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A Class of Dual-rate Sampled-data Models for Continuous-time Systems

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***Abstract*— This paper proposes a class of dual-rate sampled-data models for continuous-time systems. The proposed models are useful in digital control and in numerical simulations involving dual sampling rates which are integer related since they warrant a known time-domain performance in the limit of the sampling periods approaching zero. Designing a dual-rate control system complying to the conditions put forth in the paper, the designer warrants an intersample behavior free of undesired ripples, at least in the limit. The conditions pertain to the exogenous input signals, the discrete-time controller, and the dual-rate generalized holds and samplers. Furthermore, equivalences between dual-rate and single-rate systems are established. A numerical example illustrates the concept of dual-rate sampled-data models.**

I. INTRODUCTION

Dual-rate digital control is characterized by the fact that plant output sampling and control update rates are distinct, although usually integer-related. Dual-rate control can be used to reduce the computational burden on the digital processor; for example, in dual-stage disk drive track following [1], to optimize the allocation of processing power and to allow greater flexibility in the design of multi-channel, multi-loop control systems [2]. Using different plant output sampling and control input update rates may also be forced by system constraints; for instance, in vision-based control of autonomous vehicles [3].

The notion of a single-rate discrete-time (DT) model of a continuous-time (CT) system has been proposed in [4]. There, fundamental relationships between DT and CT systems are demonstrated. However, the concept of DT model does not address intersample behavior. It is well known that a DT control system may exhibit satisfactory performance at the sampling instants, although the inter-sample performance may be poor [5]. In [6], single-rate SD models of CT systems were proposed. A set of sufficient conditions warranting a known behavior of the SD control system in the limit, over all time instants, were given in [6]. The consequence is that, with the knowledge of a well-designed CT control system, a designer can use such CT system as a reference model and synthesize a single-rate SD control system such that its behavior approaches that of the reference model over all time instants in the limit. A few studies of dual-rate DT models are available in the literature for both linear and nonlinear systems. In [7], a method to identify fast-rate models from lifted dual-rate systems and used to capture intersample behavior for MPC synthesis is proposed. Reference [8] presents dual-rate DT models using the concept of dual-rate high-order holds and samplers. Equivalences with single-rate systems at low frequencies and recovery of the CT control systems transfer functions are demonstrated. [9] proposes to use a family of DT approximations for the single-rate DT modeling of nonlinear CT systems. The approximated model is then used for DT control design, and conditions are given in [9] such that the DT controller stabilizes the exact DT model for sufficiently short sampling periods. However, to our best knowledge, the notion of a dual-rate SD model of a CT system, which warrants uniform-in-time convergence of the signals of interest in the SD loop, has not yet been proposed.

This paper proposes a set of sufficient conditions for obtaining dual-rate SD models of linear, time-invariant CT systems, thereby guaranteeing that the dual-rate SD systems exhibit a known limiting behavior in the time domain, including intersamples. The particular class of SD systems studied are those relying on the use of dual-rate generalized holds [8] and samplers within the control loop. Such class of systems can be made equivalent to single-rate systems depending on the selection of the dual-rate generalized holds and samplers. A numerical example demonstrates the concept of dual-rate sampled-data models.

II. MATHEMATICAL PRELIMINARIES

A. Assumptions and Notation

The following assumptions are assumed to hold throughout the paper unless stated otherwise.

Assumption 1 DT signals have either period h (slow rate of $1/h$ Hz) or T (fast rate of $1/T$ Hz), which are related as $h = N \cdot T$, $N \in \mathcal{Z}^+$, where $\mathcal{Z}^+$ is the set of positive integers. T is non-pathological with respect to the plant transfer function.

Assumption 2 Zero-order hold (ZOH) H_p and ideal sampler S_p, which can each take a period p equal to T or h, are synchronized at time $t = 0$.

Assumption 3 Exogenous inputs to the control systems lie in the space $\mathcal{S}_c$, the space of bounded, uniformly continuous

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functions over $[0, \infty)$, and independent of T.

Assumption 4 CT (DT) signals are defined for time $t \geq 0$ ($k \geq 0$).

Assumption 5 The CT plant is denoted as $\overline{G}$. The plant is linear, time-invariant, has zero initial conditions and has a rational transfer function $\overline{G}(s)$ which is such that $|\overline{G}(s)| \to 0$ as $|s| \to \infty$.

