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Robust Output Feedback Tracking Control for a Class of MIMO Nonlinear Systems

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Abstract— **This paper proposes a robust output feedback controller for a class of nonlinear systems to track a desired trajectory. Our main goal is to ensure the global input-to-state (ISS) property of the tracking error nonlinear dynamics with respect to the unknown structural system uncertainties and external disturbances. Our approach consists of constructing a nonlinear observer to reconstruct the unavailable states, and then designing a discontinuous controller using a back-stepping like design procedure to ensure the ISS property. The observer design is realized through state transformation and there is only one parameter to be determined. Through solving a Hamilton-Jacoby inequality, the nonlinear control law for the first subsystem specifies a nonlinear switching surface. By virtue of nonlinear control for the first subsystem, the resulting sliding manifold in the sliding phase possesses the desired ISS property and to certain extent the optimality. Associated with the new switching surface, the sliding mode control is applied to the second subsystem to accomplish the tracking task. As a result the tracking error is bounded and the ISS property of the whole system can be ensured while the internal stability is also achieved. Finally, an example is presented to show the effectiveness of the proposed scheme.**

I. INTRODUCTION

Output feedback tracking control has been the subject of constant research over the past several decades. Despite the efforts, robust tracking of general nonlinear systems remains an open problem. In the literature, several approaches have been proposed to deal with the output feedback control in the presence of structured or unstructured uncertainties: adaptive control approach [1], variable structure control approach [2], and output dynamics controller with almost disturbance decoupling [3], etc. However, the adaptive control approach, can only deal with systems with constant parametric uncertainties. Oh and Khalil considered a nonlinear SISO system that can be represented by an input-output model and applied a tracking error estimator [2]. In [3], Marino and Tomei considered SISO systems with nonlinearities that depends on outputs only.

Assuming that it is not possible to have a sensor for each state variable, if the controller is in the state feedback form, then it is necessary to design a state observer to estimate the internal states. Several observers have been proposed for linear and nonlinear systems. In the presence of disturbances and model uncertainty, high gain observer has the advantage that they can acquire the state information while neglecting the influence of these effects. In [2] and [4], different types of observers based on high gain estimation error feedback were proposed for SISO systems and MIMO systems with both linear or highly nonlinear terms. Due to their high gains in the feedback form, these observers are effective in ensuring the convergence of estimation errors so that asymptotic states can be obtained for the controller design.

In sliding mode control (SMC), switching surface design and discontinuous reaching control law are two of the control issues. A common practice in SMC is to design a switching surface according to the null space dynamics, which must ensure a stable sliding manifold when the system is in the sliding mode [5]. However if there exist uncertaities in the null space nonlinear dynamics, switching surface design becomes extremely difficult. Traditionally the reaching control law is to force the system to reach and stay on the switching surface. Nevertheless, this feature alone is no longer sufficient in the presence of unmatched uncertainties. Due to the effect of the unmatched uncertainties, the nonlinear dynamics may become divergent in a period shorter than the reaching time, if the ISS property does not hold during the reaching phase. Hence, ISS property should be guaranteed either in the sliding phase or in the reaching phase.

In this paper, a class of nonlinear systems with null space dynamics and range space dynamics are considered for the tracking control task. Assuming that the full state is not available for measurement, the main objective of the paper is to ensure global ISS of the tracking error nonlinear dynamics while achieving a small tracking error bound. The features of our approach are the following: (i) a nonlinear observer is designed in which only one parameter needs to be determined; (ii) the resulting sliding manifold in the sliding phase possesses the desired ISS property and to certain extent the optimality through solving a Hamilton-Jacoby inequality; (iii) associated with the new switching surface, the sliding mode control applied to the second subsystem achieves the desired tracking. As a result the tracking error is bounded and the ISS property of the whole system can be ensured while the internal stability of the system states is also ensured. Usually, a discontinuous term is used to handle the matched $L_\infty[0, \infty)$ type system disturbance where the upper-bound knowledge is available [6] [7]. However, in this paper, a discontinuous term is used

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to ensure convergence of the error dynamics resulting from the observer and the desired trajectory in which there are no uncertain terms in the error dynamics. In the example, it is shown that the tracking error is small and bounded under the proposed controller.

II. Problem Formulation

A. Notation and Preliminaries

$\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximum and minimum eigenvalue of the matrix A respectively; $\{A\}_{\bar{n}\times\bar{n}}$ represents the first n rows and n columns in A, and $\{A\}_{\underline{m}\times\underline{n}}$ represents the last m rows and n columns in A; $L_\infty[0,\infty)$ is the space of uniformly bounded functions on $[0,\infty)$.

Input-to-state (ISS) stability [8]-[9]: Consider a nonlinear dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}), \tag{1}$$

where $\mathbf{x}$ and $\mathbf{u}$ are the states and the inputs of the system respectively. The system in (1) is said to be locally input-to-state-stable if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and constants k_1, $k_2 \in \mathcal{R}^+$ such that

$$\begin{aligned} \|\mathbf{x}(t)\| \le\ & \beta(\|\mathbf{x}_0\|,t) + \gamma(\|\mathbf{u}_T(\cdot)\|_{\mathcal{L}_\infty}), \\ & \forall t \ge\ 0, \quad 0 \le\ T \le t, \end{aligned} \tag{2}$$

for all $\mathbf{x}_0 \in D$ and $\mathbf{u} \in D_u$ satisfying: $\|\mathbf{x}_0\| < k_1$ and $sup_{t>0}\|\mathbf{u}_T(t)\| = \|\mathbf{u}_T\|_{\mathcal{L}_\infty} < k_2$, $0 \le\ T \le t$. It is said to be input-to-state stable or globally ISS if $D = \mathcal{R}^n$, $D_u = \mathcal{R}^m$ and (2) is satisfied for any initial state and any bounded input $\mathbf{u}$.

