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On Controllability of Switched Bilinear Systems

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***Abstract*— The controllability of switched bilinear systems (SBLS) is investigated. First, the structure of accessibility Lie algebra of SBLS is investigated. Some topological structure of (weak) controllability sub-manifolds is revealed. Then the practical controllability and the controllability of SBLS, and the controllability of state homogeneous SBLS are studied in sequence. Sets of easily verifiable sufficient conditions are obtained for each case.**

I. INTRODUCTION

In recent years, the switched systems have attracted considerable attention from the control community [12]. Controllability is one of the key issues. For switched linear system a necessary and sufficient condition for the controllable subspace was presented in [15]. [6] investigated the controllable sub-manifold which is more general than controllable subspace. Accessibility Lie algebra is essential for investigating controllability of nonlinear systems (including switched linear systems). Chow's theorem [9], [11] and generalized Frobinius' theorem [2] are two fundamental tools to connect Lie algebra with its integral manifold. They play essential role in our later approach.

As a particular kind of nonlinear systems, bilinear systems have special interest in both theoretical and practical aspects [14]. Many important systems, such as Lotka-Volterra equation for biological systems etc. [4] are in this category.

This paper considers a switched bilinear system (SBLS) on manifold M as

$$\begin{aligned} \dot{x} &= A^{\sigma(t)}x + \sum_{i=1}^{m}\left(B_i^{\sigma(t)}x + C_i^{\sigma(t)}\right)u_i, \quad x \in M \\ &:= A^{\sigma(t)}x + B^{\sigma(t)}ux + C^{\sigma(t)}u, \quad u \in \mathbb{R}^{\mathrm{m}}, \end{aligned} \tag{1}$$

where $\sigma(t) : [0, \infty) \to \Lambda$ is a right continuous measurable mapping, $\Lambda = \{1, 2, \cdots, N\}$, $u(t)$ are piecewise constant controls.

Assume $C_i^{\lambda} = 0$, $\forall \lambda \in \Lambda$ and $i = 1, \cdots, m$, then we have a state homogeneous switched bilinear system (SH-SBLS) as

$$\begin{aligned} \dot{x} &= A^{\sigma(t)}x + \sum_{i=1}^{m} u_i B_i^{\sigma(t)}x \\ &:= A^{\sigma(t)}x + B^{\sigma(t)}ux, \quad x \in M. \end{aligned} \tag{2}$$

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If, in addition, the drift terms are identically zero for all switching models, the system becomes

$$\dot{x} = \sum_{i=1}^{m} u_i B_i^{\sigma(t)}x := B^{\sigma(t)}ux, \quad x \in M, \tag{3}$$

which is called a state-control homogeneous switched bilinear system (SCH-SBLS).

Remark 1.1: *1. We assume for system (1) the state space $M = \mathbb{R}^{\mathrm{n}}$ and for system (2) or (3) the state space $M = \mathbb{R}^{\mathrm{n}}\backslash\{0\}$, because for (2) or (3) the origin, $\{0\}$, is obviously uncontrollable.*

2. For convenience, the matrix product in this paper is assumed to be semi-tensor product, which is defined and studied in [5]. A brief convenient reference is [7]. Semi-tensor product is a generalization of conventional matrix product. So when the dimensions of factor matrices meet the requirement of conventional matrix product, the product is the conventional one.

It is worth noting that for bilinear systems, each switching model is analytic. So the generalized Frobinius' theorem assures the existence of integral manifolds for the accessibility Lie algebra of system (1) (or (2) or (3)).

II. ACCESSIBILITY LIE ALGEBRA AND WEAK CONTROLLABILITY

Definition 2.1: *Consider a SBLS. (i) For a given $x \in \mathbb{R}^{\mathrm{n}}$, if there exist piecewise constant controls and a selected switching law $\sigma(t)$ such that the trajectory of the controlled switched system can be driven from x to y, then y is said to be in the reachable set of x, denoted by $y \in R(x)$.*

(ii) y is said to be weakly reachable from x, denoted as $y \in WR(x)$, if there exist spline-trajectories of the system, which connect a finite set of points, $x := x_0, x_1, \cdots, x_s := y$ pairwise in either forward or backward ways. Precisely, either $x_{k-1} \in R(x_k)$ or $x_k \in R(x_{k-1})$, $k = 1, 2, \cdots, s$.

