2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

A Further Result on Global Stabilization of Oscillators with Bounded Delayed Input

Haijun Fang and Zongli Lin

Abstract— **This paper revisits the problem of globally stabilizing an oscillator with bounded delayed input. It establishes that an oscillator with input saturation can be globally asymptotically stabilized by a linear feedback even in the presence of an arbitrarily large delay in the input. This result strengthens a recent result in the literature, which shows that such an oscillator can be globally asymptotically stabilized with a sufficiently small control input and the magnitude of the control input goes to zero as the delay increases to infinity. The controller constructed in this paper thus improves the efficiency of the closed-loop system by fully utilizing the actuator capacity.**

Keywords: **Global stabilization, Oscillator, Time delay, Actuator saturation**

I. Introduction

In this paper, we revisit the problem of globally asymptotically stabilizing the following system,

$$\dot{x}(t) = Ax(t) + B\sigma(u(t-\tau)), \qquad (1)$$

$$x(\theta) = \phi(\theta), \;\; \theta \in [-\tau, 0], \qquad (2)$$

where $x \in \mathbf{R}^2$ is the state, $u \in \mathbf{R}$ is the control input, $A \in \mathbf{R}^{2\times 2}$ has a pair of imaginary eigenvalues, (A, B) is controllable, $\tau \geq 0$ is the amount of time delay in the control input, and σ is a saturation function. This system represents an oscillator controlled by a bounded delayed input.

Without loss of generality, we will assume throughout this paper that the matrices A and B are given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad (3)$$

and that σ is a standard saturation function with unity saturation level, *i.e.*,

$$\sigma(u) = \mathrm{sign}(u) \min(1, |u|).$$

Any controllable pair (A, B) and standard saturation function can be transformed into the above simplified forms by a state transformation and a scaling on u.

Oscillation is a common phenomenon that exists in many systems such as circuits, transportation systems and chemical process. Time delay is also a common phenomenon in communications, processing and transport of information, computation, etc. Stabilization of oscillators with delayed input was studied as early as in the 1940s [1], [2], [14]. The problem of globally asymptotically stabilizing oscillators with delayed input was recently studied under the extra constraint that the input is bounded [12]. More specifically, it is shown in [12] that, for an arbitrarily large delay in the input τ, the oscillator system (1)-(2) can be globally asymptotically stabilized by a smooth feedback whose magnitude is sufficiently small. This stabilizing feedback is designed in such a way that its magnitude decreases as the delay τ increases and goes to zero as τ approaches ∞. The result of [12] was established elegantly as follows. By direct integration of the system dynamic equation, a distributed delayed control law is first derived in a straightforward way. The only distributed term in this control law is in higher order of the control input. It is thus conjectured and proven that the distributed control law still globally asymptotically stabilizes the system when the distributed term is dropped as long as the magnitude of controller input is sufficiently small.

The objective of this paper is to propose an alternative saturating feedback law that globally asymptotically stabilizes the oscillator system (1)-(2). We recall that the feedback law of [12] takes the form of

$$u = -\mu\sigma_0\Big(-\sin(\tau)x_1) + \cos(\tau)x_2\Big),$$

$$\mu \in \left(0, \min\left\{\frac{1}{2}, \frac{1}{324\tau^2}, \frac{1}{40\tau}\right\}\right),$$

where $\sigma_0 : \mathbf{R} \to \mathbf{R}$ is a smooth saturation type function. Unlike this feedback law of [12], the feedback law we propose in this paper does not require the saturation level to decrease as the amount of delay τ increases and thus allows the full utilization of the available actuator capacity. As seen in the literature on control systems with actuator saturation [17], it is possible, although nontrivial, to decompose the problem of global stabilization for a higher order linear system into those for chains of integrators and and oscillators. It is expected that the results presented in this paper on oscillators will contribute to the solution of stabiliztion problems for higher order systems with bounded delayed input.

We should note that interests in the control of time-delayed linear systems with bounded control have not been limited to the oscillator system (1)-(2). Indeed, there is a rich literature on the control of linear time-delayed systems with bounded input. A small sample of this literature

This work was supported in part by NSF grant CMS-0324329. The authors are with Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904-4743, USA. Email: hf2r, zl5y@virginia.edu

include [3], [4], [5], [6], [8], [11], [13], [15], [16].

The remainder of this paper is organized as follows. Section II contains some preliminaries that we need to establish our main result in Section III. Numerical examples in Section IV demonstrate the simplicity and efficiency of the proposed feedback law. A concluding remark is made in Section V.

