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Distributed Control Design with Robustness to Small Time Delays

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***Abstract*— Recent results by the authors have shown how to construct a class of structured controllers for large scale spatially interconnected systems via linear matrix inequalities. These controllers guarantee that the closed loop interconnected system is well-posed, stable and has $\mathcal{H}_\infty$ norm less than unity. Of paramount importance in the control of interconnected systems is the requirement that the stability and performance of the controlled system be robust to arbitrarily small communication delays between subsystems; this amounts to a continuity property. In this paper, it is shown how to realize the structured controllers obtained from the linear matrix inequalities in order to ensure this continuity property for the closed loop system.**

I. Introduction

In recent years, considerable research attention has been focused on the theory and practice of large scale spatially interconnected systems. The papers [2] and [5] proposed control design techniques for a class of interconnected systems known as spatially invariant systems; the work [10] dealt with analysis of a class of spatially interconnected systems called the multidimensional (MD) systems. The main motivation for these results is the fact that it is often not practicable to design fully centralized controllers for large scale interconnected systems due to constraints on the computational cost and allowable interconnection topology.

In recent papers [12], [11], based in turn on the results of [7] and [16], the authors have extended the results of [5] and have proposed a method to construct a class of structured controllers for spatially interconnected systems. Briefly, these results can be summarized as follows: Consider a linear time-invariant (LTI) plant consisting of 'L' subsystems G_i, $i = 1, \ldots, L$, which are interconnected over an arbitrary graph. Every node of the graph represents an LTI system and every directed edge represents the signals flowing from one node to the other. Each edge of the graph connecting subsystems G_i and G_j, $i \neq j$ is associated with the pair of numbers m_{ij}^+, m_{ij}^-, where m_{ij}^+ denotes the dimension of the output of G_i flowing towards G_j and where m_{ij}^- denotes the dimension of the output of G_j flowing towards G_i. In addition, each subsystem is affected by the exogenous disturbance d_i and outputs the signal z_i. Notice that $m_{ij}^+ = m_{ji}^-$ for all $i, j = 1, \ldots, L$.

Suppose we seek a controller consisting of subsystems K_i with the properties that:

1) Each K_i is connected to G_i.
2) Two controller subsystems K_i, K_j can communicate; that is, K_i can output a signal of dimension $m_{ij}^{\mathrm{K},+}$ to K_j, and receive an input of dimension $m_{ij}^{\mathrm{K},-}$ from K_j, so that $m_{ij}^{\mathrm{K},+} = m_{ji}^{\mathrm{K},-}$ for all $i, j = 1, \ldots, L$.
3) The inequality $\max(m_{ij}^{\mathrm{K},+}, m_{ij}^{\mathrm{K},-}) \leq \alpha \max(m_{ij}^+, m_{ij}^-)$ holds, where α is an absolute constant.
4) The closed loop system is stable and $\mathcal{H}_\infty$ norm of the system mapping $(d_1, \ldots, d_L)$ to $(z_1, \ldots, z_L)$ is less than unity.

Notice that the third condition above imposes a restriction on the possible connectivity of the controller subsystems in terms of that of the plant; in particular, no two controller subsystems can communicate unless the corresponding plant subsystems do. It was shown in [12], [11] that the above problem can be solved with a bound $\alpha = 3$, and a linear time-invariant control system explicitly computed, if a certain linear matrix inequality (LMI) condition is satisfied.

The very fact that we are trying to control a plant consisting of spatially distributed subsystems requires that we add a fifth condition to the four already given above, namely, that the closed loop system be stable, and the $\mathcal{H}_\infty$ norm of the system be less than unity, when arbitrarily small communication delays are present between any two closed-loop subsystems. It has been recognized long ago (see [18]) that seemingly innocent examples of LTI feedback systems have the property that arbitrarily small delays in the feedback loop can destroy stability. For example, a constant gain $\hat{P}(s) = q > 1$ (s denotes the Laplace transform variable) in feedback with a unity gain has this property; if $\hat{P}(s)$ is replaced by $qe^{-\epsilon s}$ (a time delay of $\epsilon > 0$ seconds), the resulting system is no longer internally stable. In the paper [13], it is shown that for an LTI feedback system, robustness with respect to small delays in the feedback loop is equivalent to the stability of the nominal system and the condition that a certain matrix structured singular value be less than unity. Thus, it is not *a priori* obvious that the closed loop system is robustly stable when small communication delays are present. The aim of this paper is to show the following: if a controller K (with subsystems K_i) is synthesized using the LMI conditions of [11], there exists another controller $\bar{K}$ (with subsystems $\bar{K}_i$) which can be explicitly computed in terms of the state-space matrices of K_i, has the same structure as K (this is of course essential, since the main contribution of [11] was to construct controllers with this structure), and guarantees robust stability and $\mathcal{H}_\infty$ performance of the closed loop system against arbitrarily small communication delays. *Note that if we directly implement the controller obtained via solving the LMIs of [11], the closed loop may not have this desirable property.*

The paper is organized as follows: In Section II, some background material on spatially interconnected systems as reported in the work [11] is given, and the problem of robustness to arbitrarily small delays is precisely formulated. In Section III, the main results of the paper are presented; a controller is constructed that achieves robust stability and performance to small time communication delays. Section IV presents the conclusions, and the proofs are given in the Appendix.