A DT signal with period p is expressed as $u(k,p)$, where the arguments are the time step $k \in \mathcal{Z}^+$ and the sampling period p, which is T or h. The z-transform of $u(k,p)$ is given by $U(z,p)$, for either sampling period T or h; that is, for simplicity, z represents the complex variable in either case. For instance, the transfer function of DT system G is given as $G(z,p)$.

B. Definitions

Several key tools and concepts used in the paper are defined as follows.

Definition 1 [10] The *CT lifting* of a signal $f(t) \in \mathcal{S}_c$, performed with period p, can be visualized as a partitioning of $f(t)$ into an infinite number of functions, each of which being a copy of $f(t)$ within the time interval $[kp, (k+1)p)$ for $k \geq 0$, $p \in \mathcal{R}^+$. The CT lifted signal is represented as a sequence $\{f_{k,p}(\tau)\}_{k=0}^{\infty}$, where $\tau \in [0,p)$. $\bowtie$

Definition 2 [11] The *DT lifting* operation L takes a fast DT signal $f(k,T)$ and converts it to a slow N-vector DT signal with period h, $f^L(l,h)$; that is, $L : l^{\infty}_{\mathcal{R}} \to l^{\infty}_{\mathcal{R}^N}$. $\bowtie$

CT and DT lifting are norm-preserving. L^{-1} is the inverse DT lifting and $L^{-1}L = I$, so no information is lost when successively applying lifting and inverse lifting. The convergence of CT signals is given as follows.

Definition 3 A CT signal $f(t)$, lifted as $\{f_{k,p}(\tau)\}_{k=0}^{\infty}$, is said to *converge uniformly in time*, as the lifting period $p \to 0$, to a CT signal $g(t)$, lifted as $\{g_{k,p}(\tau)\}_{k=0}^{\infty}$, if

$$\lim_{p \to 0} \sup_{k \in [0,\infty)} \sup_{\tau \in [0,p)} \|f_{k,p}(\tau) - g_{k,p}(\tau)\| = 0, \qquad (1)$$

where $\|\cdot\|$ is the norm.$\bowtie$

The concept of dual-rate SD model is defined as follows.

Definition 4 A dual-rate SD system is said to be a *dual-rate SD model* of a CT system if the CT output of the SD system converges uniformly in time to that of the CT system when the CT input to the SD system converges uniformly in time to that of the CT system. $\bowtie$

III. DUAL-RATE SAMPLED-DATA MODELS

The dual-rate SD systems considered in this paper are shown in Fig. 1. Without loss of generality, the reference input is assumed available at the fastest sampling rate. If not, a fictitious decimator can be inserted at the reference input channel. The digital control block can be either dynamic or static, and includes dual-rate generalized holds and samplers to ensure rate transitions. Systems with fast plant output sampling have a block diagram shown in Fig. 1(a). Fast-output sampling has been used for static output feedback warranting complete pole placement [12]. The case of fast control update rate, slow plant output sampling is shown in Fig. 1(b).

A. Dual-rate Generalized Holds and Samplers

Definition 5 A *dual-rate generalized hold* (DRGH) $H_{h,T}$ is a system that receives a bounded DT input signal with period h and outputs a bounded DT signal at period T. DRGH handles slow-to-fast rate transition, at least conceptually.$\bowtie$

For a DT scalar input $u(k,h)$, the lifted output of the DRGH, $y^L(k,h) \in R^{N \times 1}$, is given in (2), where $H_j^{i-1}(T,h) \in \mathcal{R}$, $i = 1, ..., N$ and $j = -l, ..., m-1$. DRGH has a non-zero DT impulse response at $t = -l \cdot NT, -l \cdot NT + T, ..., (m-1)NT + (N-1)T$.

$$y^L(k,h) = \sum_{j=-l}^{m-1} \begin{bmatrix} H_j^0(T,h) \\ \vdots \\ H_j^{N-1}(T,h) \end{bmatrix} u(k-j,h) \qquad (2)$$

The simplest DRGH is the DT zero-order hold (DT-ZOH), which has the following lifted output to a DT scalar input $u(k,h)$

$$y^L(k,h) = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T u(k,h). \qquad (3)$$

The lifted outputs of common DRGH are given in (4) to (7).

