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A Unified Procedure for Discrete-Time Root Locus and Bode Design

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***Abstract*—As an alternative to the numerous distinct controller design algorithms in discrete-time textbooks, a simple, unified design approach is presented for all standard discrete-time, classical compensators independent of the form of the system information. This approach is based on a simple root locus design procedure for a proportional-derivative (PD) compensator. From this procedure, design procedures for discrete-time lead, proportional-integral (PI), lag, proportional-integral-derivative (PID), and PI-lead compensators are developed. With this proposed approach, students can concentrate on the larger control system design issues, such as compensator selection and closed-loop performance, rather than the intricacies of a particular design procedure. To demonstrate this approach, an example of a lead design from a digital control system laboratory is presented.**

I. Introduction

In controls education today, there seems to be gap between the theory taught in the typical undergraduate classroom and what students are able to apply to practical systems. One obvious reason for this is the lack of undergraduate control system laboratories. The control systems community has recognized this need [1,2]. In the Systems Engineering Department at the United States Naval Academy, as well as many departments around the world, undergraduate control system laboratories are being developed [3,4].

A less obvious reason for this gap is the "cookbook" approach to compensator design found in typical classical control textbooks [5,6,7]. For example, a quick comparison reveals significant differences in the procedures for root locus lead design and root locus PI design. Even more importantly, there are significant differences in the procedures for lead compensator design using root locus techniques and Bode techniques. Furthermore, for even fairly simple systems, these design procedures may yield poor results [8]. Students can become even more confused as new methods are introduced for discrete-time control system design [9,10]. As a consequence, students concentrate on the different "recipes", which may or may not yield satisfactory results, and, as a consequence, tend to miss the big picture.

Discrete-time design methods are classified as indirect or direct. In the indirect methods, a continuous-time compensator is designed from a continuous-time model of the system and discretized for a discrete-time implementation. The indirect methods require only limited knowledge of discrete-time control. However, as Ogata [9] points out, "discretizing the continuous-time control system creates new phenomenon not present in the original continuous-time control system." To overcome this issue, the direct methods employ root locus or Bode techniques to compute the discrete-time compensator from a discrete-time model of the system. In discrete-time control textbooks [9, 10,11], direct design of classical, discrete-time compensators receives far less attention than the analogous design methods in textbooks on continuous-time control design. Furthermore, root locus techniques are not developed for each compensator and Bode techniques rely on a transformation of the pulse transfer function.

In this paper, design methods are developed that permit students to apply a simple, unified design approach for all compensators independent of the form of the system information. That is, in root locus design, the computational procedures are based on the open-loop transfer function whereas, in Bode design, the computational procedures are based on the magnitude and phase of the open-loop frequency response. With this proposed approach, students can concentrate on the larger control system design issues, such as compensator selection and closed-loop performance, rather than the intricacies of a particular design procedure. The proposed design methods (as well the continuous-time methods in [12,13]) have been applied successfully in classical and discrete-time control classes in the Systems Engineering Department of the United States Naval Academy.

Procedures for standard compensators (lead, rate feedback, proportional-integral (PI), lag, proportional-integral-derivative (PID), and PI-lead compensators) are

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developed based on a simple root locus design proportional-derivative (PD) procedure. While the procedures presented in this paper are analogous to those presented in [12,13], the presentation is self-contained and only assumes knowledge of standard classical control concepts. The remainder of the paper is organized as follows. In Section II, generalized magnitude and phase criteria are presented. A PD compensator design procedure is presented in Section III. Design procedures for lead, PI and lag, and PID and PI-Lead compensators are discussed in Sections IV through VII. An example of a lead design from an undergraduate control system laboratory is presented in Section VIII. Concluding remarks are presented in Section IX.

II. COMPENSATOR DESIGN

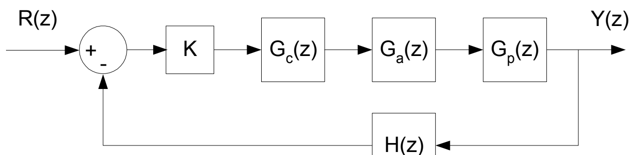

Fig. 1. Closed-loop block diagram

The integrated design procedure using time or frequency domain plant data requires a generalization of the angle criterion from root locus design. The standard closed-loop system is shown in Figure 1 where K is the control gain, $G_c(z)$ is the compensator, $G_a(z)$ represents the actuator dynamics, $G_p(z)$ represents the plant dynamics, and $H(z)$ represents the sensor dynamics.