Notation: The set of the real numbers is denoted by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, the real n - vectors by $\mathbb{R}^n$, the $m\times n$ real (complex) matrices by $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$), the $n\times n$ identity by I_n, and the real symmetric matrices by $\mathbb{R}^n_{\mathrm{S}}$. M^* is the conjugate transpose of $M \in \mathbb{C}^{m\times n}$. For $M \in \mathbb{R}^n_{\mathrm{S}}$, $M \succ 0$ means M is positive definite; $M \prec 0$ means M is negative definite. The maximum singular value of a matrix M is denoted by $\bar{\sigma}(M)$ and the spectral radius is denoted by $\rho(M)$ if M is square. The Euclidean norm of a vector is denoted $|\bullet|$. $\mathcal{L}_2$ is the space of vector signals $z(t)$ such that $\int_0^\infty |z(t)|^2 dt < \infty$ and $\mathcal{L}_{2e}$ is the space of signals $z(t)$ such that $\int_0^T |z(t)|^2 dt < \infty$ for all $T > 0$. The $\mathcal{L}_2$ norm of a signal is denoted by $\|\bullet\|$ and the $\mathcal{H}_\infty$ norm of an LTI system G is denoted by $\|G\|_\infty$; for more details, see [6]. Given matrices M_k, $k = 1, \ldots, n$, the notation $\mathbf{diag}_{k=1}^n M_k$ denotes the block-diagonal matrix with M_k along the diagonal. This is usually denoted $\mathbf{diag}_k M_k$ for brevity. Similarly, for signals or vectors x_k, the notation $\mathbf{cat}_{k=1}^n x_k$ denotes the signal or vector $(x_1, x_2, \ldots, x_n)$ formed by concatenating x_k. This is also usually denoted $\mathbf{cat}_k x_k$ for brevity. The number of scalar components of a vector or signal x is denoted by $\mathbf{dim}(x)$.

II. Problem formulation

A. Spatially interconnected systems over an arbitrary graph

The plant to be controlled consists an assembly of L possibly different linear time-invariant continuous time subsystems. Let $V := \{1, 2, \ldots, L\}$ be the set that indexes the subsystems. To each distinct pair of subsystems, indexed by i and j, we associate the following four signals - in the space $\mathcal{L}_{2e}$ - flowing between them: 1) v_{ij}, the input of i coming from j, 2) w_{ij}, the output of i flowing towards j, and similarly, 3) v_{ji} and 4) w_{ji}. As mentioned in the introduction, $\mathbf{dim}(w_{ij}) = m_{ij}^+ = \mathbf{dim}(v_{ji}) = m_{ji}^-$. Letting $m_{ij}^+ = m_{ij}^- = 0$ will denote the fact that the subsystems i and j are not interconnected - neither system outputs a signal to the other.

Define the interconnection input v_i to each subsystem and the interconnection output w_i from each subsystem as the partitioned vectors $v_i := \mathbf{cat}_j(v_{ij})$ and $w_i := \mathbf{cat}_j(w_{ij})$ respectively. Each subsystem is described by the following state-space equations:

$$\begin{bmatrix} \dot{x}_i(t) \\ w_i(t) \\ z_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} A_{\text{TT}_i} & A_{\text{TS}_i} & B^d_{\text{T}_i} & B^u_{\text{T}_i} \\ A_{\text{ST}_i} & A_{\text{SS}_i} & B^d_{\text{S}_i} & B^u_{\text{S}_i} \\ C^z_{\text{T}_i} & C^z_{\text{S}_i} & D^{zd}_i & D^{zu}_i \\ C^y_{\text{T}_i} & C^y_{\text{S}_i} & D^{yd}_i & 0 \end{bmatrix} \begin{bmatrix} x_i(t) \\ v_i(t) \\ d_i(t) \\ u_i(i) \end{bmatrix}. \tag{1}$$