(iii) An invariant sub-manifold $\mathcal{I}$ is called a controllable sub-manifold if for any two points $x, y \in \mathcal{I}$, $x \in R(y)$.

(vi) An invariant sub-manifold $\mathcal{I}$ is called a weak controllable sub-manifold if for any two points $x, y \in \mathcal{I}$, $x \in WR(y)$.

The controllable sub-manifolds are closely related to the Lie algebra generated by the vector fields extracted from the systems. Similar to the non-switched model case, we define the accessibility Lie algebra for SBLS as

Definition 2.2: *For system (1), the accessibility Lie algebra is defined as*

$$\mathcal{L}_a := \left\{ A^\lambda x, B_i^\lambda x + C_i^\lambda, \,\middle|\, \lambda \in \Lambda, i = 1, \cdots, m \right\}_{LA}. \quad (4)$$

The following result about weak controllability is a mimic of the corresponding result about general control systems [14], [16].

Proposition 2.3: *The system (1) ((2) or (3)) is globally weakly controllable, if the accessibility Lie algebra has full rank. That is,*

$$rank\mathcal{L}_a(x) = n, \quad \forall x \in M. \quad (5)$$

If (5) is satisfied, as for non-switching case, it is said that the accessibility rank condition is satisfied.

Definition 2.4: *Let V be a set of vector fields. V is said to be k-symmetric, if for any vector field $X \in V$, there is a vector field $Y \in V$ with $Y = -kX$, $k > 0$.*

Lemma 2.5: *Let Δ be an involutive analytic distribution, i.e., an involutive distribution generated by analytic vector fields. Moreover, assume $x \in WR(y)$ via spline integral curves of Δ, then*

$$rank(\Delta(y)) = rank(\Delta(x)).$$

Proposition 2.6: *1. Consider a SBLS. For a given point $x_0 \in \mathbb{R}^n$ if $rank(\mathcal{L}_a(x_0)) = k$ and there exists an open neighborhood U of x_0 such that*

$$N(x) := \{x \in U \mid rank(\mathcal{L}_a(x)) = k\}$$

is a k-th dimensional regular sub-manifold of U, then $N(x)$ is a weak controllability sub-manifold.

2. If a SBLS has (feedback) symmetric drift terms, $\{A_\lambda \mid \lambda \in \Lambda\}$, $N(x)$ is a controllable sub-manifold. Particularly for SCH-SBLS (3), $N(x)$ is a controllable sub-manifold.

Certain properties of the accessibility Lie algebra are investigated in Appendix.

III. Practical Controllability

Definition 3.1: System (1) ((2) or (3)) is said to be practically controllable at $x \in M$ if for any $y \in M$ and any given $\epsilon > 0$, there exist suitable controls and switching law such that the spline-trajectories of the switched controlled models can reach the ϵ neighborhood of y. The system is said to be practically controllable if it is practically controllable at every $x \in M$.

The constructive nonlinear decomposition technique has been used widely for bilinear systems [10]. For this approach, instead of studying the switched bilinear system (1), we consider the following set of switched systems: a linear system without control and 2 switched bilinear homogeneous control systems as:

$$\dot{x} = A^{\sigma(t)}x; \quad (6)$$

$$\begin{aligned} \dot{x} &= B^{\sigma(t)}ux + C^{\sigma(t)}u; \\ \dot{x} &= -\left(B^{\sigma(t)}ux + C^{\sigma(t)}u\right). \end{aligned} \quad (7)$$

Denote by $R_{LH}(x_0)$ the reachable set of the spline trajectories of (6) and (7). Then we have the following result, which is due to [10] for BLS. It can be extended to SBLS without any difficulties.