B. Problem Formulation

The following MIMO nonlinear cascade system with uncertainties is considered

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(t,\mathbf{x}_1) + B_1(t)\mathbf{x}_2 + G_1(t,\mathbf{x}_1)\mathbf{d}_1(t) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(t,\mathbf{x}) + B_2(t)\left[\mathbf{u} + \boldsymbol{\eta}(t,\mathbf{x})\right] \\ \qquad +G_2(t,\mathbf{x})\mathbf{d}_2(t) \\ \mathbf{y} = \mathbf{x}_1, \end{cases} \tag{3}$$

where $\mathbf{x} = [\mathbf{x}_1,\mathbf{x}_2]^T$ is the system states, $\mathbf{x}_1 \in \mathcal{R}^n$ is the null space dynamics and $\mathbf{x}_2 \in \mathcal{R}^m$ is the range space dynamics, $\mathbf{u} \in \mathcal{R}^m$ denotes the control input, $\mathbf{d}_1 \in \mathcal{R}^p$ and $\mathbf{d}_2 \in \mathcal{R}^q$ are the external disturbances. The mappings $\mathbf{f}_1(t,\mathbf{x}_1) \in \mathcal{R}^n$, $\mathbf{f}_2(t,\mathbf{x}) \in \mathcal{R}^m$, $B_1(t) \in \mathcal{R}^{n\times m}$, $B_2(t) \in \mathcal{R}^{m\times m}$, $G_1(t,\mathbf{x}_1) \in \mathcal{R}^{n\times p}$ and $G_2(t,\mathbf{x}) \in \mathcal{R}^{m\times q}$ are known. $\boldsymbol{\eta}(\mathbf{x},t) \in \mathcal{R}^m$ is the matched uncertainty. We assume that $m \le\ n$. The system (3) is assumed to satisfy the following assumptions.

Assumption 1: There exist two positive constants α_1 and α_2 such that $\forall \mathbf{x}_1 \in \mathcal{R}^n$, $\forall t > 0$,

$$0 < \alpha_1^2 I_m \le\ B_1^T(t)B_1(t) \le\ \alpha_2^2 I_m, \tag{4}$$

where I_m is the identity matrix. $B_2(t)$ is assumed to be invertible.

Assumption 2: The uncertainties $\mathbf{d}_1(t)$, $\mathbf{d}_2(t)$ and $\boldsymbol{\eta}(\mathbf{x},t)$ are bounded as

$$|\mathbf{d}_1(t)| \le\ \beta_1, \quad |\mathbf{d}_2(t)| \le\ \beta_2, \quad |\boldsymbol{\eta}(\mathbf{x},t)| \le\ \beta_\eta \tag{5}$$

where β_1, β_2 and β_η are known positive constants.

C. Control Objective

The system is required to track the known reference model: $\mathbf{y} \Rightarrow \mathbf{y}_d = \mathbf{x}_{1d}$, i.e., the $\mathbf{x}_1$ subsystem is required to track the desired reference model

$$\dot{\mathbf{x}}_{1d} = \mathbf{f}_d(\mathbf{x}_{1d},\mathbf{r}(t),t), \tag{6}$$

where $\mathbf{r}(t)$ is a smooth reference input. Define the tracking error as $\mathbf{z}_1 = \mathbf{x}_1 - \mathbf{x}_{1d}$. The control objective is to obtain ISS stability with respect to the disturbances and attenuate the disturbance influence $\mathbf{d} = [\mathbf{d}_1,\mathbf{d}_2,\boldsymbol{\eta}]^T$ on the tracking error $\mathbf{z}_1$. Furthermore, we have Assumption 3.

Assumption 3: There exists a function $\mathbf{g}_1(\cdot)$ such that the system dynamics $\dot{\boldsymbol{\xi}} = \mathbf{g}_1(\boldsymbol{\xi},t)$ is asymptotically stable. Then we can get the following equation

$$\begin{aligned} & \mathbf{f}_1(t,\mathbf{x}_1) - \mathbf{f}_d(t,\mathbf{x}_{1d},\mathbf{r}(t)) \\ = \ & \mathbf{g}_1(t,\mathbf{z}_1) + B_1(t)\boldsymbol{\zeta}(t,\mathbf{x}_1,\mathbf{x}_{1d},\mathbf{r}(t)), \end{aligned}$$

where $\boldsymbol{\zeta}(\cdot)$ is a smooth function with respect to its arguments.

From (3) and (6), the system with the error dynamics of $\mathbf{z}_1$ can be expressed as

$$\begin{cases} \dot{\mathbf{z}}_1 = \mathbf{g}_1(t,\mathbf{z}_1) + B_1(t)\left[\mathbf{x}_2 + \boldsymbol{\zeta}(\mathbf{x}_1,\mathbf{x}_{1d},\mathbf{r}(t),t)\right] \\ \qquad +G_1(t,\mathbf{x}_1)\mathbf{d}_1(t) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(t,\mathbf{x}) + B_2(t)\left[\mathbf{u} + \boldsymbol{\eta}(\mathbf{x},t)\right] + G_2(t,\mathbf{x})\mathbf{d}_2(t) \\ \mathbf{y} = \mathbf{x}_1, \end{cases} \tag{7}$$

Note that $\mathbf{z}_1 = \mathbf{y} - \mathbf{x}_{1d}$ is also available.

III. Nonlinear Observer Design

For the system (3), we first can construct a nonlinear observer according to the work in [4] and [10]. The idea is to construct an observer through a state transformation to convert the nonlinear system into a new form such that the observer gain can be designed in a straightforward manner. The observer design in this paper is an extension of the work in [4]. The extension of the observer construction is applied to the system with a more general representation of the disturbance term while $B_1(t)$ is a not a state-dependent function.

The system in (3) can be rewritten as the following

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}) + \mathbf{h}(t,\mathbf{u}) + B(t)\mathbf{x} + G(t,\mathbf{x})\mathbf{d}(\mathbf{x},t) \\ \mathbf{y} = C\mathbf{x} := \mathbf{x}_1, \end{cases} \tag{8}$$

where $\mathbf{f}(t,\mathbf{x}) = [\mathbf{f}_1(t,\mathbf{x}_1),\mathbf{f}_2(t,\mathbf{x})]^T$, and

$$\begin{aligned} \mathbf{h}(t,\mathbf{u}) &= \begin{bmatrix} \mathbf{0} \\ B_2(t)\mathbf{u} \end{bmatrix}, B(t) = \begin{bmatrix} \mathbf{0} & B_1(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ G(t,\mathbf{x}) &= \begin{bmatrix} G_1(t,\mathbf{x}_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G_2(t,\mathbf{x}) & B_2(t) \end{bmatrix}, \\ \mathbf{d}(\mathbf{x},t) &= \begin{bmatrix} \mathbf{d}_1(t) \\ \mathbf{d}_2(t) \\ \boldsymbol{\eta}(t,\mathbf{x}) \end{bmatrix}, C^T = \begin{bmatrix} I_n \\ \mathbf{0}_m \end{bmatrix}. \end{aligned}$$