Dual-rate first-order hold (DR-FOH)

$$\begin{bmatrix} 1 \\ 1 + \frac{h}{N} \\ \vdots \\ 1 + \frac{N-1}{N}h \end{bmatrix} u(k,h) + \begin{bmatrix} 0 \\ -\frac{h}{N} \\ \vdots \\ -\frac{N-1}{N}h \end{bmatrix} u(k-1,h) \qquad (4)$$

Dual-rate slewer hold (DR-SH)

$$\begin{bmatrix} 0 \\ \frac{h}{N} \\ \vdots \\ \frac{N-1}{N}h \end{bmatrix} u(k,h) + \begin{bmatrix} 1 \\ 1 - \frac{h}{N} \\ \vdots \\ 1 - \frac{N-1}{N}h \end{bmatrix} u(k-1,h) \qquad (5)$$

Dual-rate fractional-order hold (DR-FROH)

$$\begin{bmatrix} 1 \\ 1 + \frac{\varepsilon h}{N} \\ \vdots \\ 1 + \varepsilon h \frac{N-1}{N} \end{bmatrix} u(k,h) + \begin{bmatrix} 0 \\ -\frac{\varepsilon h}{N} \\ \vdots \\ -\varepsilon h \frac{N-1}{N} \end{bmatrix} u(k-1,h) \qquad (6)$$

Dual-rate moving-average hold (DR-MAH), $Q \in \mathcal{Z}^+$

$$\begin{bmatrix} \frac{1}{Q} \\ \vdots \\ \frac{1}{Q} \end{bmatrix} u(k,h) + \ldots + \begin{bmatrix} \frac{1}{Q} \\ \vdots \\ \frac{1}{Q} \end{bmatrix} u(k-Q+1,h) \qquad (7)$$

Definition 6 A *dual-rate generalized sampler* (DRGS) $S_{T,h}$ is a system that receives a bounded DT input signal with period T and outputs a bounded DT signal at period h. DRGS performs fast-to-slow rate transition. The DRGS considered in this paper is the decimator sampler (DR-DS) which outputs every Nth input. $\bowtie$

For a scalar input $u(k,T)$, lifted as $u^L(l,h)$, the output of the DR-DS is given as

$$y(l,h) = [1,0,...,0]u^L(l,h). \quad (8)$$

Remarks 1) The DRGH can be implemented as an up-sampler followed by an FIR filter [13]. 2) The DR-DS corresponds to a down-sampler. Decimation can be accomplished by preceding the DR-DS by an appropriate lowpass filter [13].

B. Some Equivalences

A dual-rate system is conceptually equivalent to a single-rate system provided certain conditions are fulfilled by the DRGH, DRGS, hold and sampler. Two such equivalences are described in Propositions 1 and 2.

Proposition 1 Assume $H_{h,T}$ is the DT-ZOH and $S_{T,h}$ is the DR-DS. Then

$$S_{T,h}H_{h,T} = I,\ H_T H_{h,T} = H_h,\ S_{T,h}S_T = S_h. \bowtie \quad (9)$$

Proof: Obvious from Definitions 5 and 6. □

Proposition 2 Assume $H_{h,T}$ is the DRGH described in (2). Furthermore, let a multi-interval generalized hold denoted as $\widetilde{H}_h$ have the following CT lifted response to a DT scalar input $u(k,h)$:

$$y_{k,h}(\tau) = \sum_{j=-l}^{m-1} \widetilde{H}_j(\tau,h)u(k-j,h) \quad (10)$$

where $\widetilde{H}_j(\tau,h)$ are piecewise-continuous functions of $\tau \in [0,h)$. Subject $H_T H_{h,T}$ and $\widetilde{H}_h$ to the unit DT impulse input

$$u(k,h) = \left\{ \begin{array}{l} 1,\ k=0 \\ 0,\ k \neq 0 \end{array} \right. . \quad (11)$$

For a fixed N,

$$\lim_{h\to 0} \sup_{k\in[0,\infty)} \sup_{\tau\in[0,h)} \left\| H_T H_{h,T}u - \widetilde{H}_h u \right\| = 0 \quad (12)$$

provided

$$\lim_{h\to 0} \max_{i,j} \sup_{\tau\in[iT,(i+1)T)} \left| H_j^i(T,h) - \widetilde{H}_j(\tau,h) \right| = 0. \bowtie \quad (13)$$

Proof: Decompose $\widetilde{H}_j(\tau,h)$ into

$$\widetilde{H}_j(\tau,h) = \left\{ \begin{array}{c} \widetilde{H}_j^0(\tau,h), \tau\in[0,T) \\ \vdots \\ \widetilde{H}_j^{N-1}(\tau,h), \tau\in[(N-1)T,NT) \end{array} \right. . \quad (14)$$