II. Preliminaries

Lemma 1: For the matrix pair (A, B) as given in (3), let

$$P(k) = \begin{bmatrix} p_1(k) & p_2(k) \\ p_2(k) & p_3(k) \end{bmatrix}$$

be the solution to the following algebraic Ricatti equation (ARE),

$$PA + A'P - \frac{2}{k}PBB'P = -\frac{1}{k}I, \quad k \geq 1. \tag{4}$$

Then, there exists an $M > 0$ such that $\|P(k)\| \leq M$, $\forall k \geq 1$, and

$$\lim_{k\to\infty} P(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5}$$

Proof: It is straightforward to verify that the solution to the ARE (5) is given by

$$p_1(k) = \sqrt{\frac{k\sqrt{k^2+2} - k^2 + 1}{2}} + \frac{\sqrt{k^2+2} - k}{k}\sqrt{\frac{k\sqrt{k^2+2} - k^2 + 1}{2}}, \tag{6}$$

$$p_2(k) = \frac{\sqrt{k^2+2} - k}{2}, \tag{7}$$

$$p_3(k) = \sqrt{\frac{k\sqrt{k^2+2} - k^2 + 1}{2}}, \tag{8}$$

from which the results of the lemma follow. □

The following lemma is adopted from [9], where this lemma was established for vector valued saturation functions.

Lemma 2: Let $u, v \in \mathbf{R}$ with $|v| \leq 1$. Let $D_1 = 0, D_2 = 1$. Then,

$$\sigma(u) \in \text{co}\{u, v\} = \text{co}\left\{D_i u + D_i^- v,\ i = 1, 2\right\}, \tag{9}$$

where co denotes the convex hull and $D_i^- = 1 - D_i$.

Consider the time-delay system,

$$\dot{x}(t) = f(x_t), \quad t \geq 0, \tag{10}$$

$$x_0(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{11}$$

Let $\mathcal{C}_{n,\tau} = \mathcal{C}([-\tau, 0], \mathbf{R}^n)$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$ with the topology of uniform convergence. We also use $x_t \in \mathcal{C}_{n,\tau}$ to denote the restriction of $x(t)$ to the interval $[t - \tau, t]$ translated to $[-\tau, 0]$, that is,

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0].$$

Assume that $\psi \in \mathcal{C}_{n,\tau}$ and the map $f(\psi) : \mathcal{C}_{n,\tau} \longmapsto \mathbf{R}^n$ is continuous and Lipschitz in ψ and $f(0) = 0$. Also denote the solution of the functional differential equation (10) with the initial condition $x_0 \in \mathcal{C}_{n,\tau}$ as $x(t, x_0)$.

Definition 1: The trivial solution $x(t) \equiv 0$ of (10) and (11) is said to be asymptotically stable if

1) for every $\delta > 0$ there exists an $\epsilon = \epsilon(\delta)$ such that, for any $\psi \in \mathcal{B}(0, \epsilon) := \{\psi(\theta) : \|\psi(\theta)\| \leq \epsilon, \forall \theta \in [-\tau, 0]\}$, the solution $x(t, \psi)$ of (10) and (11) satisfies $x_t \in \mathcal{B}(0, \delta)$ for all $t \geq 0$;
2) for every $\eta > 0$ there exist a $T(\eta)$ and a $v_0 > 0$ independent of η such that $\psi \in \mathcal{B}(0, v_0)$ implies that $||x_t||_c = \sup_{\theta\in[-\tau,0]} \|x(t + \theta)\| < \eta$, $\forall t \geq T(\eta)$.

The Razumikhin Theorem gives conditions for $x(t) \equiv 0$ to be asymptotically stable. Actually, more information about invariant set and regional stability is contained in the proof of this theorem in [7]. The additional information is incorporated in the following statement of the theorem.

Theorem 1 (Razumikhin Stability Theorem): Consider the functional differential equations (10) and (11). Suppose that $u(s), v(s), w(s)$ and $p(s) \in \mathbf{R}^+ \longmapsto \mathbf{R}^+$ are scalar, continuous and nondecreasing functions, $u(s), v(s), w(s)$ positive for $s > 0, u(0) = v(0) = 0$, and $p(s) > s$ for $s > 0$. If there is a continuous function $V : \mathbf{R}^n \longmapsto \mathbf{R}$ and a positive number ρ, such that for all

$$x_t \in M_V(\rho) := \{\psi \in \mathcal{C}_{n,\tau} : V(\psi(\theta)) \leq \rho, \forall \theta \in [-\tau, 0]\},$$

the following conditions hold,

1) $u(||x||) \leq V(x) \leq v(||x||)$,
2) $\dot{V}(x(t)) \leq -w(||x(t)||)$, if $V(x(t + \theta)) < p(V(x(t)))$, $\forall\, \theta \in [-\tau, 0]$,

then the solution $x(t) \equiv 0$ of the equation (10) and (11) is asymptotically stable. Moreover, the set $M_V(\rho)$ is an invariant set inside the domain of attraction.