Lemma 3.2: *[10]* Consider system (1) ((2) or (3)). For every $x_0 \in M$, denote the reachable set of x_0 by $R(x_0)$, then

$$R_{LH}(x_0) \subset \mathrm{cl}\{R(x_0)\}. \quad (8)$$

Here cl denotes the closure of a set. Denote

$$\mathcal{V}_c = \left\{ (B_i^\lambda, C_i^\lambda) \middle| \lambda \in \Lambda, i = 1, \cdots, m \right\}_{LA},$$

which is generated by the vectors of input channels.

Using Lemma 3.2, we have the following result immediately:

Proposition 3.3: Consider system (1), ((2) or (3)). If

$$rank\left(\Psi(\mathcal{V}_c)(x)\right) = n, \quad \forall x \in M, \quad (9)$$

then the system is practically controllable.

Here the mapping Ψ was defined in Appendix 7.

To avoid the obstacle of non-symmetry of drift terms, we consider a class of systems, which, roughly speaking, have symmetric drift term.

Definition 3.4: The system (1) ((2) or (3)) is said to have a feedback k-symmetric drift terms, if there exist controls u_0^λ, $\lambda \in \Lambda$, such that for the new drift terms under feedback

$$\tilde{A}^\lambda = A^\lambda + B^\lambda u_0^\lambda, \quad \lambda \in \Lambda \quad (10)$$

form a k-symmetric set

$$\tilde{\mathcal{A}} := \left\{ \tilde{A}^\lambda | \lambda \in \Lambda \right\}.$$

Similar to Proposition 3.3, we can prove the following:

Proposition 3.5: Consider system (1) ((2) or (3)). Assume
(i). the system has feedback k-symmetric drift terms;
(ii).

$$\dim\{\mathcal{L}_a(x)\} = n, \quad \forall x \in M.$$

Then the system is practically controllable.

Example 3.6: *Consider a bilinear switched system*

$$\dot{x} = A^{\sigma(t)}x + u\left(B^{\sigma(t)}x + C^{\sigma(t)}\right), \quad (11)$$

where $\Lambda = \{1,2\}$ and

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X_1 = (A_1, 0)$, $Y_1 = (B_1, C_1)$, $Y_2 = (B_2, 0)$, $\mathcal{V}_a$ can be easily calculated as

$$\begin{aligned} Y_3 &= \langle Y_1, Y_2 \rangle = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right); \\ Y_4 &= \langle Y_1, Y_3 \rangle = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right); \\ Y_5 &= \langle Y_2, Y_4 \rangle = \left(0, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right); \\ Y_6 &= \langle Y_3, Y_5 \rangle = \left(0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right); \\ Y_7 &= \langle X_1, Y_6 \rangle = \left(0, \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} \right). \end{aligned}$$

Now

$$\Psi^{-1}(Y_5, Y_6, Y_7) = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} \right).$$

Therefore, $rank\,(\mathcal{L}_a(x)) = 3$, $\forall x \in \mathbb{R}^3$. i.e., the accessibility rank condition is satisfied.

In addition, it is obvious that system (11) has feedback k- symmetric drift terms. According to Proposition 3.5, the system is practically controllable. □

IV. Controllability of SBLS

This section considers the global controllability. Recall a result of local controllability for general control systems first.

Consider a general control system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \tag{12}$$

where f is a C^1 mapping. Let x_0 be an equilibrium of the control system with control $u_e(x)$. i.e., $f(x_e, u_e(x_e)) = 0$. Define

$$E = \frac{\partial f}{\partial x}(x, u_e(x))\bigg|_{x_0, u_e(x_0)}, \quad D = \frac{\partial f}{\partial u}(x, u_e(x))\bigg|_{x_0, u_e(x_0)}. \tag{13}$$

We have the following sufficient condition for local controllability.

Lemma 4.1: *[13], [14]* Consider system (12). Assume there exist $x_e \in \mathbb{R}^n$ and control $u_e(x)$. such that, $f(x_e, u_e(x_e)) = 0$. Moreover, assume (E, D), defined in (13), is completely controllable. Then (12) is locally controllable at x_e. That is, there exists an open neighborhood U of x_e, such that for any $x, y \in U$, $x \in R(y)$ and $y \in R(x)$.