Then, from (2) and (10), write

$$\begin{array}{l} H_T H_{h,T}u - \widetilde{H}_h u \\ = \sum_{j=-l}^{m-1} \left[\begin{array}{c} H_j^0(T,h) - \widetilde{H}_j^0(\tau,h) \\ \vdots \\ H_j^{N-1}(T,h) - \widetilde{H}_j^{N-1}(\tau,h) \end{array} \right] u(k-j,h). \end{array} \quad (15)$$

If (13) is satisfied, it is clear from (15) that $H_T H_{h,T}u - \widetilde{H}_h u$ can be made arbitrarily small over all time instants by reducing h. □

As an example, the system composed of the DR-FOH (4) followed by the ZOH can be readily seen to approach the well-known FOH [5] in the sense of Proposition 2.

C. Convergence of Dual-rate Sampled-data Models

The main contribution of the paper is Theorem 1. The proof of Theorem 1 relies on Lemmas 1 and 2. The CT and SD control systems are shown in Figure 2, where $\overline{W}$ is the system from reference input to plant input.

Lemma 1 Let $G_T = H_{h,T}S_h\overline{G}H_T$, as shown in Fig. 2(c), and let $G = S_T\overline{G}H_T$. Subject G_T to $u(k,T) \in l_R^\infty$ and G, to $\widetilde{u}(k,T) \in l_R^\infty$, where $H_T\widetilde{u}$ can be made arbitrarily close to a continuous function $\overline{u}(t)$ by choosing a sufficiently small T. Denote the output of G_T as $y(k,T)$ and that of G as $\widetilde{y}(k,T)$. Select a fixed $N \in \mathcal{Z}^+$. If, for each fixed $t \in \mathcal{R}^+$,

$$\lim_{T\to 0} \|\widetilde{u}(\kappa,T) - u(\kappa,T)\| = 0, \quad (16)$$

where κ is an integer such that $\kappa T \le t < (\kappa+1)T$, and

$$\lim_{T\to 0} \sum_{j=-l}^{m-1} H_j^i(T,h) = 1,\ \forall i \in [0,N-1], \quad (17)$$

then, for each fixed t,

$$\lim_{\substack{T\to 0 \\ \kappa T\le t<(\kappa+1)T}} \|\widetilde{y}(\kappa,T) - y(\kappa,T)\| = 0. \bowtie \quad (18)$$

Proof: Let

$$\overline{G}: \begin{array}{l} \frac{dx(t)}{dt} = \overline{A}x(t) + \overline{B}u(t) \\ y(t) = \overline{C}x(t) + \overline{D}u(t). \end{array} \quad (19)$$

From Assumption 5, $\overline{D} = 0$. $\overline{G}$ can also be expressed by means of CT lifting at period T as

$$\begin{array}{l} x(k+1,T) = e^{\overline{A}T}x(k,T) + \int_{v=0}^{T} e^{\overline{A}(T-v)}\overline{B}u_{k,T}(v)dv \\ y_{k,T}(\tau) = \overline{C}e^{\overline{A}\tau}x(k,T) + \overline{C}\int_{v=0}^{\tau} e^{\overline{A}(\tau-v)}\overline{B}u_{k,T}(v)dv \end{array} \quad (20)$$

where $\tau \in [0,T)$, $x(k,T) = x(t)|_{t=kT}$ and $\{u_{k,T}(\tau)\}_{k=0}^\infty$ is the signal obtained with CT lifting of $u(t)$. From (20), $G = S_T\overline{G}H_T$ can be readily obtained as

$$\begin{array}{l} x(k+1,T) = \underbrace{e^{\overline{A}T}}_{=A_d}x(k,T) + \underbrace{\left(\int_{v=0}^{T} e^{\overline{A}(T-v)}\overline{B}dv\right)}_{=B_d}\widetilde{u}(k,T) \\ y(k,T) = \overline{C}x(k,T) \end{array} \quad (21)$$

where $y(k,T) = y(t)|_{t=kT}$ and $u(k,T)$ is the DT signal entering the ZOH. Performing DT lifting at period h on