III. Main Result

For the system (1)-(2), we construct the following feedback law

$$u(t) = -\frac{1}{k}B'P(k)e^{A\tau}x(t), \tag{12}$$

where $P(k)$ is the solution to the Ricatti equation (5). Under this feedback law, the closed-loop system is given by

$$\dot{x}(t) = Ax(t) - B\sigma\left(\frac{1}{k}B'P(k)e^{A\tau}x(t - \tau)\right), \tag{13}$$

$$x(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0]. \tag{14}$$

For the closed-loop system (13)-(14), we have the following results on its global asymptotic stability.

Theorem 2: For any given $\tau \geq 0$, there exists a $k^* \geq 1$, such that the closed-loop system (13)-(14) is globally asymptotically stable at the origin.

Proof: The proof for $\tau = 0$ and $\tau > 0$ differs from each other. We will consider these two cases separately. We will start with the simpler case of $\tau = 0$. This simpler case will also help to understand the proof for the case of $\tau > 0$.

In the case of $\tau = 0$, the closed-loop system (13)-(14) reduces to

$$\dot{x}(t) = Ax(t) - B\sigma\left(\frac{1}{k}B'P(k)x(t)\right). \qquad (15)$$

which we will show to be globally asymptotically stable for $\forall k \geq 1$. Let us choose the Lyapunov function $V(x) = x'P(k)x$. Denote

$$\Omega_{\mathrm{ns}} := \left\{x \in \mathbf{R}^2 : \left|\frac{1}{k}B'P(k)x\right| \leq 1\right\}.$$

This is the region where control does not saturate (see the region between the two straight lines in Fig. 1). It is then straightforward to verify that, $\forall x \in \Omega_{\mathrm{ns}} \setminus 0$,

$$\dot{V}(x) = x'\left(PA + A'P - \frac{2}{k}PBB'P\right)x = -\frac{1}{k}x'x < 0, \qquad (16)$$

Here and in the rest of the proof, we will drop the dependency on k of $P(k)$ and $p_i(k)$, $i \in [1,3]$.

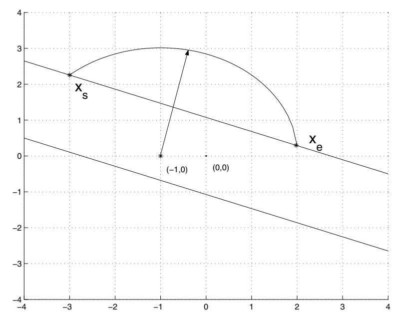

Fig. 1. System trajectory in Ω_{s}^{+}.

Let $\varepsilon(P,\rho) := \left\{x \in \mathbf{R}^2 : x'Px \leq \rho\right\}$, $\rho > 0$, be an ellipsoid lying completely inside Ω_{ns}. Then, $\varepsilon(P,\rho)$ is an invariant set for (15) and every trajectory starting from inside $\varepsilon(P,\rho)$ will go to the origin asymptotically.

We next show that, any trajectory starting from outside $\varepsilon(P,\rho)$ and thus all trajectories will enter and remain in $\varepsilon(P,\rho)$ in a finite time. It follows from (16) that $V(x)$ decreases along any trajectory segment that lies inside $\Omega_{\mathrm{ns}} \setminus \varepsilon(P,\rho)$.

We next consider the region $x \notin \Omega_{\mathrm{ns}}$, where $|\frac{1}{k}B'Px| > 1$. This is the region where control input saturates. Denote

$$\Omega_{\mathrm{s}}^{+} := \left\{x \in \mathbf{R}^2 : \frac{1}{k}B'Px > 1\right\}, \Omega_{\mathrm{s}}^{-} := \left\{x \in \mathbf{R}^2 : \frac{1}{k}B'Px < -1\right\}.$$

By (8), $p_3(k) > 0$. Hence, in Fig. 1, the region above Ω_{ns} is Ω_{s}^{+} and below is Ω_{s}^{-}. In Ω_{s}^{+}, the system (15) further simplifies to

$$\begin{aligned} & \dot{x}(t) = Ax(t) - B \\ \Longleftrightarrow \quad & \frac{d\left(x(t) - A^{-1}B\right)}{dt} = A\left(x(t) - A^{-1}B\right). \end{aligned} \qquad (17)$$