Using it and the practical controllability investigated in last section, we deduce some sufficient conditions for global controllability.

Definition 4.2: Consider a bilinear system

$$\dot{x} = Ax + Bux + Cu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m. \tag{14}$$

1. A pair $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair, if

$$Ax_e + Bu_e x_e + Cu_e = 0. \tag{15}$$

2. An equilibrium pair (x_e, u_e) is said to be stable if $A + Bu_e$ is Hurwitz, it is said to be anti-stable if $-(A + Bu_e)$ is Hurwitz.

3. An equilibrium pair (x_e, u_e) is said to be controllable, if $(A + Bu_e, B(I_m \otimes x_e) + C)$ is a controllable pair.

Theorem 4.3: Consider system (1). Assume (i) it is practically controllable;

(ii) there exist $\lambda_1 \in \Lambda$ and an equilibrium pair $(e_u, u_e^{\lambda_1})$, such that $(e_u, u_e^{\lambda_1})$ is anti-stable for the λ_1-th switching model;

(iii) there exist $\lambda_2 \in \Lambda$ ($\lambda_2 = \lambda_1$ is allowed) and an equilibrium pairs $(e_u, u_e^{\lambda_2})$, such that $(e_u, u_e^{\lambda_2})$ is controllable for the λ_2-th switching model.

Then the system (1) is globally controllable.

Proof. *Since $(e_u, u_e^{\lambda_2})$ is controllable, by Lemma 4.1 there exists a neighborhood U of e_u such that the λ_2-th switching model is controllable over U.*

Next, we show that for any $x, y \in \mathbb{R}^n$ we can drive the state from x to y. Since the system is practically controllable we can first drive x to a point $\xi \in U$. Denote the λ_1-th switching model with control $u_e^{\lambda_1}$ by V, that is,

$$\begin{aligned} V &= \left(A^{\lambda_1} + B^{\lambda_1} u_e^{\lambda_1}\right) x + C^{\lambda_1} u_e^{\lambda_1} \\ &= \left(A^{\lambda_1} + B^{\lambda_1} u_e^{\lambda_1}\right)(x - x_e). \end{aligned}$$

Since $-V$ is stable, so the integral curve of $-V$ goes from y to x_e asymptotically. Hence there is a $T > 0$ such that

$$e_T^{-V}(y) = \eta \in U.$$

Equivalently,

$$y = e_T^V(\eta).$$

To complete the proof we have only to drive the state from ξ to η. This can be done by choosing λ_2-th switching model and a suitable control u^{λ_2}, because of local controllability of this model over U. □

Example 4.4: *Recall Example 3.6. We prove that system (11) is globally controllable. Using Theorem 4.3, we have to check conditions (i)-(iii). (i) is proved in Example 3.6. Now we choose a pair as $(x_e, u_e) = (0, 0)$. Obviously, it is* *an equilibrium pair. We then show that for the first model it is anti-stable. In fact,*

$$A_1 + B_1 u_e := A_1,$$

which is anti-stable. So (ii) is satisfied. Still use this pair to the first model. We have

$$(A_1 + B_1 u_e, B_1 x_e + C) := (A_1, C_1). \tag{16}$$

It is easy to check that (16) is completely controllable, which implies (iii). The conclusion follows. □

Following the same thought of train as in the proof of Theorem 4.3, we can have the following result immediately:

Proposition 4.5: Consider system (1). Assume (i) there exist $\lambda_1 \in \Lambda$ and an equilibrium pair $(e_u, u_e^{\lambda_1})$, such that $(e_u, u_e^{\lambda_1})$ is stable for the λ_1-th switching model;

(ii) there exist $\lambda_2 \in \Lambda$ and an equilibrium pair $(e_u, u_e^{\lambda_2})$, such that $(e_u, u_e^{\lambda_2})$ is anti-stable for the λ_2-th switching model;

(iii) there exist $\lambda_3 \in \Lambda$ and an equilibrium pairs $(e_u, u_e^{\lambda_3})$, such that $(e_u, u_e^{\lambda_3})$ is controllable for the λ_3-th switching model.