(21) yields

$$x^L((k+1)N,T) = \underbrace{e^{\overline{A}TN}}_{=A_d^L} x^L(kN,T)$$
$$+\underbrace{\begin{bmatrix} A_d^{N-1}B_d & \dots & B_d \end{bmatrix}}_{=B_d^L} \widetilde{u}^L(kN,T)$$
$$y^L(kN,T) = \underbrace{\begin{bmatrix} \overline{C} \\ \overline{C}A_d \\ \vdots \\ \overline{C}A_d^{N-1} \end{bmatrix}}_{=C_d^L} x^L(kN,T) \tag{22}$$
$$+\underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ \overline{C}B_d & 0 & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 \\ \overline{C}A_d^{N-2}B_d & \dots & \overline{C}B_d & 0 \end{bmatrix}}_{=D_d^L} \widetilde{u}^L(kN,T).$$

Similarly, perform DT lifting at period h on $G_T = H_{h,T}S_h\overline{G}H_T$ using the expression (2) to obtain

$$x_{G_T}^L((k+1)N,T) = A_d^L x_{G_T}^L(kN,T) + B_d^L u^L(kN,T)$$
$$y_{G_T}^L(kN,T) = \sum_{j=-l}^{m-1} \begin{bmatrix} H_j^0(T,h) \\ \vdots \\ H_j^{N-1}(T,h) \end{bmatrix} \overline{C} x_{G_T}^L((k-j)N,T). \tag{23}$$

Fix time t and write $u^L(jN,T) = \widetilde{u}^L(jN,T) + \Delta(jN,T)$ where $\lim_{T\to 0}\sup_{0\le j<k}\|\Delta(jN,T)\| = 0$, $kNT \le t < (kN+1)T$. Using (17), (22) and (23), the norm of the difference in lifted outputs can be made arbitrarily small with T:

$$\left\|y^L(kN,T) - y_{G_T}^L(kN,T)\right\|$$
$$\le \underbrace{\left\| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \sum_{i=-l}^{m-1} H_i^0(T,h) \\ \vdots \\ \sum_{i=-l}^{m-1} H_i^{N-1}(T,h) \end{bmatrix} \right\|}_{\to 0\ (T\to 0)}$$
$$\cdot \underbrace{\left\| \overline{C} \sum_{j=0}^{k-1} (A_d^L)^{k-j-1} B_d^L \widetilde{u}^L(jN,T) \right\|}_{\text{finite for each fixed } t}$$
$$+\underbrace{\left\| \sum_{j=0}^{k-1}\sum_{i=-l}^{m-1} \begin{bmatrix} H_i^0(T,h) \\ \vdots \\ H_i^{N-1}(T,h) \end{bmatrix} \overline{C}(A_d^L)^{k-j-1} B_d^L \Delta(jN,T) \right\|}_{\to 0\ (T\to 0)}$$
$$+\underbrace{\left\| \begin{bmatrix} 0 \\ \overline{C}B_d \\ \vdots \\ \overline{C}A_d^{N-2}B_d + \dots + \overline{C}B_d \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \right\|}_{\to 0\ (T\to 0)}$$
$$+\underbrace{\left\| \begin{bmatrix} 0 \\ \overline{D}\Delta(N,T) \\ \vdots \\ \overline{C}A_d^{N-3}B_d\Delta(N,T) + \dots + \overline{D}\Delta(N-1,T) \end{bmatrix} \right\|}_{\to 0\ (T\to 0)}. \square \tag{24}$$

Lemma 2 Let $G_h = S_{T,h}\beta S_T\overline{G}H_h$, as shown in Fig. 2(b), and let $G = S_h\overline{G}H_h$. Subject G_h to $u(k,h) \in l_R^\infty$ and G, to $\widetilde{u}(k,h) \in l_R^\infty$, where $H_h\widetilde{u}$ can be made arbitrarily close to a continuous function $\overline{u}(t)$ by making h short enough. Denote the output of G_h as $y(k,h)$ and that of G as $\widetilde{y}(k,h)$. Select a fixed $N \in \mathcal{Z}^+$. If, for each fixed $t \in \mathcal{R}^+$,

$$\lim_{\substack{h\to 0 \\ \kappa h\le t<(\kappa+1)h}} \|\widetilde{u}(\kappa,h) - u(\kappa,h)\| = 0 \tag{25}$$

and β is an FIR filter such that

$$\beta(z,T) = \frac{a_0 z^N + a_1 z^{N-1} + \dots + a_N}{z^N},\ \lim_{T\to 0}\sum_{i=0}^{N} a_i = 1, \tag{26}$$

then, for each fixed t,

$$\lim_{\substack{h\to 0 \\ \kappa h\le t<(\kappa+1)h}} \|\widetilde{y}(\kappa,h) - y(\kappa,h)\| = 0.\ \bowtie \tag{27}$$