Thus, any trajectory in Ω_{s}^{+} is an arc of a circle centered at $A^{-1}B = (-1,0)'$. Let x_{t} be the point on the line $\frac{1}{k}B'Px = 1$ such that the line segment connecting x_{t} and $(-1,0)'$ is perpendicular to the line $\frac{1}{k}B'Px = 1$. It can be easily verified that $x_{\mathrm{t}} = (x_{\mathrm{t}1}, x_{\mathrm{t}2})' = \left(\frac{p_2k - p_3^2}{p_3^2 + p_2^2}, \ \frac{p_3(k+p_2)}{p_3^2+p_2^2}\right)'$. Thus, a trajectory starting from any point x on the line $\frac{1}{k}B'Px = 1$ with $x_1 < x_{\mathrm{t}1}$ will go into Ω_{s}^{+}, any one with $x_1 \geq x_{\mathrm{t}1}$ will go into Ω_{ns}. Let $x_{\mathrm{s}} = (x_{\mathrm{s}1}, x_{\mathrm{s}2})'$ be a point on the line $\frac{1}{k}B'Px = 1$ with $x_{\mathrm{s}1} < x_{\mathrm{t}1}$. Then, the trajectory starting from x_{s} will travel along an arc in Ω_{s}^{+} and reach another point x_{e} on the line $\frac{1}{k}B'Px = 1$. Clearly,

$$\frac{x_{\mathrm{s}} + x_{\mathrm{e}}}{2} = x_{\mathrm{t}} \Longrightarrow x_{\mathrm{e}} = 2x_{\mathrm{t}} - x_{\mathrm{s}}.$$

Let

$$x_{\mathrm{s}} = x_{\mathrm{t}} - \begin{bmatrix} a \\ -a\frac{p_2}{p_3} \end{bmatrix}, \quad a > 0.$$

Then,

$$\begin{aligned} V(x_{\mathrm{e}}) - V(x_{\mathrm{s}}) &= x_{\mathrm{e}}'Px_{\mathrm{e}} - x_{\mathrm{s}}'Px_{\mathrm{s}} = 4x_{\mathrm{t}}'Px_{\mathrm{t}} - 4x_{\mathrm{t}}'Px_{\mathrm{s}} \\ &= 4x_{\mathrm{t}}'P\begin{bmatrix} a \\ -a\frac{p_2}{p_3} \end{bmatrix} = a\frac{p_1p_3 - p_2^2}{p_3}\left(p_2k - p_3^2\right) < 0. \end{aligned} \qquad (18)$$

The last inequality holds because, by (6)-(8), $p_2k - p_3^2 = -0.5$, $p_1p_3 - p_2^2 > 0$ and $p_3 > 0$.

Similar conclusion can be drawn for the trajectories in Ω_{s}^{-}. Thus, along any trajectory starting outside of $\varepsilon(P,\rho)$, the value of $V(x)$ will eventually decreases (not necessarily monotonically) to and remain below the value of ρ. We thus conclude global asymptotic stability of the closed-loop system for the case of $\tau = 0$.

We now consider the case of $\tau > 0$. The reasoning follows the same step as in the case of $\tau = 0$. The complexity lies in that it is now necessary to establish $\dot{V} < 0$ in a region larger than Ω_{ns}, the region where control input does not saturate.

We start with

$$\begin{aligned} x(t) &= e^{A\tau}x(t-\tau) \\ &+ \int_{-\tau}^{0} e^{A(-s)}B\sigma\left(\frac{1}{k}B'Pe^{A\tau}x(s+t-\tau)\right)ds, \end{aligned} \qquad (19)$$

from which it follows that

$$\frac{1}{k}B'Pe^{A\tau}x(t-\tau) = \frac{1}{k}B'Px(t) - \frac{1}{k}L(x(t),\tau),$$

$$L(x(t),\tau) = B'P\int_{-\tau}^{0} e^{A(-s)}B\sigma\left(\frac{1}{k}B'Pe^{A\tau}x(s+t-\tau)\right)ds.$$

And let $\Delta(\tau)$ be such that $|L(x(t),\tau)| \leq \Delta(\tau), \forall x \in \mathbf{R}^2$. Such a $\Delta(\tau)$ exists because $P(k)$ is bounded by Lemma 1.