In root locus design, the compensator must satisfy the well-known angle and magnitude criteria

$$\begin{aligned} &\angle G_c(z_d) + \angle G_{sys}(z_d) = \pm 180^\circ \\ &K\left|G_c(z_d)G_{sys}(z_d)\right| = 1 \end{aligned} \tag{1}$$

where $G_{sys}(z) := G_a(z)G_p(z)H(z)$. The discrete-time design point $z_d = e^{s_d T}$, where T is the sampling period, is determined from the continuous-time design point $s_d = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2}$ where ζ is the damping ratio and ω_n is the natural frequency.

In Bode design methods, the specifications are incorporated through the desired phase margin PM and gain crossover frequency ω_{gc} and result in another set of angle and magnitude constraints

$$\begin{aligned} &\angle G_c\left(e^{j\omega_{gc}T}\right) + \angle G_{sys}\left(e^{j\omega_{gc}T}\right) = \pm 180^\circ + PM \\ &K\left|G_c\left(e^{j\omega_{gc}T}\right)G_{sys}\left(e^{j\omega_{gc}T}\right)\right| = 1 \end{aligned} \tag{2}$$

Using standard 2nd order assumptions, such as those found in [5], the PM and ω_{gc} can also be determined from the continuous-time design point as

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right) \tag{3}$$

$$\omega_{gc} = \frac{2\zeta\omega_n}{\tan(PM)} \tag{4}$$

To write the angle and magnitude constraints in (1) and (2) in a common form, the notions of the design point and the desired angle must be generalized. Comparing (1) and (2), a reasonable definition for the design point is

$$z_0 = \begin{cases} e^{s_d T}, & \text{root locus} \\ e^{j\omega_{gc}T}, & \text{Bode} \end{cases} \tag{5}$$

Using this definition, the generalized angle and magnitude constraints take the form

$$\begin{aligned} &\angle G_c(z_0) + \angle G_{sys}(z_0) = \phi \\ &K\left|G_c(z_0)G_{sys}(z_0)\right| = 1 \end{aligned} \tag{6}$$

where the desired angle in the angle constraint is

$$\phi = \begin{cases} \pm 180^\circ, & \text{root locus} \\ \pm 180^\circ + PM, & \text{Bode} \end{cases} \tag{7}$$

Using the generalized constraints in (6) and (7), the computations for the design of discrete-time compensators are identical using root locus or Bode methods. Furthermore, given the generalized design point (5) and the angle constraint defined in (6) and (7), the desired compensator angle is computed from

$$\angle G_c(z_0) = \phi - \angle G_{sys}(z_0) =: \theta_c \tag{8}$$

where the desired compensator angle θ_c can be computed from the plant information and the design point without knowledge of the compensator type. In root locus methods, θ_c determines a geometric relationship between the design point and the compensator poles and zeros. In Bode methods, θ_c is the phase the must be added at the gain crossover frequency.

III. PD COMPENSATOR DESIGN

As in the continuous-time case, the design procedures for all compensators are based on the PD design procedure [12,13]. The PD compensator has a transfer function

$$G_c(z) = \frac{z-\alpha}{z} \tag{9}$$

The angle of the PD compensator at the design point z_0 is

$$\angle G_c(z_0) = \angle(z_0 - \alpha) - \angle(z_0) \tag{10}$$

and the PD zero is computed from

$$\theta_c = \angle(z_0 - \alpha) - \angle(z_0) \tag{11}$$

where the desired angle θ_c is defined in (8),

$$\angle(z_0 - \alpha) = \tan^{-1}\left(\frac{y_0}{x_0 - \alpha}\right) \tag{12}$$

and $z_0 = x_0 + jy_0$. Using (11) and (12), the compensator zero is given by

$$\alpha = x_0 - \frac{y_0}{\tan(\theta_\alpha)} \tag{13}$$

where $\theta_\alpha = \theta_{c,PD} := \theta_c + \angle(z_0)$. For this compensator, and each compensator to follow, the gain K is computed using the magnitude constraint in (6).