Proof: Consider the plant given in (19). Fix time t, let $u(j,h) = \widetilde{u}(j,h) + \Delta(j,h)$ where $\lim_{h\to 0}\sup_{0\le j<k}\|\Delta(j,h)\| = 0$, $kh \le t < (k+1)h$ from the conditions on the input signals, and write the norm of the difference in the outputs of $S_{T,h}\beta S_T\overline{G}H_h$ and $S_h\overline{G}H_h$ as

$$\|\widetilde{y}(k,h) - y(k,h)\| \le \Psi_1\|u(k,h)\| + \Psi_2$$
$$+\Psi_3 \left\| \sum_{j=0}^{k-1} (e^{\overline{A}h})^{k-j-1} \int_{v=0}^{h} e^{\overline{A}(h-v)}\overline{B}dv\,u(j,h) \right\| \tag{28}$$

where

$$\Psi_1 = \left\|\overline{D} - D_h^L\right\|,$$
$$D_h^L = a_0\overline{D} + \begin{bmatrix} a_N & \dots & a_1 \end{bmatrix}$$
$$\times \begin{bmatrix} \overline{D} \\ \overline{C}B_d + \overline{D} \\ \vdots \\ \overline{C}A_d{}^{(N-2)}B_d + \dots + \overline{C}B_d + \overline{D} \end{bmatrix} q^{-1}, \tag{29}$$

$$\Psi_2 = \left\| C_h^L \sum_{j=0}^{k-1} (e^{\overline{A}h})^{k-j-1} \int_{v=0}^{h} e^{\overline{A}(h-v)}\overline{B}dv\,\Delta(j,h) \right\|$$
$$+\left\|D_h^L\Delta(k,h)\right\|, \tag{30}$$

$$\Psi_3 = \left\|\overline{C} - C_h^L\right\|,$$
$$C_h^L = \begin{bmatrix} a_N & \dots & a_1 \end{bmatrix} \begin{bmatrix} \overline{C} \\ \overline{C}A_d \\ \vdots \\ \overline{C}A_d{}^{(N-1)} \end{bmatrix} q^{-1} + a_0\overline{C}. \tag{31}$$

Each Ψ_1, Ψ_2 and Ψ_3 approaches zero as $h \to 0$ (fixed N) from (26) and the fact the effect of the delay operator q^{-1} vanishes with $h \to 0$. $\square$

Theorem 1 Consider the control systems shown in Fig. 2, where $\overline{G}$ is as defined in Lemma 1. Either the control systems are subjected to the same reference input in $\mathcal{S}_c$, or the reference inputs to the SD control systems converge uniformly in time, as the sampling periods approach zero, to that of the CT control system, which lies in $\mathcal{S}_c$. The control systems in Fig. 2 (b) and (c) are dual-rate SD models of the CT control system in Fig. 2(a) if the following conditions are satisfied:

Condition 1 (Stability): The DT controllers internally stabilize the plant at the sampling instants.

Condition 2 (FIR Filter): β satisfies (26).

Condition 3 (DRGH): $H_{h,T}$ fulfills (17).

Condition 4 (DT Controller): With

$$\overline{\Omega}: \begin{array}{l} \frac{dx_{\overline{\Omega}}(t)}{dt} = \overline{A}_{\overline{\Omega}} x_{\overline{\Omega}}(t) + \overline{B}_{\overline{\Omega}} e(t) \\ u(t) = \overline{C}_{\overline{\Omega}} x_{\overline{\Omega}}(t) + \overline{D}_{\overline{\Omega}} e(t), \end{array} \tag{32}$$

where $e(t)$ is controller input, $x_{\overline{\Omega}}(t)$ is controller state and $u(t)$ is plant input, the DT controllers have at least one realization in the δ-form

$$\Omega: \begin{array}{l} \delta x_{\Omega}(k,p) = A_{\Omega} x_{\Omega}(k,p) + B_{\Omega} e(k,p) \\ u(k,p) = C_{\Omega} x_{\Omega}(k,p) + D_{\Omega} e(k,p),\ p \in \{T, h\} \end{array} \tag{33}$$

such that $\lim_{p\to 0} A_{\Omega} = \overline{A}_{\overline{\Omega}}$, $\lim_{p\to 0} B_{\Omega} = \overline{B}_{\overline{\Omega}}$, $\lim_{p\to 0} C_{\Omega} = \overline{C}_{\overline{\Omega}}$ and $\lim_{p\to 0} D_{\Omega} = \overline{D}_{\overline{\Omega}}$. $\bowtie$