It is then clear that $\frac{1}{k}B'Pe^{A\tau}x(t-\tau) > 1$ if $\frac{1}{k}B'Px(t) > 1 + \Delta(\tau)/k$ and $\frac{1}{k}B'Pe^{A\tau}x(t-\tau) < -1$ if $\frac{1}{k}B'Px(t) < -1 - \Delta(\tau)/k$. Denote

$$\Omega_{\tau,k} := \left\{x \in \mathbf{R}^2 : \left|\frac{1}{k}B'Px(t)\right| \leq 1 + \frac{\Delta(\tau)}{k}\right\} \supset \Omega_{\mathrm{ns}}.$$

For every $x(t) \in \Omega_{\tau,k}$,

$$\left|\frac{1}{k(1+\Delta(\tau)/k)}B'Px(t)\right| \leq 1,$$

and, by Lemma 2,

$$\sigma\left(\frac{1}{k}B'Px(t-\tau)\right) \in \text{co}\{H_i(x(t),\tau), i=1,2\}, \quad (20)$$
$$H_i(x(t),\tau) = D_i\left(\frac{1}{k}B'Px(t) - \frac{1}{k}L(x(t),\tau)\right) + D_i^- \frac{1}{k(1+\Delta(\tau)/k)}B'Px(t).$$

Hence,

$$\begin{aligned}\dot V &= x'(t)P\left(Ax(t) - B\sigma\left(\frac{1}{k}B'Pe^{A\tau}x(t-\tau)\right)\right) + \left(Ax(t) - B\sigma\left(\frac{1}{k}B'Pe^{A\tau}x(t-\tau)\right)\right)'Px(t)\\
&\le \max_{i\in\{1,2\}}\{x'(t)P\left[Ax(t) - BH_i(x(t),\tau)\right] + \left[Ax(t) - BH_i(x(t),\tau)\right]'Px(t)\}\\
&= \max_{i\in\{1,2\}}\left\{x'(t)\left(PA + A'P - \frac{2}{k}PBB'P\right)x(t) + \frac{2\Delta(\tau)}{k^2(1+\Delta(\tau)/k)}x'(t)PBD_i^-B'Px(t) - \frac{2}{k}x'(t)PBD_iL(x(t),\tau)\right\}. \quad (21)\end{aligned}$$

Similar to (20), we have

$$\sigma\left(\frac{1}{k}B'Px(s+t-\tau)\right) \in \text{co}\{J_i(x(t),\tau), i=1,2\}, \quad (22)$$
$$J_i(x(t),\tau) = D_i\frac{1}{k}B'Px(s+t-\tau) + D_i^- \frac{1}{k(1+\Delta(\tau)/k)}B'Px(t).$$

by which and (5), (21) can be continued as follows,

$$\begin{aligned}\dot V &\le \max_{i,j\in\{1,2\}}\left\{-\frac{1}{k}x'(t)x(t) + \frac{2\Delta(\tau)}{k^2(1+\Delta(\tau)/k)}x'(t)PBD_iB'Px(t) + \frac{2}{k^2}x'(t)PBD_i^-B'P \times\left[\int_{-\tau}^0 e^{A(-s)}B\left(D_jB'Pe^{A\tau}x(s+t-\tau) + \frac{1}{(1+\Delta(\tau)/k)}D_j^-B'Px(t)\right)ds\right]\right\}\\
&\le \max_{i,j\in\{1,2\}}\left\{-\frac{1}{k}x'(t)x(t) + \frac{\Delta(\tau)}{k^2(1+\Delta(\tau)/k)}x'(t)W_i^1(P)x(t) + \frac{1}{k^2}\left(x'(t)W_{ij}^2(P,\tau)x(t) + \int_{-\tau}^0 V(t+s-\tau)ds + \frac{1}{(1+\Delta(\tau)/k)}x'(t)W_{ij}^3(P,\tau)x(t)\right)\right\}, \quad (23)\end{aligned}$$

where

$$\begin{aligned}W_i^1(P) &= 2PBD_iBP,\\
W_{ij}^2(P,\tau) &= 2PBD_i^-B'P\int_{-\tau}^0\left(e^{A(-s)}BD_jB'Pe^{A\tau} \times P^{-1}e^{A'\tau}PBD_jB'e^{A'(-s)}\right)dsPBD_iB'P,\\
W_{ij}^3(P,\tau) &= 2PBD_i^-B'P\int_{-\tau}^0 e^{A(-s)}BD_j^-B'Pds,\\
&\qquad i\in[1,2], j\in[1,2].\end{aligned}$$

Here and elsewhile $V(t) := V(x(t))$ and $\dot V := \dot V(t) = \dot V(x(t))$.