The local state is x_i; each subsystem outputs the sensor measurements y_i and is equipped with the control signal u_i. The local exogenous disturbance is d_i and the signals z_i represent the performance output in the standard $\mathcal{H}_\infty$ formulation [20] (that is, we would like the signal $z := \mathbf{cat}_i z_i$ to have small $\mathcal{L}_2$ norm when $d := \mathbf{cat}_i d_i$ satisfies $\|d\| = 1$). Once the relationships between the inputs and outputs at each vertex have been defined, the distributed system can be described by closing all loops by imposing the constraints of interconnection. In other words, the condition that

$$v_{ij}(t) = w_{ji}(t) \tag{2}$$

is imposed at all times t, for all $i, j \in V$. We make the following assumptions on the plant matrices:

1) The control and measurement matrices of the closed loop system satisfy
$$B^u_{\text{S}_i} = D^{zu}_i = C^y_{\text{S}_i} = D^{yd}_i = 0. \tag{3}$$
These assumptions can always be satisfied by placing low-pass filters of sufficiently high bandwidth at the local measurement y_i and control input u_i as has been done in the linear parameter varying (LPV) control literature [1].
2) A subsystem must not have feed-through from neighboring subsystems:
$$A_{\text{SS}_i} = 0. \tag{4}$$
Thus the large-scale system must not have algebraic loops. This condition can be relaxed somewhat; see Remark 4.
3) The identity
$$\bar{\sigma}(F_i) < 1 \tag{5}$$
is assumed to hold, where
$$F_i := \begin{bmatrix} 0 & B^d_{\text{S}_i} \\ C^z_{\text{S}_i} & D^{zd}_i \end{bmatrix}. \tag{6}$$
Equation (5) is not required for stability robustness; see Remarks 2 and 3. Therefore, if this assumption is not satisfied, we can still guarantee that the closed loop system is robustly stable to arbitrarily small delays.

A controller for this system has subsystems K_i given by

$$\begin{bmatrix} \dot{x}^{\text{K}}_i(t) \\ w^{\text{K}}_i(t) \\ u_i(t) \end{bmatrix} = \begin{bmatrix} A^{\text{K}}_{\text{TT}_i} & A^{\text{K}}_{\text{TS}_i} & B^{\text{K}}_{\text{T}_i} \\ A^{\text{K}}_{\text{ST}_i} & A^{\text{K}}_{\text{SS}_i} & B^{\text{K}}_{\text{S}_i} \\ C^{\text{K}}_{\text{T}_i} & C^{\text{K}}_{\text{S}_i} & D^{\text{K}}_i \end{bmatrix} \begin{bmatrix} x^{\text{K}}_i(t) \\ v^{\text{K}}_i(t) \\ y_i(t) \end{bmatrix}. \tag{7}$$

It is important to note that no assumptions are made on the state-space matrices of the controller subsystems since there seems to be no way to impose sparsity of these matrices as a convex optimization problem. Notice that controller subsystems can communicate via the signals v^{K}_i and w^{K}_i. Each interconnection edge of the controller's graph is associated with the signals w^{K}_{ij} and v^{K}_{ij} such that $v^{\text{K}}_i = \mathbf{cat}_j v^{\text{K}}_{ij}$ and $w^{\text{K}}_i = \mathbf{cat}_j w^{\text{K}}_{ij}$. As explained in the introduction, $\mathbf{dim}(w^{\text{K}}_{ij}) = m^{\text{K},+}_{ij}$ and $\mathbf{dim}(v^{\text{K}}_{ij}) = m^{\text{K},-}_{ij}$. The relation $v^{\text{K}}_{ij} = w^{\text{K}}_{ji}$, and consequently, $m^{\text{K},+}_{ij} = m^{\text{K},-}_{ji}$ always holds by definition. The closed loop subsystems are given by

$$\begin{bmatrix} \dot{x}^{\text{C}}_i(t) \\ w^{\text{C}}_i(t) \\ z_i(t) \end{bmatrix} = \begin{bmatrix} A^{\text{C}}_{\text{TT}_i} & A^{\text{C}}_{\text{TS}_i} & B^{\text{C}}_{\text{T}_i} \\ A^{\text{C}}_{\text{ST}_i} & A^{\text{C}}_{\text{SS}_i} & B^{\text{C}}_{\text{S}_i} \\ C^{\text{C}}_{\text{T}_i} & C^{\text{C}}_{\text{S}_i} & D^{\text{C}}_i \end{bmatrix} \begin{bmatrix} x^{\text{C}}_i(t) \\ v^{\text{C}}_i(t) \\ d_i(t) \end{bmatrix}, \tag{8}$$

where

$$x^{\text{C}}_i := (x_i\,,\, x^{\text{K}}_i), v^{\text{C}}_i := (v_i\,,\, v^{\text{K}}_i), w^{\text{C}}_i := (w_i\,,\, w^{\text{K}}_i) \tag{9}$$