Define the transformation matrix $T(t)$, the matrices Δ_θ, A and $\bar{C}$ as

$$T(t)\Big|_{2n\times(n+m)} = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & B_1(t) \end{bmatrix}, \Delta_\theta = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \frac{I_n}{\theta} \end{bmatrix},$$
$$A = \begin{bmatrix} \mathbf{0} & I_n \\ \mathbf{0}_n & \mathbf{0} \end{bmatrix}, \bar{C}^T = \begin{bmatrix} I_n \\ \mathbf{0}_n \end{bmatrix}.$$

Hence $\Delta_\theta A \Delta_\theta^{-1} = \theta A$, $\bar{C}^T\bar{C}\Delta_\theta = \bar{C}^T\bar{C}$ and

$$\mathbf{w}(t) = T(t)\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ B_1(t)\mathbf{x}_2 \end{bmatrix}, \quad T(t)B(t) = AT(t).$$

Denote $T^+(t)$ as the left inverse of the matrix $T(t)$. The $\mathbf{w}$ system can be written as

$$\begin{aligned} \dot{\mathbf{w}}(t) &= T(t)\dot{\mathbf{x}} + \dot{T}(t)\mathbf{x} \\ &= T(t)\left[\mathbf{f}(t,\mathbf{x}) + \mathbf{h}(t,\mathbf{u}) + B(t)\mathbf{x}\right. \\ &\quad \left.+G(t,\mathbf{x})\mathbf{d}(\mathbf{x},t)\right] + \dot{T}(t)\mathbf{x} \\ &= A\mathbf{w} + T\left[\mathbf{f}(t,T^+\mathbf{w}) + \mathbf{h}(t,\mathbf{u})\right. \\ &\quad \left.+G(t,T^+\mathbf{w})\mathbf{d}(T^+\mathbf{w},t)\right] + \dot{T}(t)T^+\mathbf{w} \quad (9) \\ \mathbf{y} &= \bar{C}\mathbf{w}. \end{aligned}$$

Then the observer for the transformed $\mathbf{w}$ system in (9) can be constructed as

$$\begin{aligned} \dot{\hat{\mathbf{w}}}(t) &= A\hat{\mathbf{w}} + T\left[\mathbf{f}(t,T^+\hat{\mathbf{w}}) + \mathbf{h}(t,\mathbf{u})\right] + \dot{T}(t)T^+\hat{\mathbf{w}} \\ &\quad +\theta\Delta_\theta^{-1}P^{-1}\bar{C}^T(\mathbf{y} - \bar{C}\hat{\mathbf{w}}) \end{aligned} \quad (10)$$

where P is the symmetric positive definite solution of the following algebraic Lyapunov equation

$$P + A^TP + PA - \bar{C}^T\bar{C} = 0. \quad (11)$$

Theorem 1: Assume that the system in (8) satisfies Assumptions $1-2$. Then the estimation error of the states has the following property

$$\|\mathbf{e}_x(t)\| = \|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \leq k_\theta\|\mathbf{e}_x(0)\| + \beta_d\delta, \quad (12)$$

where $k_\theta = \mu_t^+\theta\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}e^{-\left(\frac{\theta-c_1}{2}\right)t}\mu_t$, $\beta_d = \mu_t^+\frac{c_2\theta}{(\theta-c_1)\sqrt{\lambda_{min}(P)}}$, $\delta = \sqrt{\beta_1^2+\beta_2^2+\beta_\eta^2}$ and $c_1 < \theta$, c_2, μ_t^+, μ_t are positive constants.

Proof: See Appendix. ■

From (9) and $\dot{\hat{\mathbf{w}}} = T(t)\dot{\hat{\mathbf{x}}} + \dot{T}(t)\hat{\mathbf{x}}$, the observer to the original coordinate is

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= T^+\left[\dot{\hat{\mathbf{w}}} - \dot{T}(t)\hat{\mathbf{x}}\right] \\ &= T^+\Big\{A\hat{\mathbf{w}} + T\left[\mathbf{f}(t,T^+\hat{\mathbf{w}}) + \mathbf{h}(t,\mathbf{u})\right] + \dot{T}T^+\hat{\mathbf{w}} \\ &\quad +\theta\Delta_\theta^{-1}P^{-1}\bar{C}^T(\mathbf{y} - \bar{C}\hat{\mathbf{w}}) - \dot{T}\hat{\mathbf{x}}\Big\} \\ &= T^+AT\hat{\mathbf{x}} + \mathbf{f}(t,\hat{\mathbf{x}}) + \mathbf{h}(t,\mathbf{u}) \\ &\quad +\theta\Delta_\theta^{-1}P^{-1}\bar{C}^T(\mathbf{y} - C\hat{\mathbf{x}}) \\ &= B(t)\hat{\mathbf{x}} + \mathbf{f}(t,\hat{\mathbf{x}}) + \mathbf{h}(t,\mathbf{u}) \\ &\quad +\theta T^+\Delta_\theta^{-1}P^{-1}\bar{C}^T(\mathbf{y} - C\hat{\mathbf{x}}), \quad (13) \\ \mathbf{y} &= \mathbf{x}_1. \end{aligned}$$

Hence the estimation error dynamics in the $\mathbf{x}-$coordinate with $\mathbf{e}_x(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ becomes

$$\begin{aligned} \dot{\mathbf{e}}_x &= B(t)\mathbf{e}_x + \mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\hat{\mathbf{x}}) + G(t,\mathbf{x})\mathbf{d}(\mathbf{x},t) \\ &\quad -\theta T^+(t)\Delta_\theta^{-1}P^{-1}\bar{C}^TC\mathbf{e}_x. \end{aligned}$$