It follows from Lemma 1 that, for any constant $\epsilon > 0$, there exists a $k_1^*(\epsilon) > 0$ such that, for all $k_1 \ge k^*(\epsilon)$,

$$-\frac{1}{k}I + \max_{i,j\in\{1,2\}}\frac{1}{k^2}\left(\frac{\Delta(\tau)}{1+\Delta(\tau)/k}W_i^1 + W_{ij}^2(P,\tau) + \epsilon\tau P + \frac{1}{(1+\Delta(\tau)/k)}W_{ij}^3(P,\tau)\right) \le -\frac{1}{2k}I. \quad (24)$$

Thus, under the condition that

$$V(t+\theta) < \epsilon V(t), \quad \forall\,\theta\in[-\tau,0], \quad (25)$$

we have,

$$\begin{aligned}\dot V &\le \max_{i,j\in\{1,2\}} x'(t)\left[-\frac{1}{k}I + \frac{1}{k^2}\left(\frac{\Delta(\tau)}{(1+\Delta(\tau)/k)}W_i^1(P) + W_{ij}^2(P,\tau) + \tau\epsilon V(x(t)) + \frac{1}{(1+\Delta(\tau)/k)}W_{ij}^3(P,\tau)\right)\right]x(t)\\
&\le -\frac{1}{2k}x'(t)x(t), \forall x\in\Omega_{\tau,k}, \forall k\ge k_1^*(\epsilon). \quad (26)\end{aligned}$$

Now, for $\epsilon = 2$, let $\rho > 0$ be such that $\varepsilon(P,\rho) \subset \Omega_{\tau,k}$ for all $k \ge k_1^*(2)$. Thus, by Theorem 1, the closed-loop system (13) is asymptotically stable and $M_V(\rho)$ is an invariant set inside the domain of attraction.

Next, for all $x \in \Omega_{\tau,k}\backslash\varepsilon(P,\rho)$, $V(x) \ge \rho$. It thus follows from [10, Exercise 3.6] that, within the region $\Omega_{\tau,k}\backslash\varepsilon(P,\rho)$, the condition (25) holds for some $\epsilon_1 > 0$. Hence,

$$\dot V < 0, \quad \forall x \in \Omega_{\tau,k}\setminus\varepsilon(P,\rho), \forall k \ge k_1^*(\epsilon_1). \quad (27)$$

We now pick $k^* = \max\{k_1^*(2), k_1^*(\epsilon_1)\}$ and assume that $k \ge k^*$.

Finally, for $x(t) \notin \Omega_{\tau,k}$, $|\frac{1}{k}B'Px(t)| > 1 + \frac{\Delta(\tau)}{k}$ which implies $|\frac{1}{k}B'Pe^{A\tau}x(t-\tau)| > 1$, and the closed-loop system (13) simplifies to (17) if $\frac{1}{k}B'Px(t) > 1 + \frac{\Delta(\tau)}{k}$. For this system, it has been shown earlier that any trajectory starting from a point to the right of x_t will enter the region $\Omega_{\tau,k}$, and any trajectory starting from a point x_s to the left of x_t will travel along an arc to reach x_e in a finite time, with $V(x_e) < V(x_s)$. We can draw similar conclusion if $\frac{1}{k}B'Px(t) < -1 - \frac{\Delta(\tau)}{k}$. So every trajectory of the closed-loop system (13)-(14) will enter and remain in $\varepsilon(P,\rho)$ in a finite time. We conclude that the closed-loop system with any $k \geq k^*$ is globally asymptotically stable at the origin. $\square$

IV. Examples

Example 1: Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 5 \end{bmatrix} \sigma(u(t-\tau)). \qquad (28)$$

Following the design procedure given in Section III, we obtain the following feedback law

$$\begin{aligned} u(t) = -\frac{1}{k}\{&[(p_1(k) + 5p_2(k))\cos(2\tau) \\ &-(p_2(k) + 5p_3(k))\sin(2\tau)]\, x_1(t) \\ &+ [(p_1(k) + 5p_2(k))\sin(2\tau) \\ &+ (p_2(k)+5p_3(k))\cos(2\tau)]\, x_2(t)\}\,, \end{aligned}$$

where p_1, p_2 and p_3 are defined in (6), (7) and (8) respectively. Shown in Figs. 2-4 are some simulation results for $\tau = 3$ seconds with a choice of $k = 10$. We note that $V(x(t))$ does not decrease to zero monotonically. The oscillation in the value of $V(x(t))$ is caused by the fact that $\dot{V} < 0$ cannot be guaranteed in a region where the control input is saturated.