Proof: From Lemmas 1 and 2 (*Conditions 2* and *3*), $H_{h,T} S_h \overline{G} H_T$ and $S_{T,h} \beta S_T \overline{G} H_h$ behave like the single-rate systems $S_T \overline{G} H_T$ and $S_h \overline{G} H_h$, respectively, for each time instant as the sampling periods approach zero. With *Conditions 1* and *4* satisfied, the closed-loop DT systems are stable at the sampling instants and there exists at least one realization of W approaching that of $\overline{W}$ as the sampling periods approach zero. Then, from [6], uniform-in-time convergence of the CT plant input and output in Fig. 2(b) and (c) to those respective signals in Fig. 2(a) is obtained.$\square$

Remarks 1) With Lemma 1, the output of $H_{h,T} S_h \overline{G} H_T$ can be made arbitrarily close to that of $S_T \overline{G} H_T$ for each fixed time instant. A consequence of Lemma 1 is that $H_{h,T} S_h \overline{G} H_T$ is a DT model of $\overline{G}$ [4]. 2) When $\beta(z,T) = 1$, the system of Fig. 2(b) reduces to a single-rate DT system with period h according to (9).

IV. NUMERICAL EXAMPLE

Consider the double-integrator plant $\overline{G}$ in closed-loop with $\overline{\Omega}(s) = (s+0.5)/(s+3)$ [14]. The CT control system has the structure of Fig. 2(a). $\overline{\Omega}(s)$ is discretized using two well-known techniques; that is, Tustin's method, yielding

$$\Omega_{Tustin}(z,p) = \frac{(2/p + 1/2)z + (1/2 - 2/p)}{(2/p+3)z + (3 - 2/p)}, \tag{34}$$

and the step-invariant model

$$\Omega_{SIM}(z,p) = \frac{z - (e^{-3p}+5)/6}{z - e^{-3p}}. \tag{35}$$

It can be shown that there exist realizations of (34) and (35) in the δ-form approaching a realization of $\overline{\Omega}(s)$ in the sense of *Condition 4* in Theorem 1. Controllers (34) and (35) are placed in closed-loop with $\overline{G}$ according to Fig. 2(b)-(c). The DRGH $H_{h,T}$ is given by (3)-(7), where $\varepsilon = 0.5$ and $Q = 4$, $S_{T,h}$ is the DR-DS and the filter β is assumed to be any of the following three transfer functions:

$$\beta_1(z,T) = \frac{\frac{1}{4}z^3 + \frac{1}{4}z^2 + \frac{1}{4}z + \frac{1}{4}}{z^3}, \tag{36}$$

$$\beta_2(z,T) = \frac{-10z^3 + 10z^2 + \frac{1}{2}z + \frac{1}{2}}{z^3}, \tag{37}$$

$$\beta_3(z,T) = \frac{z^3 + z^2 + 0.1z + 10}{z^3}. \tag{38}$$

β_1 and β_2 satisfy *Condition 2* of Theorem 1 whereas β_3 violates the condition. The responses obtained with the various control systems subjected to a fixed sinusoidal reference input are shown in Fig. 3, 4. Fig. 3 presents the responses of the closed-loop systems having block diagrams shown in Fig. 2(a)-(b). In Figure 3, the legend indicates the DT controller used, either (34) or (35), and the FIR filter β found in the feedback loop, which is either β_1, β_2 or β_3. Fig. 4 shows the responses obtained with systems having the structures of Fig. 2(a), (c) for various DRGH. For brevity, only the responses obtained with Ω_{Tustin} are shown; however, similar results were obtained with Ω_{SIM}. It is clear from the figures that dual-rate control systems fulfilling the conditions of Theorem 1 have plant inputs and outputs approaching, uniformly in time, those respective signals of the CT control system.

V. CONCLUSIONS

A set of sufficient conditions to obtain dual-rate sampled-data models of linear, time-invariant continuous-time systems were proposed. Such models guarantee a known time-domain behavior for the dual-rate sampled-data systems, including intersamples, in the limit of the sampling periods approaching zero. Furthermore, equivalences between dual-rate generalized holds and well-known multi-interval, single-rate holds, such as the first-order and the slewer holds, were established. In the future, the proposed sampled-data models could be extended to include other types of linear dual-rate control systems, multivariable systems and classes of nonlinear systems.