IV. Controller Design and Stability Analysis

Before the controller design, we would like to rewrite the observer dynamics in (13) as

$$\begin{cases} \dot{\hat{\mathbf{x}}}_1 = \mathbf{f}_1(t,\hat{\mathbf{x}}_1) + B_1(t)\hat{\mathbf{x}}_2 + \boldsymbol{\psi}_n \\ \dot{\hat{\mathbf{x}}}_2 = \mathbf{f}_2(t,\hat{\mathbf{x}}) + B_2(t)\mathbf{u} + \boldsymbol{\psi}_m, \end{cases} \quad (14)$$

where

$$\begin{aligned} \boldsymbol{\psi}_n &= \left\{\theta T^+(t)\Delta_\theta^{-1}P^{-1}\bar{C}^T\right\}_{\bar{n}\times\bar{n}}(\mathbf{y} - C\hat{\mathbf{x}}) \\ &\triangleq E_n(\mathbf{y} - \bar{C}\hat{\mathbf{w}}) = E_n\bar{C}\mathbf{e} = E_n\bar{C}\Delta_\theta^{-1}\bar{\mathbf{e}}, \end{aligned}$$

denotes the $n-$vector with the first n elements in the vector $\theta T^+(t)\Delta_\theta^{-1}P^{-1}\bar{C}^T(\mathbf{y} - C\hat{\mathbf{x}})$ and

$$\begin{aligned} \boldsymbol{\psi}_m &= \left\{\theta T^+(t)\Delta_\theta^{-1}P^{-1}\bar{C}^T\right\}_{\underline{m}\times\underline{n}}(\mathbf{y} - C\hat{\mathbf{x}}) \\ &\triangleq E_m(\mathbf{y} - \bar{C}\hat{\mathbf{w}}) = E_m\bar{C}\mathbf{e} = E_m\bar{C}\Delta_\theta^{-1}\bar{\mathbf{e}}, \end{aligned}$$

represents the $m-$vector with the last m elements in the vector $\theta T^+(t)\Delta_\theta^{-1}P^{-1}\bar{C}^T(\mathbf{y} - C\hat{\mathbf{x}})$.

According to the structure in (7), and comparing the desired target in (6) and the observer dynamics in (14), we have the following error dynamics

$$\begin{cases} \dot{\hat{\mathbf{z}}}_1 = \mathbf{g}_1(t,\hat{\mathbf{z}}_1) + B_1(t)\left[\hat{\mathbf{x}}_2 + \boldsymbol{\zeta}(\hat{\mathbf{x}}_1,\mathbf{x}_{1d},\mathbf{r}(t),t)\right] \\ \qquad +\boldsymbol{\psi}_n \\ \dot{\hat{\mathbf{x}}}_2 = \mathbf{f}_2(t,\hat{\mathbf{x}}) + B_2(t)\mathbf{u} + \boldsymbol{\psi}_m, \end{cases} \quad (15)$$

where $\hat{\mathbf{z}}_1 \triangleq \hat{\mathbf{x}}_1 - \mathbf{x}_{1d}$.

The controller design is separated into two steps: (1) Design a desired $\hat{\mathbf{x}}_2^*$ for the error dynamics of the null space dynamics $\hat{\mathbf{z}}_1$ to facilitate the switching surface design; (2) Design a controller $\mathbf{u}$ for the whole system in (15) with ISS stability.

Step 1: The task in this step is to find the desired $\hat{\mathbf{x}}_2^*$.

Theorem 2: The tracking error norm, $\|\mathbf{z}_1(t)\|$, tends, in finite time, to a ball B_r defined as

$$B_r = \left\{\mathbf{z}_1(t) : \|\mathbf{z}_1(t)\| \leq \rho_3^2\delta^2 = r\right\},$$

where ρ_3 is a positive constant and $\delta = \sqrt{\beta_1^2+\beta_2^2+\beta_\eta^2}$, if the following sliding mode holds,

$$\begin{aligned} \boldsymbol{\sigma}(\hat{\mathbf{x}},\mathbf{x}_{1d},t) = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_2^* = \hat{\mathbf{x}}_2 + \frac{B_1^T(t)}{r_1(t,\hat{\mathbf{x}}_1,\mathbf{x}_{1d})}(D_{\hat{\mathbf{z}}_1}V)^T \\ +\boldsymbol{\zeta}(\hat{\mathbf{x}}_1,\mathbf{x}_{1d},\mathbf{r}(t),t), \end{aligned} \quad (16)$$

where $V(\hat{\mathbf{z}}_1,t)$, $\forall\hat{\mathbf{z}}_1 \in \mathcal{R}^n$ and $t \geq 0$ is a positive definite smooth solution of the following Hamilton-Jacoby inequality

$$\begin{aligned} D_tV + (D_{\hat{\mathbf{z}}_1}V)\mathbf{g}_1 - (D_{\hat{\mathbf{z}}_1}V)\frac{B_1B_1^T}{r_1}(D_{\hat{\mathbf{z}}_1}V)^T \\ +\frac{1}{4\rho_1^2}(D_{\hat{\mathbf{z}}_1}V)HH^T(D_{\hat{\mathbf{z}}_1}V)^T + \hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 \leq 0, \end{aligned} \quad (17)$$

with $r_1(\hat{\mathbf{z}}_1, \mathbf{x}_{1d}, t) > 0$.

Proof: Construct a second Lyapunov function as $V_1 = V_0 + V$, where $V(\hat{\mathbf{z}}_1, t)$, $\forall \hat{\mathbf{z}}_1 \in \mathcal{R}^n$, $t \geq 0$ is a positive definite smooth Lyapunov function to be determined. Design

$$\hat{\mathbf{x}}_2^* = -\frac{B_1^T(t)}{r_1(t, \hat{\mathbf{x}}_1, \mathbf{x}_{1d})}(D_{\hat{\mathbf{z}}_1}V)^T - \boldsymbol{\zeta}(\cdot).$$

The derivative of $V(\cdot)$ is

$$\begin{aligned}
\dot{V} &= D_tV + (D_{\hat{\mathbf{z}}_1}V)\left[\mathbf{g}_1 + B_1(\hat{\mathbf{x}}_2 + \boldsymbol{\zeta}) + \boldsymbol{\psi}_n\right] \\
&= D_tV + (D_{\hat{\mathbf{z}}_1}V)\mathbf{g}_1 - (D_{\hat{\mathbf{z}}_1}V)\frac{B_1B_1^T}{r_1}(D_{\hat{\mathbf{z}}_1}V)^T \\
&\quad +(D_{\hat{\mathbf{z}}_1}V)H\bar{\mathbf{e}} \\
&= D_tV + (D_{\hat{\mathbf{z}}_1}V)\mathbf{g}_1 - (D_{\hat{\mathbf{z}}_1}V)\frac{B_1B_1^T}{r_1}(D_{\hat{\mathbf{z}}_1}V)^T \\
&\quad +\rho_1^2\bar{\mathbf{e}}^T\bar{\mathbf{e}} + \frac{1}{4\rho_1^2}(D_{\hat{\mathbf{z}}_1}V)HH^T(D_{\hat{\mathbf{z}}_1}V)^T \\
&\quad -\|\frac{1}{2\rho_1}H^T(D_{\hat{\mathbf{z}}_1}V)^T - \rho_1\bar{\mathbf{e}}\|^2 \\
&\leq D_tV + (D_{\hat{\mathbf{z}}_1}V)\mathbf{g}_1 - (D_{\hat{\mathbf{z}}_1}V)\frac{B_1B_1^T}{r_1}(D_{\hat{\mathbf{z}}_1}V)^T \\
&\quad +\rho_1^2\bar{\mathbf{e}}^T\bar{\mathbf{e}} + \frac{1}{4\rho_1^2}(D_{\hat{\mathbf{z}}_1}V)HH^T(D_{\hat{\mathbf{z}}_1}V)^T. \qquad (18)
\end{aligned}$$