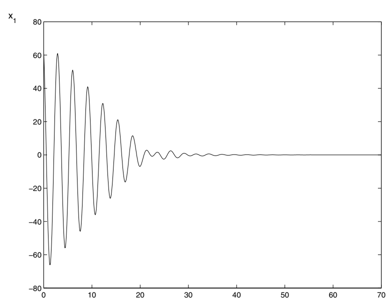

Fig. 2. Example 1. The states x_1 of the closed-loop system.

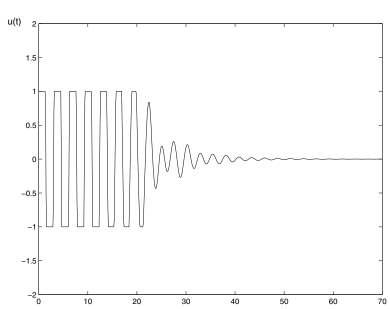

Fig. 3. Example 1. The control $u(t)$.

Example 2: Consider the example given in [12]. The system is (1)-(2) with (A, B) as given in (3). It is shown in [12] that, for $\tau = 3\pi/4$, the magnitude of the control input must be small than 0.1. Indeed, it was shown that, under the controller of [12] with a magnitude of 0.1, the trajectories

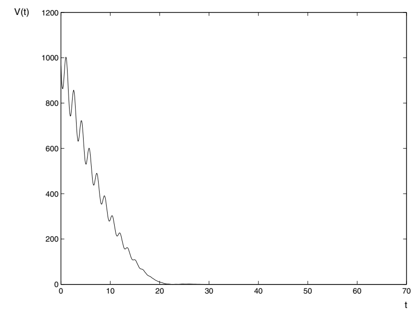

Fig. 4. Example 1. The Lyapunov function $V(x(t))$.

of the closed-loop system with $\tau = 3\pi/4$ and relatively small initial conditions diverge.

Following the design procedure of Section III, we obtain the following feedback law

$$\begin{aligned} u(t) = -\frac{1}{k}\,\{&[p_2(k)\cos(\tau) - p_3(k)\sin(\tau)]\, x_1(t) \\ &+ [p_2(k)\sin(2\tau) + p_3(k)\cos(2\tau)]\, x_2(t)\}\,, \end{aligned}$$

where p_2 and p_3 are defined in (7) and (8) respectively. Simulation results (Figs. 5-8) show that the controller (29) with $k = 5$ is able to stabilize the system with $\tau = \pi/4$, and the controller (29) with $k = 10$ is able to stabilize the system with $\tau = 3\pi/4$ while making full use of the allowable level of the controller input. Simulation results plotted in Fig. 9,10 show also that the state response of the closed-loop system, with $\tau = \pi/4$, under the controller (29) is much faster than that under the controller of [12].

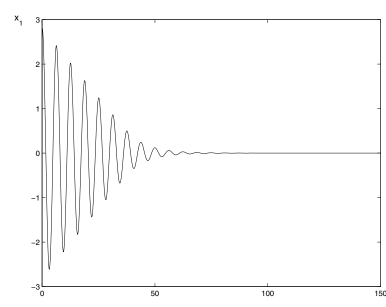

Fig. 5. Example 2. The states x_1 of the closed-loop system with delay $\tau = \frac{\pi}{4}$.

V. Conclusions

This paper revisits the problem of global stabilizaiton for the oscillator with arbitrary time-delay and saturation in the input. An alternative design was proposed that strengthens an existing globally stabilizing feedback law by not requiring the control input to be sufficiently small. This improves the efficiency and the performance of the closed-loop system.

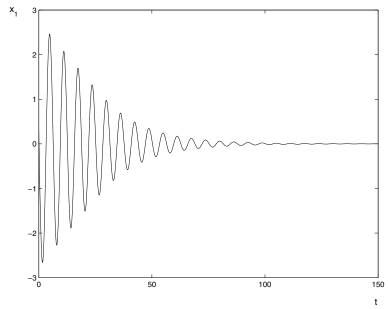

Fig. 6. Example 2. The states x_1 of the closed-loop system with delay $\tau = \frac{3\pi}{4}$.

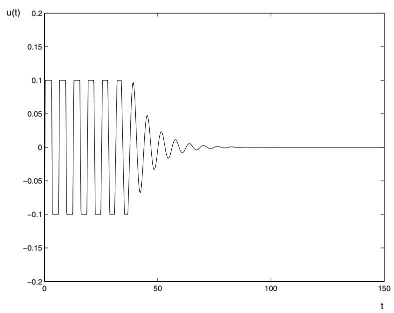

Fig. 7. Example 2. The control $u(t)$ of the closed-loop system with delay $\tau = \frac{\pi}{4}$.