Then the system (1) is globally controllable.

Remark 4.6: In fact, In Theorem 4.3 e_u in condition (ii) (specified as e_u^2 to distinguish it from e_i in condition (iii)) can be different from the e_u (e_u^3) in condition (iii). It is enough that $e_u^2 \in R(e_u^3)$. Particularly, since $R(e_u^3)$ contains a controllable open neighborhood, U, of e_u^3, so it suffices that $e_u^2 \in U$. Similarly, for Proposition 4.5, $e_u^3 \in R(e_u^1)$ and $e_u^2 \in R(e_u^3)$ are enough. Particularly, when U is a controllable open neighborhood of e_u^3, and $e_u^1, e_u^2 \in U$ is enough.

V. CONTROLLABILITY OF SH-SBLS

State homogeneous bilinear systems have some special properties, which are heavily depending on the transfer matrices. These properties make them different from general SBLS. So, it is worth discussing them separately. Recall Remark 1.1, in this section $M = \mathbb{R}^n \backslash \{0\}$.

We first give the following result.

Theorem 5.1: Consider system (2). Assume

(i) the system has feedback k-symmetric drift terms;

(ii) for the $\tilde{A}^\lambda$ in the k-symmetric set, $\tilde{A}^\lambda$ is commutative with B_i^λ, $i = 1, \cdots, m$;

(iii).

$$\dim\{\mathcal{L}_a(x)\} = n, \quad \forall x \in M.$$

Then the system is globally controllable.

proof. It is easy to show that

$$\mathcal{L}_a(x) = \left\{ A^\lambda x + \sum_{i=1}^m u_i B_i^\lambda x \,\middle|\, u_i = constant,\ \lambda \in \Lambda \right\}_{LA}.$$

Without loss of generality we can assume A^λ, $\lambda \in \Lambda$ are k-symmetric and each one is commutative with its corresponding B_i^λ, $i = 1, \cdots, m$.

By condition (iii) and the Chow's Theorem, for any two points $x, y \in M$ there exit $X_1, \cdots, X_s$, where X_i are vector fields of the form $A^\lambda x + \sum_{i=1}^m u_i B_i^\lambda x$, such that (when the negative time is allowed)

$$y = e_{t_s}^{X_s} \cdots e_{t_1}^{X_1}(x). \tag{17}$$

Now in (17) assume $t_i = -d < 0$ and the corresponding vector field is

$$X_i = A^\lambda x + Bux.$$

Then we replace this segment of integral curve by the following spline integral curves:

$$\begin{aligned} e_{t_i}^{X_i} &= exp\left[-d\left(A^\lambda + B^\lambda u\right)\right] \\ &= exp\left[d\left(-A^\lambda - B^\lambda u\right)\right] \\ &= exp\left[-2dA^\lambda\right] exp\left[d\left(A^\lambda - B^\lambda u\right)\right] \\ &= exp\left[\tfrac{2d}{k}(-A^{k\lambda})\right] exp\left[d\left(A^\lambda - B^\lambda u\right)\right]. \end{aligned}$$

The above deduction is legal because of the commutativity. According to the k-symmetric property, it is easily seen that the last row in the above equation is physically realizable. The conclusion follows. □

Using the same argument about the partition of the controllable sub-manifolds as in Section 2, we have the following:

Corollary 5.2: Consider system (2). If the assumption (i) and (ii) of Theorem 5.1 hold, then for any $x_0 \in M$ the reachable set of x_0 is the largest integral manifold of $\mathcal{L}_a$, passing through x_0. Moreover, the controllable sub-manifolds are composed of all the largest integral sub-manifolds of $\mathcal{L}_a$.