If there is a solution of V such that the inequality in (17) is satisfied, then (18) becomes

$$\dot{V} \leq -\hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 + \rho_1^2\bar{\mathbf{e}}^T\bar{\mathbf{e}}. \qquad (19)$$

Using (19) and (32), the derivative of V_1 becomes

$$\begin{aligned}
\dot{V}_1 &= \dot{V}_0 + \dot{V} \\
&\leq -\theta\bar{\mathbf{e}}^TP\bar{\mathbf{e}} + \bar{\mathbf{e}}^TP\Delta_\theta T\left[\mathbf{f}(t, T^+\mathbf{w}) - \mathbf{f}(t, T^+\hat{\mathbf{w}})\right] \\
&\quad +\bar{\mathbf{e}}^TP\Delta_\theta TG\mathbf{d} + \bar{\mathbf{e}}^TP\Delta_\theta\dot{T}T^+\Delta_\theta^{-1}\bar{\mathbf{e}} \\
&\quad -\hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 + \rho_1^2\bar{\mathbf{e}}^T\bar{\mathbf{e}} \\
&\leq -\theta\lambda_{min}(P)\|\bar{\mathbf{e}}\|^2 + \lambda_{max}(P)(l_f + l_t)\|\bar{\mathbf{e}}\|^2 \\
&\quad +\bar{\mathbf{e}}^TP\Delta_\theta TG\mathbf{d} - \hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 + \rho_1^2\bar{\mathbf{e}}^T\bar{\mathbf{e}} \\
&\leq -\left[\theta\lambda_{min}(P) - \rho_1^2 - \lambda_{max}(P)(l_f + l_t)\right]\|\bar{\mathbf{e}}\|^2 \\
&\quad -\hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 + \frac{1}{4\rho_2^2}\bar{\mathbf{e}}^TP\Delta_\theta TG(P\Delta_\theta TG)^T\bar{\mathbf{e}} \\
&\quad +\rho_2^2\|\mathbf{d}\|^2 \\
&\leq -\Big[\theta\lambda_{min}(P) - \rho_1^2 - \lambda_{max}(P)(l_f + l_t) \\
&\quad -\frac{1}{4\rho_2^2}\lambda_{max}(P)^2l_g\Big]\|\bar{\mathbf{e}}\|^2 - \hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 + \rho_2^2\|\mathbf{d}\|^2 \\
&\leq -\hat{\mathbf{z}}_1^T\hat{\mathbf{z}}_1 + \rho_2^2\|\mathbf{d}\|^2, \qquad (20)
\end{aligned}$$

where θ is selected such that

$$\begin{aligned}
&\theta\lambda_{min}(P) - \rho_1^2 - \lambda_{max}(P)(l_f + l_t) \\
&-\frac{1}{4\rho_2^2}\lambda_{max}(P)^2l_g^2 \geq 0. \qquad (21)
\end{aligned}$$

Using $\mathbf{e}_{x,1} = \mathbf{x}_1 - \hat{\mathbf{x}}_1 = \mathbf{z}_1 - \hat{\mathbf{z}}_1$, $\|\hat{\mathbf{z}}_1\|^2 \geq \|\mathbf{z}_1\|^2 - \|\mathbf{e}_{x,1}\|^2$, and $\mathbf{e}_{x,1} = \bar{\mathbf{e}}_1$ according to $\mathbf{e}_x = T^+\Delta_\theta^{-1}\bar{\mathbf{e}}$, (20) becomes

$$\begin{aligned}
\dot{V}_1 &\leq -\mathbf{z}_1^T\mathbf{z}_1 + \|\mathbf{e}_{x,1}\|^2 + \rho_2^2\|\mathbf{d}\|^2 \\
&= -\mathbf{z}_1^T\mathbf{z}_1 + \|\bar{\mathbf{e}}_1\|^2 + \rho_2^2\|\mathbf{d}\|^2 \\
&\leq -\mathbf{z}_1^T\mathbf{z}_1 + \|\bar{\mathbf{e}}\|^2 + \rho_2^2\|\mathbf{d}\|^2. \qquad (22)
\end{aligned}$$

From (33), $\|\bar{\mathbf{e}}\|$ is bounded as

$$\|\bar{\mathbf{e}}\| \leq \frac{\lambda_{max}(P)l_g}{[\lambda_{min}(P)\theta - \lambda_{max}(P)(l_f + l_t)]}\|\mathbf{d}\|, \qquad (23)$$

where $\lambda_{min}(P)\theta - \lambda_{max}(P)(l_f + l_t) > 0$ according to (21). Hence (22) becomes

$$\dot{V}_1 \leq -\mathbf{z}_1^T\mathbf{z}_1 + \rho_3^2\|\mathbf{d}\|^2. \qquad (24)$$

where $\rho_3 = \sqrt{\rho_2^2 + k_d}$ and $k_d = \dfrac{l_g^2}{\left[\frac{\lambda_{min}(P)}{\lambda_{max}(P)}\theta - (l_f + l_t)\right]^2}$.

Note that $\lim_{\theta\to\infty} k_d = 0$. Equation (24) shows that the tracking error norm in the $\mathbf{z}_1$ subsystem, $\|\mathbf{z}_1(t)\|$, tends, in finite time, to a ball B_r defined by

$$B_r = \left\{\mathbf{z}_1(t) : \|\mathbf{z}_1(t)\| \leq \rho_3^2\delta^2 = r\right\},$$

where $\delta = \sqrt{\beta_1^2 + \beta_2^2 + \beta_\eta^2}$. ∎

Step 2: The task in this step is to design the controller to ensure the ISS stability.