There is a limit to the improvement that the PD compensator can achieve. In general, the compensator zero should not be placed outside the unit circle because a non-minimum phase compensator can lead to poor performance and/or instability in the closed-loop system. Under this assumption, the maximum value for $\theta_{c,PD}$ is $\angle(z_0 - 1)$ and is achieved by the discrete-time derivative compensator $G_c(z) = \frac{z-1}{z}$. The PD compensator reduces to a proportional controller if $\alpha = 0$ and, therefore, the minimum value for $\theta_{c,PD}$ is $\angle z_0$. It follows that the design point can be achieved or, equivalently, that the PD compensator design problem is feasible if and only if $\angle(z_0) < \theta_{c,PD} < \angle(z_0 - 1)$ or, equivalently, $0 < \theta_c < \theta_{c,\max}$ where

$$\theta_{c,\max} = \angle(z_0 - 1) - \angle(z_0) \tag{14}$$

Feasibility relationships are shown in Table I for the other compensators (lead, PI, PID, PI-lead) using the relationships between these compensators and the PD compensator developed in the sequel.

TABLE I
FEASIBILITY OF COMPENSATOR DESIGNS

Compensator	Feasibility relationship
PD, lead	$0 < \theta_c < \theta_{c,\max}$
PI, lag	$-\theta_{c,\max} < \theta_c < 0$
PID, PI-lead	$-\theta_{c,\max} < \theta_c < \theta_{c,\max}$

To illustrate the computations introduced in this section, a simple root locus example is provided. For the transfer function $G(z) = (z+0.8)/(z^2 - 1.5z + 0.5)$, suppose a design point $z_0 = 0.5 + j0.5$ is given. The desired compensator angle is computed from (8) and $\theta_c = 24°$. For the given data, the design for the basic compensator is feasible because $0 < \theta_c < \theta_{c,\max} = 90°$ where $\theta_{c,\max}$ is computed from (14). From (13), the compensator zero is $\alpha = 0.308$ and, from (6), the gain is $K = 0.335$.

IV. Lead Compensator Design

The lead compensator has a transfer function $G_c(z) = \frac{z-\alpha}{z-\beta}$ where $\alpha > \beta$. As in the continuous-time case [12,13], the PD compensator design provides the limits on the lead compensator design as discussed below.

The angle of the lead compensator at the design point z_0 is $\angle G_c(z_0) = \angle(z_0 - \alpha) - \angle(z_0 - \beta) = \theta_\alpha - \theta_\beta$ and the lead pole and zero must be selected to satisfy the angle constraint (8) or, equivalently,

$$\theta_c = \theta_\alpha - \theta_\beta \tag{15}$$

In general, it is not desirable to place the compensator pole on the negative real axis (even inside the unit circle) because this pole location can lead to an oscillatory control signal. As a result, $\beta \geq 0$ and it follows that $\theta_\beta = \theta_\alpha - \theta_c \geq \angle(z_0)$ as illustrated in Figure 2. Furthermore, it follows that $\theta_\alpha \geq \theta_c + \angle(z_0) = \theta_{c,PD}$ and that

$$1 \geq \alpha \geq \alpha_{PD} \tag{16}$$

as illustrated in Figure 2. Note that this constraint is analogous to the constraint obtained in the continuous-time case [12,13]. It follows from the relationship between the lead and PD compensators that the lead compensator is feasible if and only if the PD compensator is feasible.

The lead compensator design has three unknowns and only two constraints. As in the continuous-time case, the lead compensator zero is chosen using the constraint in (16). After the lead zero is chosen the lead pole is computed from

$$\beta = x_0 - \frac{y_0}{\tan\left(\theta_\beta\right)} \tag{17}$$

where $\theta_\beta = \theta_\alpha - \theta_c$.

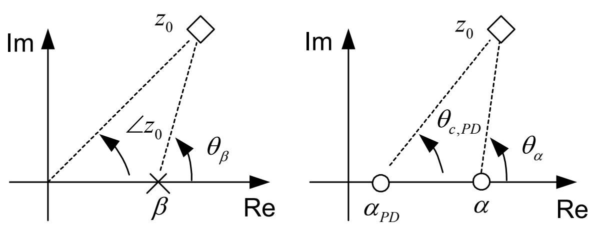

Fig. 2. Relationship between the PD compensator zero and the lead compensator pole and zero

V. PI COMPENSATOR DESIGN

The PI compensator has a transfer function $G_c(z) = \dfrac{z-\alpha}{z-1}$. The angle of the PI compensator at the design point z_0 is $\angle G_c(z_0) = \angle\left(z_0-\alpha\right) - \angle\left(z_0-1\right)$ and the PI zero is computed from $\angle\left(z_0-\alpha\right) - \angle\left(z_0-1\right) = \theta_c$. The above design expression can be rewritten to collect the known terms on one side of the equation $\angle\left(z_0-\alpha\right) = \theta_c + \angle\left(z_0-1\right) =: \theta_{c,PI}$ where $\theta_{c,PI}$ can be computed from the plant information and the design point assuming a PI compensator is desired. The compensator zero, α_{PI}, is computed using (13) with $\theta_\alpha = \theta_{c,PI}$.