Example 5.3: *Consider the following switched system*

$$\dot{x} = A^{\sigma(t)} + uB^{\sigma(t)}x, \tag{18}$$

where $\Lambda = 2$ and

$$A^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix};$$

$$B^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using $u_0^1 = -2$ and $u_0^2 = -4$, we have

$$\tilde{A}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now it is easy to check that the conditions (i) and (ii) of Theorem 5.1 are satisfied. Through a straightforward computation, a basis of $\mathcal{L}_a$ is obtained as

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

A normal routing verification shows that there are five controllable sub-manifolds: two one-dimensional sub-manifolds:

$$\{x \in \mathbb{R}^3 \,|\, x_1 > 0, x_2 = x_3 = 0\},$$
$$\{x \in \mathbb{R}^3 \,|\, x_1 < 0, x_2 = x_3 = 0\};$$

one two-dimensional sub-manifold:

$$\left\{x \in \mathbb{R}^3 \,\middle|\, \mathrm{x}_1 = 0, (\mathrm{x}_2, \mathrm{x}_3) \in \mathbb{R}^2\backslash\{0\}\right\};$$

and two three-dimensional sub-manifolds:

$$\{x \in \mathbb{R}^3 \,|\, \mathrm{x}_1 > 0, (\mathrm{x}_2, \mathrm{x}_3) \in \mathbb{R}^2\backslash\{0\}\},$$
$$\{x \in \mathbb{R}^3 \,|\, \mathrm{x}_1 < 0, (\mathrm{x}_2, \mathrm{x}_3) \in \mathbb{R}^2\backslash\{0\}\}.$$

□

Next, we deduce another set of sufficient conditions, which are based on the transitive properties of $GL(n, \mathbb{R})$. First we consider a single SH bilinear system

$$\dot{x} = Ax + Bux, \quad x \in M,\ u \in \mathbb{R}^{\mathrm{m}}. \tag{19}$$

We still use $\mathcal{L}_a$ for the Lie algebra generated by $\{A, B_i, i = 1, \cdots, m\}$. Denote by $\mathcal{G}$ the Lie group generated by $\mathcal{L}_a$, (precisely, the smallest connected Lie group with $\mathcal{L}_a$ as its Lie algebra). That is,

$$\mathcal{G} = \left\{ \prod_{i=1}^{k} e^{X_i} \,\middle|\, X_i \in \mathcal{L}_a, k < \infty \right\}. \tag{20}$$

An auxiliary system over $GL(n, \mathbb{R})$ is constructed as

$$\dot{\Theta} = A\Theta + Bu\Theta, \quad \Theta \in GL(n, \mathbb{R}), \mathrm{u} \in \mathbb{R}^{\mathrm{m}}. \tag{21}$$

Denote the reachable set of (21) with initial state I_n by $R(I)$. Using this auxiliary system, we can, instead of Lie algebra approach, use Lie group approach to investigate the controllability problem. The following result is due to Elliott and Tarn [3]:

Lemma 5.4: System (2) is globally controllable, iff for the system (21) the reachable set of the identity, $R(I)$, is transitive on M.

Using this, we have

Theorem 5.5: Consider system (2). Assume

(i) the system has feedback k-symmetric drift terms;

(ii) for each $\tilde{A}^\lambda$ in the k-symmetric set: $\tilde{A}^\lambda$ is commutative with B_i^λ, $i = 1, \cdots, m$;

(iii). The Lie group $\mathcal{G}$, generated by $\mathcal{L}_a$, is transitive on M.

Then the system (2) is globally controllable.

Proof. Note that $[A, B] = 0$ is equivalent to $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^B\mathrm{e}^A$. Then using same technique as in the proof of Theorem 5.1 we can prove that for the corresponding auxiliary switching system

$$\dot{\Theta} = A^{\sigma(t)}\Theta + B^{\sigma(t)}u\Theta, \quad \Theta \in GL(n, \mathbb{R}), \mathrm{u} \in \mathbb{R}^{\mathrm{m}}, \tag{22}$$

the reachable set $R(I)$ is transitive. The conclusion follows immediately. □

Example 5.6: *Consider the following switched system*

$$\dot{x} = A^{\sigma(t)}x + uB^{\sigma(t)}x, \quad x \in \mathbb{R}^3\backslash\{0\},\ \mathrm{u} \in \mathbb{R}, \tag{23}$$

where $\Lambda = \{1, 2\}$ and

$$A^1 = I_3, \quad A^2 = -I_3;$$

$$B^1 = \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}, \quad B^2 = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