Theorem 3: With the switching surface in (16) and the following sliding mode control law,

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_c + \mathbf{u}_s, \qquad (25) \\
\mathbf{u}_c &= -B_2^{-1}\left[D_t\boldsymbol{\sigma} + (D_{\mathbf{x}_{1d}}\boldsymbol{\sigma})\dot{\mathbf{x}}_{1d}\right. \\
&\quad \left.+S(\mathbf{f}_1 + B_1\hat{\mathbf{x}}_2 + \boldsymbol{\psi}_n) + \mathbf{f}_2 + \boldsymbol{\psi}_m\right], \qquad (26) \\
\mathbf{u}_s &= -k_\delta\frac{B_2^T\boldsymbol{\sigma}}{\|B_2^T\boldsymbol{\sigma}\|}, \qquad (27)
\end{aligned}$$

where $S(\hat{\mathbf{x}}_1, \mathbf{x}_{1d}, t) = D_{\hat{\mathbf{x}}_1}\boldsymbol{\sigma} \in \mathcal{R}^{m\times n}$, and $k_\delta > 0$ is a positive constant, the system is globally ISS stable with respect to the external disturbance inputs and the tracking error norm, $\|\mathbf{z}_1\|$ is bounded in B_r as in Theorem 2.

Proof: Construct a Lyapunov function $V_2 = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\sigma}$. Define $S(\hat{\mathbf{x}}_1, \mathbf{x}_{1d}, t) = D_{\hat{\mathbf{x}}_1}\boldsymbol{\sigma} \in \mathcal{R}^{m\times n}$ and $D_{\hat{\mathbf{x}}_2}\boldsymbol{\sigma} = I_m$ holds. Then

$$\begin{aligned}
\dot{V}_2 &= \boldsymbol{\sigma}^T\left[D_t\boldsymbol{\sigma} + (D_{\mathbf{x}_{1d}}\boldsymbol{\sigma})\dot{\mathbf{x}}_{1d} + S(\mathbf{f}_1 + B_1\hat{\mathbf{x}}_2 + \boldsymbol{\psi}_n)\right. \\
&\quad \left.+\mathbf{f}_2 + B_2\mathbf{u} + \boldsymbol{\psi}_m\right] \\
&\leq -k_\delta\|B_2^T\boldsymbol{\sigma}\|. \qquad (28)
\end{aligned}$$

Define a new Lyapunov function $V_3(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{x}_{1d}, t) = V_1(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{x}_{1d}, t) + V_2(\hat{\mathbf{x}}, \mathbf{x}_{1d}, t)$. From (22) and (28),

$$\begin{aligned}
\dot{V}_3 &= \dot{V}_1 + \dot{V}_2 \leq -\mathbf{z}_1^T\mathbf{z}_1 + \rho_3^2\|\mathbf{d}\|^2 - k_\delta\|B_2^T\boldsymbol{\sigma}\| \\
&\leq -\mathbf{z}_1^T\mathbf{z}_1 + \rho_3^2\|\mathbf{d}\|^2, \qquad (29)
\end{aligned}$$

which implies that the system is globally ISS stable with respect to the external disturbance input and the tracking error norm, $\|\mathbf{z}_1\|$, is bounded in B_r in finite time. ∎

Remark 1: In the nonlinear uncertain system in (15), if $\mathbf{g}_1(\hat{\mathbf{z}}_1, t)$ can be expressed as $F_1(\hat{\mathbf{z}}_1, t)\hat{\mathbf{z}}_1$, when $F_1(\hat{\mathbf{z}}_1, t)$ is a matrix-valued smooth function, then the HJ inequality in (17) can be simplified into the following differential Riccati inequality

$$\frac{1}{2}\dot{Q} + \frac{1}{2}(QF_1 + F_1^T Q) - Q\left[\frac{B_1 B_1^T}{r_1} - \frac{1}{4\rho_1^2}G_1 G_1^T\right]Q + I_{n\times n} \leq 0, \quad (30)$$

where $Q(\hat{\mathbf{z}}_1, t) \in \mathcal{R}^{n\times n}$ is a symmetric positive definite smooth matrix. The $\mathbf{z}_1$ subsystem warrants the tracking error norm, $\|\mathbf{z}_1\|$, to be bounded in B_r (as defined in Theorem 2) by the nonlinear control law

$$\hat{\mathbf{x}}_2^* = -\frac{1}{r_1}B_1^T Q\hat{\mathbf{z}}_1 - \boldsymbol{\zeta}, \quad (31)$$

which also specifies the switching surface as $\boldsymbol{\sigma} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_2^*$.

Remark 2: Note that the value of ρ_3 determines the bound of the tracking error and it depends on ρ_2 and θ. According to the expression of ρ_3 as shown in the proof of Theorem 2, for a fixed ρ_2, a larger θ value in the observer design results a smaller tracking error bound ρ_3.

Remark 3: In the observer design, the parameter θ is the only key parameter to be determined. It should be designed to satisfy the two conditions in (21) and $\theta = max\{1, c_1\}$ simultaneously. In the case of existing noise existing in the output measurement, there is a trade-off between fast convergence of the observer and the sensitivity to noise.

V. Numerical Example

Consider a nonlinear uncertain cascaded system as in (3) where

$$\mathbf{f}_1(t, \mathbf{x}_1) = F_1 \cdot \mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix},$$

$$G_1(t, \mathbf{x}_1) = \begin{bmatrix} cos(x_{12}) & sin(x_{12}) \\ sin(x_{11}) & cos(x_{11}) \end{bmatrix},$$

$$G_2(t, \mathbf{x}) = \begin{bmatrix} sin(x_{11}) & cos(x_{12}) \\ cos(x_{21}) & sin(x_{22}) \end{bmatrix},$$

$$\mathbf{w}_1 = \left[e^{-0.1t}, -e^{-0.5t}\right]^T, \mathbf{w}_2 = \left[-e^{-0.2t}, e^{-0.3t}\right]^T,$$

$$\mathbf{f}_2(t, \mathbf{x}) = [x_{11}sin(x_{21}), x_{12}sin(x_{22})]^T,$$

$$B_1(t) = (1 + 0.5sin(t))\, I_2, B_2(t) = (1 + 0.5cos(t))\, I_2,$$

$$\boldsymbol{\eta}(t, \mathbf{x}) = [sin(x_{11}) + sin(x_{12}), -cos(x_{21}) - cos(x_{22})]^T.$$

The initial condition is as $\mathbf{x}_1(0) = [5, 5]^T$, $\mathbf{x}_2(0) = [3, 3]^T$.