Note that the PI compensator is just a special case of the lag compensator. The lag compensator has the same form as the lead compensator, but $\alpha < \beta$ for the lag compensator. The design procedure for the lag compensator is identical to that of the lead compensator, except that, instead of α satisfying (16), it must chosen such that $0 \le \alpha \le \alpha_{PI}$.

VI. PID COMPENSATOR DESIGN

The PID compensator has a transfer function $G_c(z) = \dfrac{\left(z-\alpha_1\right)\left(z-\alpha_2\right)}{z\left(z-1\right)}$. The angle of the PID compensator at the design point z_0 is $\angle G_c\left(z_0\right) = \angle\left(z_0-\alpha_1\right) + \angle\left(z_0-\alpha_2\right) - \angle\left(z_0-1\right) - \angle\left(z_0\right)$ and the PID zeros are computed from

$$\begin{aligned}\theta_{\alpha_1} + \theta_{\alpha_2} &= \angle\left(z_0-\alpha_1\right) + \angle\left(z_0-\alpha_2\right) \\ &= \theta_c + \angle\left(z_0-1\right) + \angle\left(z_0\right) =: \theta_{c,PID}\end{aligned} \tag{18}$$

Since there are three unknown parameters and only two constraints, there is a degree of freedom in selecting the PID parameters. Two design methods are considered. In the first method, one of the PID zeros is chosen, most likely to cancel a plant pole or shape the loop response. The remaining PID zero is then computed from the angle constraint in (18). That is, given the PID zero α_1, the remaining PID zero is computed using (13) with $\theta_\alpha = \theta_{c,PID} - \theta_{\alpha_1}$. In the second method, the two PID zeros are assumed to be co-located and computed directly from the angle constraint. If $\alpha_1 = \alpha_2 =: \alpha$, the angle constraint in (18) becomes $2\theta_\alpha = \theta_{c,PID}$ and the PID zeros are computed using (13) with $\theta_\alpha = \theta_{c,PID}/2$.

VII. PI-LEAD COMPENSATOR DESIGN

The PI-lead compensator has a transfer function $G_c(z) = \dfrac{\left(z-\alpha_1\right)\left(z-\alpha_2\right)}{\left(z-\beta\right)\left(z-1\right)}$. The angle of the PID compensator at the design point z_0 is $\angle G_c\left(z_0\right) = \angle\left(z_0-\alpha_1\right) + \angle\left(z_0-\alpha_2\right) - \angle\left(z_0-1\right) - \angle\left(z_0-\beta\right)$ and the PI-lead pole and zeros are computed from

$$\begin{aligned}\theta_{\alpha_1} + \theta_{\alpha_2} - \theta_\beta &= \angle\left(z_0-\alpha_1\right) + \angle\left(z_0-\alpha_2\right) \\ -\angle\left(z_0-\beta\right) &= \theta_c + \angle\left(z_0-1\right) = \theta_{c,PI}\end{aligned} \tag{19}$$

Since there are four unknown parameters and only two constraints, there are two degrees of freedom in selecting the PI-lead parameters. As with the PID design, two design methods are considered.

In the first method, one of the PI-lead zeros is chosen, most likely to cancel a plant pole or shape the loop response. Given the PI-lead zero α_1, the remaining PI-lead zero and pole must satisfy

$$\theta_{\alpha_2} - \theta_\beta =: \theta_{c,PI} - \theta_{\alpha_1} \tag{20}$$

where the quantities on the right side of (20) are known and the quantities on the left side of (20) are unknown. Comparing (20) with the angle constraint (15) for the lead compensator design, the selection of α_2 and β are equivalent to a lead compensator design for the desired compensator angle $\hat{\theta}_c := \theta_{c,PI} - \theta_{\alpha_1}$. Using this analogy, the selection of α_2 must satisfy a constraint similar to (16) $1 \ge \alpha_2 \ge x_0 - \dfrac{y_0}{\tan\left(\hat{\theta}_c + \angle(z_0)\right)}$. After α_2 is selected, β is computed from $\beta = x_0 - \dfrac{y_0}{\tan\left(\theta_{\alpha_2} - \hat{\theta}_c\right)}$.