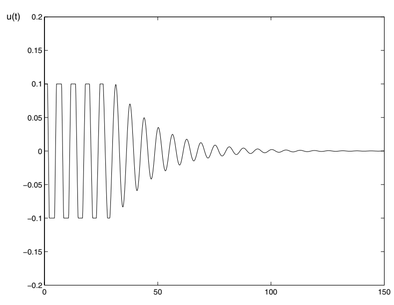

Fig. 8. Example 2. The control $u(t)$ of the closed-loop system with delay $\tau = \frac{3\pi}{4}$.

REFERENCES

[1] H.I. Ansoff, "Stability for linear oscillating systems with constant time lag," *J. Appl. Mech.*, Vol. 16, pp. 158-164, 1949.

[2] H.I. Ansoff and J.A. Krumhansl, "A general stability criterion for linear oscillating systems with constant time lag," *Quart. Appl. Math.*, Vol. 6, pp. 337-341, 1948.

[3] Y.-Y. Cao, Z. Lin and T. Hu, "Stability analysis of linear time-delay systems subject to input saturation," *IEEE Transactions on Circuits and Systems*, Vol. 49, pp. 233-240, 2002.

[4] B.-S. Chen, S.-S. Wang and H.-C. Lu, "Stabilization of time-delay system containing saturating actuators," *Int. J. Contr.*, Vol. 47, pp. 867-881, 1988.

[5] J.-H. Chou, I.-R. Horng and B.-S. Chen, "Dynamical feedback compensator for uncertain time-delay systems containing saturating actuators," *Int. J. Contr.*, Vol. 49, pp. 961-968, 1989.

[6] E. Fridman, A. Pila and U. Shaked, "Regional stabilization and H_∞ control of time-delay systems with saturating actuators," *International Journal of Robust and Nonlinear Control*, Vol. 13, pp. 885-907, 2003.

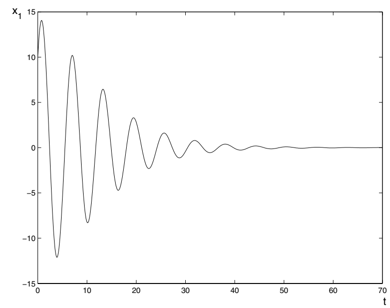

Fig. 9. Example 2. The state response of the closed-loop system under controller of this paper.

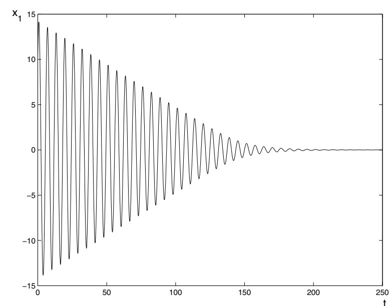

Fig. 10. Example 2. The state response of the closed-loop system under the controller of [12].

[7] J. Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977.

[8] Q.L. Han and B. Ni, "Delay-dependent robust stabilization for uncertain constrained systems with pointwise and distributed time-varying delays," *Proc. 38th IEEE Conf. Decision and Control*, pp. 215-220, 1999.

[9] T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhäuser, Boston, 2001. khalil

[10] H.K. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, Upper Saddle River, NJ, 2002.

[11] L. Levaggi and E. Punta, "On a second order discontinuous control system with delayed input," *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 4062-4067, 2003.

[12] F. Mazenc, S. Mondie and S-I Niculescu, "Global stabilization of oscillators with bounded delayed input," *Systems and Control Letters*, to appear in 2004.

[13] F. Mazenc, S. Mondie and S-I Niculescu, "Global asymptotic stabilization for chains of integrators with a delay in the input," *IEEE Transactions on Automatic Control*, Vol. 48, pp. 57-63, 2003.

[14] N. Minorsky, "Self-excited oscillations in dynamical systems possessing retarded actions," *J. Appl. Mech.*, Vol. 9, pp. A65-A71, 1942.

[15] S. Oucheriah, "Global stabilization of a class of linear continuous time-delay systems containing saturating controls," *IEEE Trans. Circuits Syst. I*, Vol. 43, pp. 1012-1015, 1996.

[16] S. Tarbouriech and J.M. Gomes da Silva Jr., "Synthesis of controllers for continuous-time delay systems with saturating controls via LMI's," *IEEE Trans. Auto. Contr.*, Vol. 45, pp. 105-111, 2000.

[17] H.J. Sussmann, E.D. Sontag and Y. Yang, "A general result on the stabilization of linear systems using bounded controls", *IEEE Transactions on Automatic Control*, Vol. 39, pp. 2411-2425, 1994.