The description of the closed loop system is completed by the equations

$$v^{\text{C}}_{ij}(t) = w^{\text{C}}_{ji}(t) \tag{10}$$

for each $i, j \in V$, where of course,

$$v^{\text{C}}_{ij} := (v_{ij}, v^{\text{K}}_{ij}), \quad w^{\text{C}}_{ij} := (w_{ij}, w^{\text{K}}_{ij}) \tag{11}$$

and

$$A^{\text{C}}_{\text{TT}_i} := \begin{bmatrix} A_{\text{TT}_i} + B^u_{\text{T}_i} D^{\text{K}}_i C^y_{\text{T}_i} & B^u_{\text{T}_i} C^{\text{K}}_{\text{T}_i} \\ B^{\text{K}}_{\text{T}_i} C^y_{\text{T}_i} & A^{\text{K}}_{\text{TT}_i} \end{bmatrix}, \tag{12}$$

$$A^{\text{C}}_{\text{TS}_i} := \begin{bmatrix} A_{\text{TS}_i} & B^u_{\text{T}_i} C^{\text{K}}_{\text{S}_i} \\ 0 & A^{\text{K}}_{\text{TS}_i} \end{bmatrix}, A^{\text{C}}_{\text{ST}_i} := \begin{bmatrix} A_{\text{ST}_i} & 0 \\ B^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} & A^{\text{K}}_{\text{ST}_i} \end{bmatrix}, \tag{13}$$

$$A^{\text{C}}_{\text{SS}_i} := \begin{bmatrix} 0 & 0 \\ 0 & A^{\text{K}}_{\text{SS}_i} \end{bmatrix}, B^{\text{C}}_{\text{T}_i} := \begin{bmatrix} B^d_{\text{T}_i} \\ 0 \end{bmatrix}, B^{\text{C}}_{\text{S}_i} := \begin{bmatrix} B^d_{\text{S}_i} \\ 0 \end{bmatrix}, \tag{14}$$

$$C^{\text{C}}_{\text{T}_i} := [\, C^z_{\text{T}_i} \;\; 0 \,], C^{\text{C}}_{\text{S}_i} := [\, C^z_{\text{S}_i} \;\; 0 \,], D^{\text{C}}_i := D^{zd}_i. \tag{15}$$

Notice that in the above equations for the closed loop state space matrices, we have made the assumptions 1) and 2). Let the dimensions of $x^{\text{C}}_i(t)$, $w^{\text{C}}_{ij}(t)$ and $v^{\text{C}}_{ij}(t)$ be denoted respectively by $n^{\text{C}}_i, m^{\text{C},+}_{ij}$ and $m^{\text{C},-}_{ij}$. Define the matrices $X_{\text{T}_i} \in \mathbb{R}^{n^{\text{C}}_i}_{\mathbf{S}}$, $X_{ij} \in \mathbb{R}^{m^{\text{C},+}_{ij}}_{\mathbf{S}}$, $Z_{ij} \in \mathbb{R}^{m^{\text{C},-}_{ij}}_{\mathbf{S}}$ and $Y_{ij} \in \mathbb{R}^{m^{\text{C},+}_{ij} \times m^{\text{C},-}_{ij}}$ satisfying the conditions

$$X_{\text{T}_i} \succ 0, X_{ij} = -Z_{ji}, Y^*_{ij} = -Y_{ji}. \tag{16}$$

(The identity $m^{\text{C},+}_{ij} = m^{\text{C},-}_{ji}$ ensures that the latter two conditions in (16) can hold).

The following simple analysis result for spatially interconnected systems is essentially from [11].

Theorem 1: [11] With the matrices X_{T_i}, X_{ij}, Y_{ij} and Z_{ij} as above, the closed loop system (8), (10) is well-posed [8], stable and satisfies the $\mathcal{H}_\infty$ performance bound

$$\sup_{\|d\|=1} \|z\|^2 \leq (1 - \kappa) \tag{17}$$

for some $\kappa > 0$ if the quadratic form

$$x^{\text{C}}_i(t)^* X_{\text{T}_i} \dot{x}^{\text{C}}_i(t) + \dot{x}^{\text{C}}_i(t)^* X_{\text{T}_i} x^{\text{C}}_i(t) + \sum_{j=1}^{L} \begin{bmatrix} w^{\text{C}}_{ij}(t) \\ v^{\text{C}}_{ij}(t) \end{bmatrix}^* \begin{bmatrix} X_{ij} & Y_{ij} \\ Y^*_{ij} & Z_{ij} \end{bmatrix} \times \begin{bmatrix} w^{\text{C}}_{ij}(t) \\ v^{\text{C}}_{ij}(t) \end{bmatrix} + z^*_i(t) z_i(t) - (1 - \kappa) d^*_i(t) d_i(t) \tag{18}$$

is negative definite for each $i \in V$. Here $x^{\text{C}}_i(t)$, $v^{\text{C}}_{ij}(t)$ and $d_i(t)$ are free to take values in their respective linear spaces while $\dot{x}^{\text{C}}_i(t)$, $w^{\text{C}}_{ij}(t)$ and $z_i(t)$ are computed from the subsystem equations (8).