In the second method, the two PI-lead zeros are assumed to be co-located, $\alpha_1 = \alpha_2 =: \alpha$, and the angle constraint in (19) becomes $2\theta_\alpha - \theta_\beta = \theta_{c,PI}$. This angle constraint is analogous to the angle constraint (15) for the lead compensator. Mimicking the lead design procedure, the PI-lead zeros must be greater than the PID zeros for the co-located case, i.e., $1 \ge \alpha \ge \alpha_{PID}$. After the PI-lead zeros are chosen, the PI-lead pole is computed from $\beta = x_0 - \dfrac{y_0}{\tan\left(\theta_\beta\right)}$ where $\theta_\beta = 2\theta_\alpha - \theta_{c,PI}$.

VIII. EXAMPLE

The example from [13] is revisited and a discrete-time compensator is designed to regulate the shaft position of a SRV-02 DC motor from Quanser Consulting, Incorporated. In this case, the angular position data is sampled at a rate of 50 samples/second $(T = 0.02\,\text{sec})$. The closed loop step response is specified to have an overshoot of less than *5%* and a settling time of less than *0.2 seconds*. This example is indicative of students' designs in a senior-level digital control systems laboratory course.

The open loop information is represented by the Bode plot in Figure 3 and describes the relationship between the input voltage and the shaft position (in degrees). The Simulink-based, real-time dynamic signal analyzer described in [4] was used to experimentally measure the Bode plot in Figure 3 at the required sample rate of 50 samples/second. The frequency-domain specifications have been determined from the given time-domain specifications using (4) and (5) and are listed in Table II.

From the Bode plot in Figure 3, these specifications cannot be achieved using proportional control. If the gain is computed to achieve $\omega_{gc} = 19.5$ *rad/sec*, the uncompensated phase margin is $49°$. It follows that phase must be added to the system and a PD or lead compensator must be used. A lead compensator is chosen to illustrate the proposed design methods. Given the desired phase margin and gain crossover frequency, $\theta_c = 24°$ because the desired phase at ω_{gc} is $-116°$ and the actual phase is $-140°$ (from Figure 3). The lead compensator design are feasible since $0 < \theta_c < \theta_{c,\max} = 79°$ where $\theta_{c,\max}$ is computed from (14) using $z_0 = e^{j(19.5)(0.02)}$.

Using the methods introduced in Section III, the PD zero is computed to be $\alpha_{PD} = 0.563$ and the lead zero α_{lead} must be chosen to satisfy $1 \ge \alpha_{lead} \ge 0.563$. The initial lead design is summarized in Table III.

From (4), the settling time will be reduced if the desired gain crossover frequency is increased to $\omega_{gc} = 21$ *rad/sec*. Repeating the above steps, a lead compensator $G_c(z) = \dfrac{z - 0.65}{z - 0.183}$ and gain $K = 0.404$ are obtained. The closed-loop step response is shown in Figure 4 and has a settling time of 0.199 *seconds* and no overshoot. The response meets both specifications.

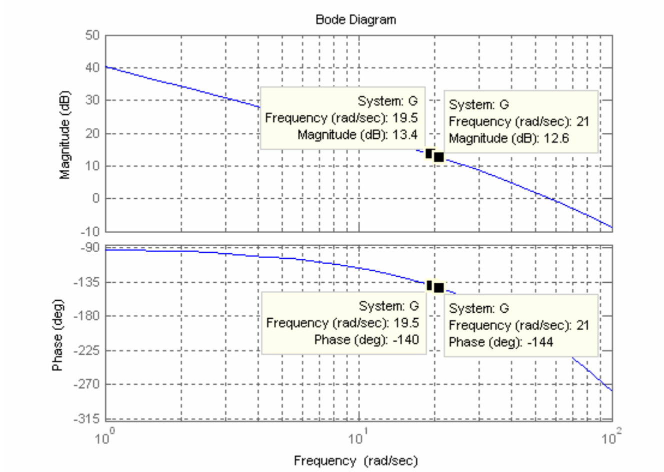

Fig. 3. Bode plot of DC motor for a sampling rate of 50 samples/second

TABLE II
DESIGN SPECIFICATIONS

Percent overshoot (*P.O.*)	5.0 %
Settling time (t_s)	0.20 sec
Damping ratio (ζ)	0.69
Undamped natural frequency (ω_n)	29 rad/sec
Phase margin (*PM*)	64°
Gain crossover frequency (ω_{gc})	19.5 rad/sec