Conditions (i) and (ii) of Theorem 5.5 are obviously true. Moreover, it is easy to calculate that

$$\mathcal{L}_a = o(3, \mathbb{R}) \oplus \{\mathrm{rI} \mid \mathrm{r} \in \mathbb{R}\}.$$

Then the connected Lie group of this Lie algebra is

$$\mathcal{G} = SO(3, \mathbb{R}) \oplus \{\mathrm{e^r I} | \, \mathrm{r} \in \mathbb{R}\}.$$

Where $o(3, \mathbb{R})$ is the 3 dimensional linear orthogonal algebra, $SO(3, \mathbb{R})$ is the 3 dimensional linear orthogonal Lie group, and $\oplus$ is a direct sum.

It is easy to see that $\mathcal{G}$ is transitive. So system (23) is completely controllable. □

A detailed transitive group approach and a list of transitive matrix algebras can be found in [1].

VI. CONCLUSION

In this paper the controllability of switched bilinear control systems was considered. First, the accessibility Lie algebra of switched bilinear systems was defined and used to investigate the weak controllability and weak controllable sub-manifolds. Then the practical controllability, the controllability of general SBLS, and the controllability of SH-SBLS were investigated in detail separately. For each case several easily verifiable sufficient conditions were provided. Some properties of accessibility Lie algebra were studied in Appendix.

It was shown in the paper that switched bilinear systems have rich algebraic structure and can be manipulated easily.

VII. APPENDIX

In this appendix, we explore structure and some properties of the accessibility Lie algebra $\mathcal{L}_a$ of the SBLS (1).

Consider a space $\mathcal{V}_n := M_{n\times n} \oplus \mathbb{R}^{\mathrm{n}}$, where $M_{n\times n}$ is the set of $n \times n$ matrices, $\oplus$ is a direct sum of two vector *spaces. Define the addition and scalar multiplication in conventional way as: for $(A,\xi),(B,\eta)\in\mathcal{V}_n$,*

$$\begin{cases}(A,\xi)\pm(B,\eta):=(A\pm B,\xi\pm\eta);\\ r(A,\xi):=(rA,r\xi),\quad r\in\mathbb{R}.\end{cases}\tag{24}$$

Then $\mathcal{V}_n$ becomes a vector space. Define a product $\langle,\rangle$ on $\mathcal{V}_n$ as

$$\langle(A,\xi),(B,\eta)\rangle:=[A,B]+A\eta-B\xi,\tag{25}$$

where $[A,B]:=AB-BA$ is the conventional Lie bracket on $gl(n,\mathbb{R})$.

In this way we produced a Lie algebra.

Proposition 7.1: The vector space $\mathcal{V}_n$ with the product $\langle,\rangle$ is a Lie algebra.

Now consider the set of analytic vector fields on $\mathbb{R}^n$, denoted by $V^\omega(\mathbb{R}^n)$. Define its linear subset as

$$L(\mathbb{R}^n):=\{\mathrm{X}=\mathrm{Ax}+\mathrm{C}|\,\mathrm{A}\in\mathrm{M_{n\times n}},\mathrm{C}\in\mathbb{R}^n\}\subset\mathrm{V}^\omega(\mathbb{R}^n);$$

and its homogeneous linear subset as

$$LH(\mathbb{R}^n):=\{\mathrm{X}=\mathrm{Ax}|\,\mathrm{A}\in\mathrm{M_{n\times n}}\}\subset\mathrm{L}(\mathbb{R}^n).$$

The following two propositions are straightforward verifiable.

Proposition 7.2: 1. Both $LH(\mathbb{R}^n)$ and $L(\mathbb{R}^n)$ are Lie sub-algebra of $V^\omega(\mathbb{R}^n)$;

2. Both $LH(\mathbb{R}^n)$ and $L(\mathbb{R}^n)$ are invariant under a linear coordinate transformation. Precisely, under a linear coordinate transformation the new LH and L are Lie algebra isomorphic to their predecessors respectively.