Remark 1: A special case of the results in this paper appeared in [4]. The results of [4] assumed that $Y_{ij} = 0$ (which can be restrictive), and also only addressed the problem of stabilization (as opposed to the $\mathcal{H}_\infty$ control problem addressed herein).

Theorem 2: [11] Given a plant (1), (2), a controller which renders the quadratic form (18) negative definite (for each $i \in V$) for some choice of suitable matrices X_{T_i}, X_{ij}, Y_{ij} and Z_{ij} can be computed as the solution of an LMI. Moreover, the dimensions of the controller interconnection signals can be bounded by $\max(m^{\text{K},+}_{ij}, m^{\text{K},-}_{ij}) \leq 3 \max(m^+_{ij}, m^-_{ij})$.

B. Feedback loops with small delays: an abstract analysis problem

The interconnections described by equation (10) may be termed *ideal*, that is, they allow an instantaneous flow of information between any two subsystems. Since this assumption is rarely justified in practice, we would like to guarantee the stability and performance robustness of the closed loop system in the face of arbitrarily small delays between subsystems.

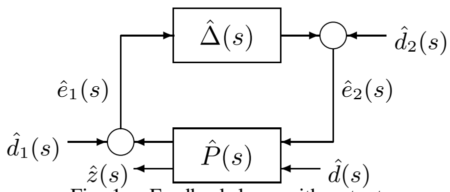

Fig. 1. Feedback loop with output

Definition 1: Consider a closed loop system with subsystems given by (8) and with the interconnections between subsystems described by

$$\hat{v}^{\mathrm{C}}_{ij}(s) = \hat{\Delta}_{ij}(s)\hat{w}^{\mathrm{C}}_{ji}(s) \tag{19}$$

instead of (10), where

$$\hat{\Delta}_{ij}(s) := \mathbf{diag}_{k=1}^{m^{\mathrm{C},+}_{ji}} e^{-\epsilon_{ijk}s}. \tag{20}$$

($\hat{}$ denotes taking the Laplace transform). The system is said to exhibit robust stability and performance to small communication delays if there exists an $\epsilon > 0$ such that the system is internally stable [8] and satisfies (17) for some $\kappa > 0$ and for all $\epsilon_{ijk} \leq \epsilon$.

In this subsection, we describe the analysis results which will enable us to tackle the problem of synthesizing controllers that guarantee stability and performance of an interconnected system in the face of small communication delays between any two subsystems. The idea is to consider a linear fractional representation [15] consisting of an LTI system $\hat{P}(s)$ in feedback with a diagonal delay $\hat{\Delta}(s) := \mathbf{diag}_{k=1}^{N} e^{-\epsilon_k s}$ (see Figure 1), and to develop analysis conditions which will guarantee robust stability and performance of this feedback system in the presence of any delay satisfying $\epsilon_k \leq \epsilon$ (for some $\epsilon > 0$). Since one can readily express an interconnected system with delays between subsystems as a linear fractional representation (by "pulling out the delays" as is done in robust control theory), such a result would be useful for synthesizing a controller which ensures this robustness property for the closed loop interconnected system given by (8), (19).

The analysis result presented in this subsection for feedback systems with small delays in the loop is essentially from [13].

Consider the feedback loop shown in Figure 1. Suppose the nominal feedback loop (i.e., with $\hat{\Delta}(s) = I$) is stable and satisfies (17) for some $\kappa > 0$. We would like to guarantee the existence of $\epsilon > 0$ such that the system shown in Figure 1 is internally stable (i.e., has uniformly bounded gain as an operator from $(d_1\,,\,d_2)$ to $(e_1\,,\,e_2)$) and, with $(d_1\,,\,d_2)$ set to zero, continues to satisfy (17) for all time delays $\hat{\Delta}(s) =: \mathbf{diag}_{k=1}^{N} e^{-\epsilon_k s}$ such that $\epsilon_k \leq \epsilon$. The solution to this problem involves the computation of a structured singular value. Assume that $A \in \mathbb{C}^{(N+n)\times(N+m)}$. Define, for $\gamma > 0$, the set

$$\mathcal{D}_\gamma := \left\{ \begin{bmatrix} \mathbf{diag}_k \delta_k & 0 \\ 0 & \Theta \end{bmatrix} : \delta_k \in \mathbb{C}, \Theta \in \mathbb{C}^{m\times n}, \max(|\delta_k|, \bar{\sigma}(\Theta)) \leq \gamma \right\} \tag{21}$$

and the structured singular value

$$\mu(A) := \frac{1}{\sup\{\gamma > 0 : \det(I + AA') \neq 0, \forall A' \in \mathcal{D}_\gamma\}}. \tag{22}$$

The following theorem will suffice for our purposes.