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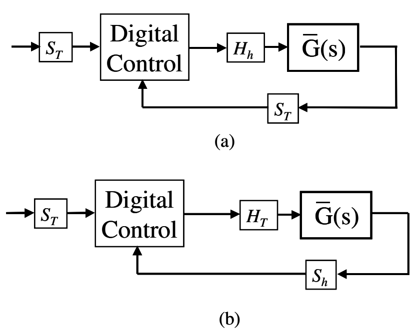

Fig. 1: Dual-rate control systems

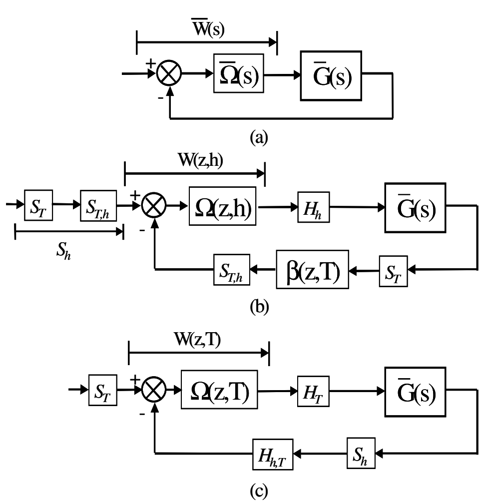

Fig. 2: CT and SD control systems

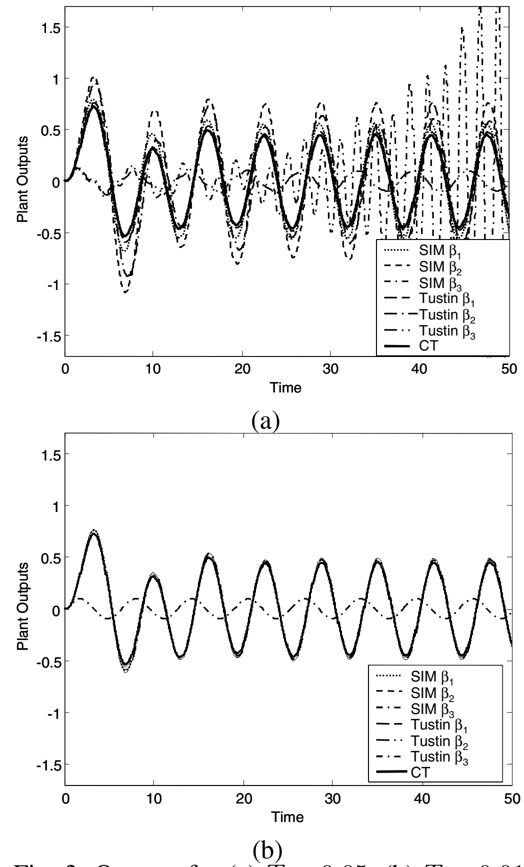

Fig. 3: Outputs for (a) $T = 0.05$, (b) $T = 0.01$ $(N = 3)$

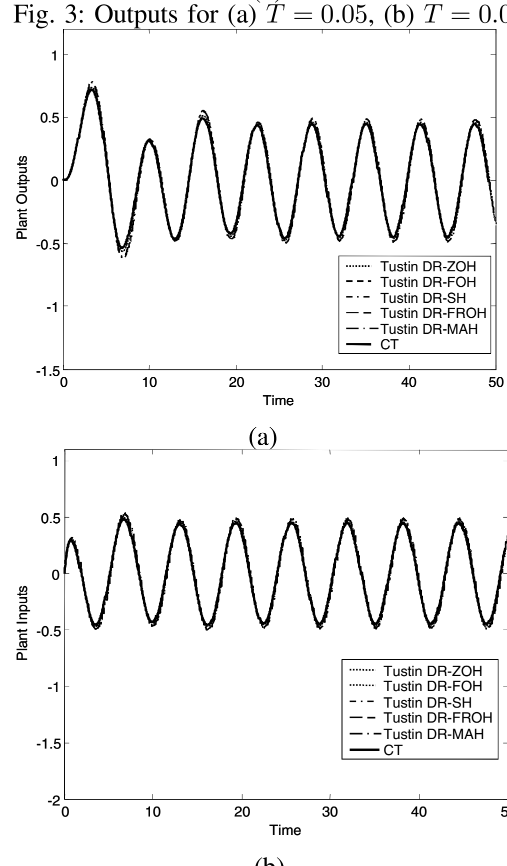

Fig. 4: (a) Plant outputs and (b) plant inputs for $T = 0.05$ $(N = 3)$