The observer is designed as in (13). According to (23), we have the symmetric positive definite solution

$$P = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

$\theta = 80$ is selected according to Remark 3 with $l_f = 4.5622$, $l_t = 1$, $l_g = 1.0028$ and $\rho_2 = 1$.

The target trajectory is $x_{11d} = 0.2sin(\pi t)$ and $x_{12d} = \dot{x}_{11d} = 0.2\pi cos(\pi t)$. The error dynamics of the $\mathbf{x}_1$ subsystem in (15) can be expressed as

$$\dot{\mathbf{z}}_1 = F_1\mathbf{z}_1 + B_1(t)\left[\mathbf{x}_2 + \boldsymbol{\zeta}(t)\right] + G_1\mathbf{w}_1,$$

where $\boldsymbol{\zeta}(t) = [0, -\dot{x}_{12d} - 2x_{11d} - 4x_{12d}]^T$. In the $\hat{\mathbf{z}}_1$ subsystem, according to $Remark$ 1, we first choose $V(\hat{\mathbf{z}}_1, t) = \frac{1}{2}\hat{\mathbf{z}}_1^T Q\hat{\mathbf{z}}_1$, where Q is determined by the differential Riccati inequality (30). When $\dot{Q} = 0$, from the linear algebraic matrix inequality (30) and using the singular values of the matrices B_1 and $H = \left\{\theta T^+(t)\Delta_\theta^{-1}P^{-1}\bar{C}^T\right\}_{\bar{n}\times n}\bar{C}\Delta_\theta^{-1}$, which are 0.5 and 160 respectively, we can get a symmetric positive definite smooth matrix $Q = \begin{bmatrix} 0.0106 & 0 \\ 0 & 0.0108 \end{bmatrix}$, $\rho_1 = 0.9659$ and $r_1 = 0.001$. Thus from (31), we have $\hat{\mathbf{x}}_2^*(\mathbf{x}_1, \mathbf{x}_{1d}, t) = -\frac{1}{r_1}B_1^T Q\hat{\mathbf{z}}_1 - \boldsymbol{\zeta}(t)$. The switching surface is $\boldsymbol{\sigma} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_2^*$. Then the controller is constructed according to (25) in Theorem (2).

Simulation results are shown as follows. In $Fig.1$, $\mathbf{u} = \mathbf{0}$ is first applied. It is shown that the tracking task can not been realized without any controller though the system is stable. Hence it is necessary to design an output feedback controller. In $Fig.2$, the tracking errors of the states x_{11} and x_{12} are bounded with fast convergence.

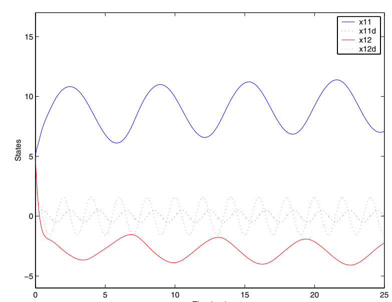

Fig. 1. The evolution of the states $\mathbf{x}_1(t)$ and the desired trajectory $\mathbf{x}_{11d}(t)$.

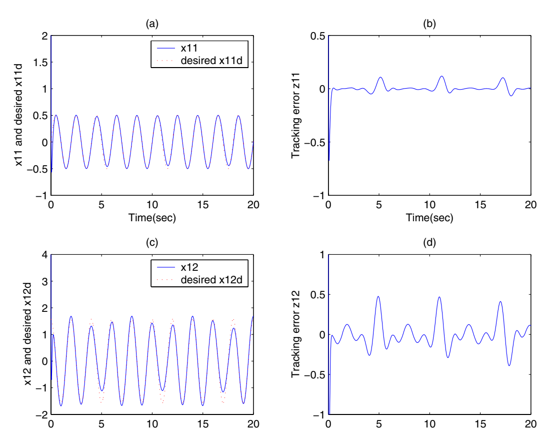

Fig. 2. (a) $x_{11}(t)$ and desired $x_{11d}(t)$; (b) Tracking error $z_{11}(t) = x_{11}(t) - x_{11d}(t)$; (c) $x_{12}(t)$ and desired $\hat{x}_{12d}(t)$; (d) Tracking error $z_{12}(t) = x_{12}(t) - x_{12d}(t)$.

VI. Conclusions

We have considered the tracking control problem for a class of nonlinear systems with unknown system uncertainties and external disturbances, and proposed a robust output feedback control law based on a nonlinear observer that achieves input-to-stability. The design procedure is based on a back-stepping like procedure. First a stable switching surface is allocated. Then a discontinuous controller is constructed to ensure the convergence of the Lyapunov function. As a result, the tracking error is bounded.

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Appendix: Proof of Theorem 1

Define $\mathbf{e}(t) = \mathbf{w}(t) - \hat{\mathbf{w}}(t)$. According to (9) and (10), the estimation error dynamics becomes

$$\begin{aligned}\dot{\mathbf{e}} &= \theta\Delta_\theta^{-1}(A - P^{-1}\bar{C}^T\bar{C})\Delta_\theta\mathbf{e}\\ &\quad + T\left[\mathbf{f}(t, T^+\mathbf{w}) - \mathbf{f}(t, T^+\hat{\mathbf{w}})\right]\\ &\quad + TG(t, T^+\mathbf{w})\mathbf{d}(T^+\mathbf{w}, t) + \dot{T}(t)T^+\mathbf{e}\end{aligned}$$

Consider a transformation on the error as $\bar{\mathbf{e}} = \Delta_\theta\mathbf{e}$. Then

$$\begin{aligned}\dot{\bar{\mathbf{e}}} &= \theta(A - P^{-1}\bar{C}^T\bar{C})\bar{\mathbf{e}}\\ &\quad + \Delta_\theta T\left[\mathbf{f}(t, T^+\mathbf{w}) - \mathbf{f}(t, T^+\hat{\mathbf{w}})\right]\\ &\quad + \Delta_\theta TG(t, T^+\mathbf{w})\mathbf{d}(T^+\mathbf{w}, t) + \Delta_\theta\dot{T}(t)T^+\Delta_\theta^{-1}\bar{\mathbf{e}}\end{aligned}$$