Theorem 3: Assume that the feedback loop of Figure 1 is internally stable in the nominal case ($\hat{\Delta}(s) = I$) and satisfies (17). Then there exists an $\epsilon > 0$ such that

1) the induced norm from $(d_1\,,\,d_2)$ to the internal signal $(e_1\,,\,e_2)$ is uniformly bounded for all $\epsilon_k \leq \epsilon$
2) upon setting $(d_1\,,\,d_2)$ is to zero, equation (17) continues to be satisfied by the feedback loop

if $\mu(\hat{P}(\infty)) < 1$.

Proof: The proof is a simple modification of the "if" part of Theorem 2.2 in the paper [13]; one uses the main loop theorem [15] to cast the robustness problem as a structured singular value test. The details are omitted for brevity. ■

Note that the condition that the induced norm from $(d_1\,,\,d_2)$ to $(e_1\,,\,e_2)$ is uniformly bounded amounts to *internal* stability of the delay system [8].

C. Linear fractional representation for interconnected systems

As mentioned in the previous subsection, we need to express the interconnected system as a feedback loop (Figure 1) by "pulling out the delays". For each $i \in V$, let $G^{\mathrm{C}}_i : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ be the subsystem represented by the state-space equations (8). Thus,

$$\begin{bmatrix} w^{\mathrm{C}}_i \\ z_i \end{bmatrix} = G^{\mathrm{C}}_i \begin{bmatrix} v^{\mathrm{C}}_i \\ d_i \end{bmatrix} \tag{23}$$

(recall that v^{C}_i and w^{C}_i are interconnection signals concatenated as in (9)). We need the following Lemma, the proof of which is simple and is omitted.

Lemma 4: Suppose the interconnected system containing subsystems G^{C}_i, coupled as in Equations (19), (20), is represented as the feedback loop in Figure 1, where

$$\hat{\Delta} := \mathbf{diag}_{i=1}^{L}\mathbf{diag}_{j=1}^{L}\hat{\Delta}_{ij}(s). \tag{24}$$

Then

$$\hat{P}(s) = Q_1(\mathbf{diag}_i G^{\mathrm{C}}_i(s))Q_2, \tag{25}$$

where Q_1 and Q_2 are permutation matrices.

D. The control problem

It is now possible to precisely formulate our control problem as follows: *Given a plant whose subsystems are represented in state space by (1), find a controller represented by the equations (7) such that the closed loop is stable and satisfies (17) when the ideal (nominal) interconnection relation (11) is satisfied. In addition, ensure that the structured singular value condition $\mu(\hat{P}(\infty)) < 1$ is satisfied.* The following lemma allows us to reduce the problem to one involving the maximum singular value of the matrix $\hat{G}^{\mathrm{C}}_i(\infty)$ rather than $\mu(\hat{P}(\infty))$ (this is convenient since the structured singular value is hard to compute [17]).

Lemma 5: The condition $\mu(\hat{P}(\infty)) < 1$ is satisfied provided $\bar{\sigma}(\hat{G}^{\mathrm{C}}_i(\infty)) < 1$ for each $i \in V$.

Proof: Merely observe that permutation matrices are orthogonal, so that $\bar{\sigma}(\mathbf{diag}_i\hat{G}^{\mathrm{C}}_i(\infty)) = \bar{\sigma}(\hat{P}(\infty))$. Since $\mu(A) \leq \bar{\sigma}(A)$ for any matrix A [15], and since $\bar{\sigma}(\mathbf{diag}_i\hat{G}^{\mathrm{C}}_i(\infty)) = \max_i \bar{\sigma}(\hat{G}^{\mathrm{C}}_i(\infty))$, the lemma follows immediately. ■

The observations of this section yield

Theorem 6: Suppose we can find a distributed controller that renders the closed loop system (8), (10) stable, with $\mathcal{H}_\infty$ norm less than unity. Suppose also that the closed loop subsystems G^{C}_i satisfy $\bar{\sigma}(\hat{G}^{\mathrm{C}}_i(\infty)) < 1$. From equations (14), (15), this last condition is equivalent to

$$\bar{\sigma}(\Xi_i) < 1, \tag{26}$$

where

$$\Xi_i := \hat{G}^{\mathrm{C}}_i(\infty) = \begin{bmatrix} 0 & 0 & B^d_{\mathrm{S}_i} \\ 0 & A^{\mathrm{K}}_{\mathrm{SS}_i} & 0 \\ C^z_{\mathrm{S}_i} & 0 & D^{zd}_i \end{bmatrix}, \tag{27}$$

for each $i \in V$. Then the closed loop exhibits robust stability and performance to small delays.

