

$$
\frac{1}{2} D_a^+ V_0(\phi) r(\phi),
\tag{36}
$$

with

$$
D_a^+ V_0(\phi) = \limsup_{h\to 0^+} \frac{V_0(\phi_h^a) - V_0(\phi)}{h}
\tag{37}
$$

Now, in order to prove the input-to-state stability of the closed loop system (*17*), let us apply Theorem 2, by using the Liapunov-Krasovskii functional V_0 and the M_2 norm as a M_a functional. Let $D^+V_0(\phi)$ be the derivative of the functional V_0 with respect to the system (*17*), according to (3). Taking into account (21), (35), (36), the following equalities hold

$$
\begin{aligned}
&D^+V_0(\phi,d) = \limsup_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g(p(\phi)+d)) - V_0(\phi)}{h} = \\
&\lim_{h\to 0^+} \frac{V_0(\phi_h^a + \delta_h^g(p(\phi)+d)) - V_0(\phi_h^a)}{h} + \\
&\lim_{h\to 0^+} \frac{V_0(\phi_h^a) - V_0(\phi)}{h} = \\
&D_a^+V_0(\phi) + r^T(\phi)(p(\phi)+d) = \\
&D_a^+V_0(\phi) + r^T(\phi)\frac{1}{2}D_a^+V_0(\phi)r(\phi) + r^T(\phi)d
\end{aligned}
\tag{38}
$$

For $|d| \leq \rho\|\phi\||_{M_2}$, ρ positive real, taking into account (20), the following inequalities hold

$$
\begin{aligned}
D^+V_0(\phi,d) \leq\ & -s|\phi(0)|^2 + q_2 s|\phi(0)|^2 - q_1 s|\phi(-\Delta)|^2 - \\
& - \frac{q_2-q_1}{\Delta} s \int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau + \\
& \frac{1}{2}\left(-s|\phi(0)|^2 + q_2 s|\phi(0)|^2 - q_1 s|\phi(-\Delta)|^2 - \right.\\
& \left. - \frac{q_2-q_1}{\Delta} s \int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau\right) |r(\phi)|^2 + \\
& \frac{\rho}{2}|r(\phi)|^2\|\phi\|_{M_2}^2 + \frac{\rho}{2}\|\phi\|_{M_2}^2 \leq \\
& - s\left(1 - q_2 - \frac{\rho}{2s}\right)|\phi(0)|^2 - \\
& - \left(\frac{q_2-q_1}{\Delta}s - \frac{\rho}{2}\right)\int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau - \\
& - \frac{s}{2}\left(\left(1 - q_2 - \frac{\rho}{2s}\right)|\phi(0)|^2 - \right.\\
& \left. - \left(\frac{q_2-q_1}{\Delta}s - \frac{\rho}{2}\right)\int_{-\Delta}^{0} |\phi(\tau)|^2 d\tau\right)|r(\phi)|^2
\end{aligned}
\tag{39}
$$

Therefore, by choosing ρ sufficiently small, taking into account that $0 < q_1 < q_2 < 1$, the following inequality holds for a suitable positive real k

$$
D^+V_0(\phi) \leq -k\|\phi\|_{M_2}^2
\tag{40}
$$

and, by Theorem (2), the input-to-state stability of the closed loop system (*17*) is proved. □

CONCLUSIONS

In this paper, it is proved that a retarded nonlinear system which is linearizable and stabilizable, is also input-to-state stabilizable with respect to disturbances adding to the control law. The converse Liapunov-Krasovskii theorem for linear retarded functional differential equations is used to find the input-to-state stabilizing feedback. An example taken from the past literature is investigated in details, showing the effectiveness and the applicability of the methodology here proposed.

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