Construct $V_0 = \frac{1}{2}\bar{\mathbf{e}}^T P\bar{\mathbf{e}}$, where P is the solution of (11),

$$\begin{aligned}\dot{V}_0 &= -\theta V_0 - \frac{\theta}{2}\bar{\mathbf{e}}^T\bar{C}^T\bar{C}\bar{\mathbf{e}}\\ &\quad + \bar{\mathbf{e}}^T P\Delta_\theta T\left[\mathbf{f}(t, T^+\mathbf{w}) - \mathbf{f}(t, T^+\hat{\mathbf{w}})\right]\\ &\quad + \bar{\mathbf{e}}^T P\Delta_\theta TG(t, T^+\mathbf{w})\mathbf{d}(T^+\mathbf{w}, t)\\ &\quad + \bar{\mathbf{e}}^T P\Delta_\theta\dot{T}(t)T^+\Delta_\theta^{-1}\bar{\mathbf{e}}\\ &\le -\theta V_0 + \bar{\mathbf{e}}^T P\Delta_\theta T\left[\mathbf{f}(t, T^+\mathbf{w}) - \mathbf{f}(t, T^+\hat{\mathbf{w}})\right]\\ &\quad + \bar{\mathbf{e}}^T P\Delta_\theta TG(t, T^+\mathbf{w})\mathbf{d}(T^+\mathbf{w}, t)\\ &\quad + \bar{\mathbf{e}}^T P\Delta_\theta\dot{T}(t)T^+\Delta_\theta^{-1}\bar{\mathbf{e}}. \qquad (32)\end{aligned}$$

For any $\theta \ge 1$, we have $\|\Delta_\theta\dot{T}T^+\Delta_\theta^{-1}\| \le l_t$, $\|\Delta_\theta T\left[\mathbf{f}(t, T^+\mathbf{w}) - \mathbf{f}(t, T^+\hat{\mathbf{w}})\right]\| \le l_f\|\bar{\mathbf{e}}\|$, $\|\Delta_\theta TG\| \le l_g$ and $\|\mathbf{d}(T^+\mathbf{w}, t)\| \le \delta$, where l_f, l_t and l_g do not depend on θ. Then (32) can be

$$\begin{aligned}\dot{V}_0 &\le -\theta V_0 + \lambda_{max}(P)(l_f + l_t)\|\bar{\mathbf{e}}\|^2\\ &\quad + \lambda_{max}(P)l_g\|\bar{\mathbf{e}}\|\|\mathbf{d}\| \qquad (33)\\ &\le -\theta V_0 + \lambda_{max}(P)(l_f + l_t)\|\bar{\mathbf{e}}\|^2\\ &\quad + \lambda_{max}(P)l_g\delta\|\bar{\mathbf{e}}\|,\\ \Rightarrow \dot{V}_0 &\le -(\theta - c_1)V_0 + c_2\delta\sqrt{V_0}, \qquad (34)\end{aligned}$$

where $c_1 = 2\frac{\lambda_{max}(P)}{\lambda_{min}(P)}(l_f + l_t)$ and $c_2 = \lambda_{max}(P)l_g\frac{\sqrt{2}}{\sqrt{\lambda_{min}(P)}}$. If $\theta > max\{1, c_1\}$ is selected, then (34) becomes

$$\begin{aligned}\frac{d\sqrt{V_0}}{dt} &\le -\left(\frac{\theta - c_1}{2}\right)\sqrt{V_0} + \frac{c_2\delta}{2},\\ \Rightarrow \sqrt{V_0(t)} &\le e^{-\left(\frac{\theta - c_1}{2}\right)t}\sqrt{V_0(0)}\\ &\quad + \frac{c_2\delta}{\theta - c_1}\left[1 - e^{-\left(\frac{\theta - c_1}{2}\right)t}\right],\\ \Rightarrow \|\bar{\mathbf{e}}(t)\| &\le \sqrt{\lambda_{max}(P)/\lambda_{min}(P)}e^{-\left(\frac{\theta - c_1}{2}\right)t}\|\bar{\mathbf{e}}(0)\|\\ &\quad + \frac{c_2\delta}{(\theta - c_1)\sqrt{\lambda_{min}(P)}}, \qquad (35)\end{aligned}$$

Using $\|\bar{\mathbf{e}}(t)\| \le \|\mathbf{e}(t)\| \le \theta\|\bar{\mathbf{e}}(t)\|$, (35) becomes

$$\begin{aligned}\|\mathbf{e}(t)\| &\le \theta\sqrt{\lambda_{max}(P)/\lambda_{min}(P)}e^{-\left(\frac{\theta - c_1}{2}\right)t}\|\mathbf{e}(0)\|\\ &\quad + \frac{c_2\delta\theta}{(\theta - c_1)\sqrt{\lambda_{min}(P)}}\\ &\le k_\theta^{'}\|\mathbf{e}(0)\| + \beta_d^{'}\delta,\end{aligned}$$

where $k_\theta^{'} = \theta\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}e^{-\left(\frac{\theta - c_1}{2}\right)t}$ and $\beta_d^{'} = \frac{c_2\theta}{(\theta - c_1)\sqrt{\lambda_{min}(P)}}$. Hence $\|\mathbf{e}(t)\| = \|\mathbf{w}(t) - \hat{\mathbf{w}}(t)\| \le k_\theta^{'}\|\mathbf{e}(0)\| + \beta_d^{'}$. Furthermore, from Assumption 1, $\|T^+\| \le \mu_t^+$ and $\|T\| \le \mu_t$ where μ_t^+ and μ_t are constants. According to $\mathbf{w}(t) = T(t)\mathbf{x}(t)$ and $\hat{\mathbf{w}}(t) = T(t)\hat{\mathbf{x}}(t)$, we have

$$\begin{aligned}\|\mathbf{e}_x(t)\| &= \|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| = \|T^+\mathbf{e}(t)\|\\ &\le \mu_t^+ k_\theta^{'}\mu_t\|\mathbf{e}_x(0)\| + \mu_t^+\beta_d^{'}\delta\\ &\triangleq k_\theta\|\mathbf{e}_x(0)\| + \beta_d\delta,\end{aligned}$$

where $k_\theta = \mu_t^+ k_\theta^{'}\mu_t$ and $\beta_d = \mu_t^+\beta_d^{'}$.