The following corollary for robust *stability* follows as a special case of the above result, noting that the closed loop feed-through matrix is $\mathbf{diag}_i A^{\mathrm{C}}_{\mathrm{SS}_i}$, modulo some permutation matrices, when the exogenous inputs and outputs (d_i and z_i respectively) are absent.

Corollary 7: Given that the closed loop system exhibits nominal stability and performance, a sufficient condition for robust *stability* to small delays is the requirement $\bar{\sigma}(A^{\mathrm{C}}_{\mathrm{SS}_i}) < 1$.

Remark 2: The matrix Ξ_i is similar, via a permutation matrix, to

$$\widetilde{\Xi}_i := \begin{bmatrix} A^{\mathrm{K}}_{\mathrm{SS}_i} & 0 \\ 0 & F_i \end{bmatrix} \tag{28}$$

(see equation (6)). Equation (26) can therefore be satisfied *only if* the assumption of equation (5) is satisfied. The precise condition on the matrix F_i which is necessary for robust performance in the presence of small delays involves the computation of a structured singular value; this point is not pursued further. However, Corollary 7 shows that robust stability to small delays can be obtained even if (5) is not satisfied.

We already know that a controller that stabilizes the nominal system can be found using the LMI analysis conditions of [11]; in addition, (26) must be satisfied. In the next section, we shall show how to obtain such a controller from the controller synthesized using the LMIs of [11].

III. Stability and Performance Robustness Against Small Communication Delays

The main results of this paper are presented in this section. The proofs are collected in the Appendix.

A. Rewriting a controller for robustness against small delays

Theorem 3 provides the insight that if algebraic loops of gain ≥ 1 (in the sense of the structured singular value) are absent in the closed loop system, then it is robust to arbitrarily small time delays. This confirms and generalizes the intuition based on the simple example (a unity feedback around a gain of $q > 1$) described in the Introduction.

Now, we have assumed that the feed-through matrices A_{SS_i} between the plant subsystems are zero; see (4). This is justified in practice as a well-posedness assumption on the plant model and can sometimes be relaxed; see Remark 4. Algebraic loops in the closed loop are therefore caused by the controller feed-through matrices $A^{\text{K}}_{\text{SS}_i}$. However, one cannot simply assume that $A^{\text{K}}_{\text{SS}_i}$ are zero, since the controller matrices are obtained from the LMI conditions of [11], and imposing sparsity on controller matrices seems to be an intractable problem in general (the failure to cast decentralized control problems as convex optimization problems in general cases [3] seems to confirm this intuition). With the insight that algebraic loops of gain ≥ 1 destroy stability or performance in the presence of small delays however, one could try to *rewrite* the controller obtained from the *convex* synthesis conditions of [11] so that the hypothesis of Theorem 6 are satisfied by the *new* controller feed-through matrices. This is the procedure adopted in the sequel. Needless to say, the *plant should not be modified in any way* and the rewritten controller must also be distributed.

B. Controller construction

Suppose, therefore, that the controller (7) has been computed such that the quadratic form (18) is negative definite for each $i \in V$. Partition X_{ij} and Z_{ij} as

$$X_{ij} =: \begin{bmatrix} X^{\text{G}}_{ij} & X^{\text{GK}}_{ij} \\ (X^{\text{GK}}_{ij})^* & X^{\text{K}}_{ij} \end{bmatrix},\ Z_{ij} =: \begin{bmatrix} Z^{\text{G}}_{ij} & Z^{\text{GK}}_{ij} \\ (Z^{\text{GK}}_{ij})^* & Z^{\text{K}}_{ij} \end{bmatrix} \quad (29)$$

so that the blocks conform to $w^{\text{C}}_{ij} = (w_{ij}\,,\, w^{\text{K}}_{ij})$ and $v^{\text{C}}_{ij} = (v_{ij}\,,\, v^{\text{K}}_{ij})$ respectively. Correspondingly, partition Y_{ij} as

$$Y_{ij} =: \begin{bmatrix} Y^{\text{G}}_{ij} & Y^{\text{GK}}_{ij} \\ Y^{\text{KG}}_{ij} & Y^{\text{K}}_{ij} \end{bmatrix} \quad (30)$$

(note that Y_{ij} is not necessarily symmetric or even square).