TABLE III
INITIAL LEAD COMPENSATOR DESIGN

Lead zero (α_{lead})	0.6
Lead pole (β_{lead})	0.127
$\left\lvert G_{sys}\left(e^{j(19.5)(0.02)}\right)\right\rvert$	4.68
$\left\lvert G_c\left(e^{j(19.5)(0.02)}\right)\right\rvert$	0.705
K	0.378
$(P.O.)_{measured}$	0 %
$(t_s)_{measured}$	0.220 sec

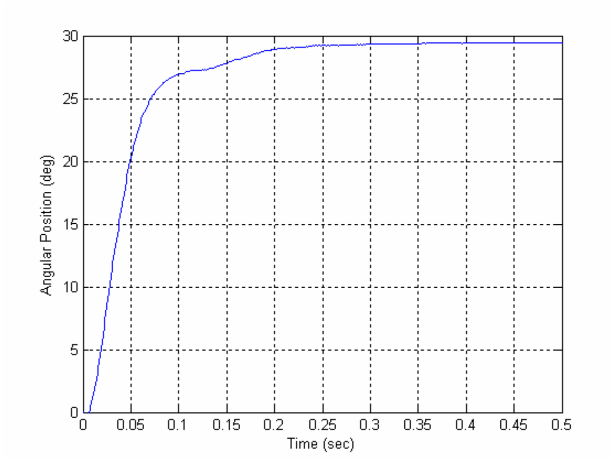

Fig. 4. Step response of lead-compensated DC motor

IX. Conclusion

Discrete-time compensator design methods have been streamlined with the objective of moving the students' focus from the computational procedures of the algorithms to the more important issues of control system design such as compensator selection and closed-loop performance. Established discrete-time control concepts were presented in a logical progression that facilitates comprehension for students in a first course in discrete-time control. The design procedures for five compensators: lead, PI, lag, PID, and PI-lead were developed from a PD design procedure. These procedures are analogous to the continuous-time design procedures presented in [12,13]. This common design approach helps to bridge the gaps between the more intuitive continuous-time design and the practical direct discrete-time design.

References

[1] P. Dorato, "Undergraduate Control Education in the U.S.," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 38-39, 1999.

[2] P. Antsaklis, T. Basar, R. DeCarlo, N.H. McClamroch, M. Spong, and S. Yurkovich, "Report on the NSF/CSS Workshop on New Directions in Control Engineering Education," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 53-58, 1999.

[3] B. Armstrong and R. Perez, "Controls Laboratory Program with an Accent on Discovery Learning," *IEEE Control Systems Magazine*, vol. 21, no. 1, pp. 14-21, 2001.

[4] J. M. Watkins and R. T. O'Brien, Jr., "A Novel Approach to a Control Systems Laboratory," *Proceedings of the 2003 ASME International Mechanical Engineering Congress & Exposition*, Washington, DC, November, 2004.

[5] N. S. Nise, *Control Systems Engineering*, 3rd Edition, John Wiley & Sons, New York, 2000.

[6] K. Ogata, *Modern Control Engineering*, 4th Edition, Prentice-Hall, New Jersey, 2002.

[7] R. C. Dorf and R. H. Bishop, *Modern Control Systems,* 9th Edition, Prentice-Hall, New Jersey, 2001.

[8] R. C. Garcia and B. S. Heck, "An Interactive Tool for Classical Control Design Education," *Proceedings of the American Control Conference*, pp. 1460-1464, San Diego, CA, June 1999.

[9] K. Ogata, *Discrete-Time Control Systems*, Prentice Hall, New Jersey, 1987.

[10] K. Astrom and B. Wittenmark, *Computer Control Systems: Theory and Design*, 3rd edition, Prentice Hall, New Jersey, 1997.

[11] G. Franklin, J. Powell and M. Workman, *Digital Control of Dynamic Systems*, 3rd edition, Addison Wesley Longman, California, 1998.

[12] R. T. O'Brien, Jr. and J. M. Watkins, "A Streamlined Approach for Teaching Root Locus Compensator Design," *Proceedings of the Conference on Decision and Control*, Orlando, FL, December 2001.

[13] R. T. O'Brien, Jr. and J. M. Watkins, "A Unified Approach For Teaching Root Locus And Bode Compensator Design," *Proceedings of the American Control Conference*, Denver, CO, June 2003.