Proposition 7.3: Define a mapping $\Psi: L(\mathbb{R}^n)\to\mathcal{V}_n$ as:

$$\Psi(Ax+C)=-(A,C)$$

Then Ψ is a Lie algebra isomorphism.

Define by $\mathcal{V}_a\subset\mathcal{V}_n$ the Lie sub-algebra generated by the vector fields produced by system (1) with constant controls. That is,

$$\mathcal{V}_a=\left\{(A^\lambda,0),(B_i^\lambda,C_i^\lambda)\,\middle|\,\lambda\in\Lambda,i=1,\cdots,m\right\}_{LA}.\tag{26}$$

Then the following is obvious.

Corollary 7.4: 1.

$$\mathcal{L}_a=\Psi^{-1}\left(\mathcal{V}_a\right).\tag{27}$$

2.

$$\dim(\mathcal{L}_a)\le n(n+1).\tag{28}$$

This paper is closely related to the paper [8], which contains some details, while in this paper we emphasized on the algebraic structure of the bilinear systems.

REFERENCES

[1] *Boothby, W.M., E.N., Wilson, Determination of the transitivity of bilinear systems,* **SIAN J. Contr. Opt.**, *Vol. 17, No. 2, pp. 212-221, 1979.*

[2] *Boothby,* **An Introduction to Differential Manifolds and Riemannian Geometry**, *2nd ed. Academic Press Inc., Orlando, 1987.*

[3] *Bruni, C., G. Dipillo, G. Koch, Bilinear systems: An appealing class of nearly linera systems in theory and applications,* **IEEE Trans. Aut. Contr.**, *Vol. 19, No. 4, pp. 334-348, 1974.*

[4] *Casti, J.L.,* **Nonlinear System Theory**, *Academic Press, Orlandor, 1985.*

[5] *Cheng, D.,* **Matrix and Polynomial Approach to Dynamic Control Systems**, *Science Press, Beijing, 2002.*

[6] *Cheng, D., Accessibility of switched linear systems, Proc. 42nd IEEE CDC, Mauii, pp. 5709-5714, 2003.*

[7] *Cheng, D., X. Hu, Y. Wang, Non-regular Feedback Linearization of Nonlinear Systems via a Normal Form Algorithm, Automatica, Vol. 40, No. 3, 439-447, 2004.*

[8] *Cheng, D., Controllability of switched bilinear systems, to appear in IEEE Trans. Aut. Contr.*

[9] *Hermann, R.,* **Differential Geometry and the Calculus of Variations**, *Academic Press, 1968.*

[10] *Khapalov, A.Y., R.R. Mohler, Reachable sets and controllability of bilinear time-invariant systems: A qualitative Appraoch,* **IEEE Trans. Aut. Contr.**, *Vol. 41, Mo. 9, pp. 1342-1346, 1996.*

[11] *Krener, A.J., A generalization of Chow's theorem and the bang-bang theorem to nonlinear control problems,* **SIAM J. Contr.**, *Vol. 12, No. 1, 1974.*

[12] *Liberzon, D., and A.S. Morse, Basic problems on stability and design of switched systems,* **IEEE Control Systems Magazine**, *Vol.19, No. 5, pp. 59-70, 1999.*

[13] *Lee. E.B., Markus, L., Optimal control for optimal processes,* **Arch, Rational Mech. Anal.**, *Vol. 8, pp.36-58, 1961.*

[14] *Mohler, R.R.,* **Nonlinera Systems, Volume II, Applications to Bilinear Control**, *Prentice Hall, New Jersey, 1991.*

[15] *Sun, Z., S.S. Ge, T.H. Lee, Controllability and reachability criteria for switched linear systems,* **Automatica**, *Vol. 38, pp. 775-786, 2002.*

[16] *Susmann, H.J., A general theorem on local controllability,* **SIAM J. Contr. Opt.**, *Vol. 25, 158-194, 1987.*