The first step in the construction is to compute certain transformation matrices. Assume without loss of generality that the symmetric matrices

$$U^{\text{K}}_{ij} := \begin{bmatrix} X^{\text{K}}_{ij} & Y^{\text{K}}_{ij} \\ (Y^{\text{K}}_{ij})^* & Z^{\text{K}}_{ij} \end{bmatrix} \quad (31)$$

are invertible and still result in (18) being negative definite (by a small perturbation if necessary). Compute matrices T_{ij} via a factorization

$$U^{\text{K}}_{ij} =: T^*_{ij} R_{ij} T_{ij}, \quad (32)$$

where

$$R_{ij} = \mathbf{diag}(I, -I). \quad (33)$$

Note that if R_{ij} has n_p positive entries and n_n negative entries, the inertia (the ordered triple consisting of the number of positive, zero and negative eigenvalues respectively) of U^{K}_{ij} is $(n_p, 0, n_n)$. Make the co-ordinate transformation

$$\begin{bmatrix} w^{\text{K}}_{ij} \\ v^{\text{K}}_{ij} \end{bmatrix} =: T^{-1}_{ij} \begin{bmatrix} \breve{w}^{\text{K}}_{ij} \\ \breve{v}^{\text{K}}_{ij} \end{bmatrix}, \quad (34)$$

where the partition $(\breve{w}^{\text{K}}_{ij},\ \breve{v}^{\text{K}}_{ij})$ is conformable with the principal submatrices (I and $-I$ respectively) of R_{ij}. From the closed loop equations (8) and matrices (13), (14), we have

$$w^{\text{K}}_i = B^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} x_i + A^{\text{K}}_{\text{ST}_i} x^{\text{K}}_i + A^{\text{K}}_{\text{SS}_i} v^{\text{K}} \quad (35)$$

$$\Longrightarrow \begin{bmatrix} I & -A^{\text{K}}_{\text{SS}_i} \end{bmatrix} \mathcal{P}^{\text{K}}_i \mathbf{diag}_j T^{-1}_{ij} \left(\mathbf{cat}_j \begin{bmatrix} \breve{w}^{\text{K}}_{ij} \\ \breve{v}^{\text{K}}_{ij} \end{bmatrix} \right) = B^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} x_i + A^{\text{K}}_{\text{ST}_i} x^{\text{K}}_i, \quad (36)$$

where $\mathcal{P}^{\text{K}}_i$ is a permutation matrix. Define $\tilde{\mathcal{P}}^{\text{K}}_i$ by

$$\tilde{\mathcal{P}}^{\text{K}}_i : \begin{bmatrix} \mathbf{cat}_j \breve{w}^{\text{K}}_{ij} \\ \mathbf{cat}_j \breve{v}^{\text{K}}_{ij} \end{bmatrix} \mapsto \mathbf{cat}_j \begin{bmatrix} \breve{w}^{\text{K}}_{ij} \\ \breve{v}^{\text{K}}_{ij} \end{bmatrix}, \quad (37)$$

and define the matrix $\mathcal{A}_i$ and the vector ζ_i by

$$\mathcal{A}_i := \begin{bmatrix} I & -A^{\text{K}}_{\text{SS}_i} \end{bmatrix} \mathcal{P}^{\text{K}}_i \mathbf{diag}_j T^{-1}_{ij} \tilde{\mathcal{P}}^{\text{K}}_i \quad (38)$$

$$\zeta_i := B^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} x_i + A^{\text{K}}_{\text{ST}_i} x^{\text{K}}_i \quad (39)$$

to get

$$\mathcal{A}_i \begin{bmatrix} \mathbf{cat}_j \breve{w}^{\text{K}}_{ij} \\ \mathbf{cat}_j \breve{v}^{\text{K}}_{ij} \end{bmatrix} = \zeta_i \quad (40)$$

$$\Longrightarrow \mathcal{A}^w_i \mathbf{cat}_j \breve{w}^{\text{K}}_{ij} = \mathcal{A}^v_i \mathbf{cat}_j \breve{v}^{\text{K}}_{ij} + \zeta_i, \quad (41)$$

where $\mathcal{A}_i =: \begin{bmatrix} \mathcal{A}^w_i & -\mathcal{A}^v_i \end{bmatrix}$ has been partitioned conformably with the vector $(\mathbf{cat}_j \breve{w}^{\text{K}}_{ij},\ \mathbf{cat}_j \breve{v}^{\text{K}}_{ij})$.

Thus if the matrix $\mathcal{A}^w_i$ has a left inverse (it is in general rectangular), we can solve uniquely for $\mathbf{cat}_i \breve{w}^{\text{K}}_{ij}$. The following theorem shows that this is indeed the case. For convenience, define $\breve{w}^{\text{K}}_i := \mathbf{cat}_i \breve{w}^{\text{K}}_{ij}$, $\breve{v}^{\text{K}}_i := \mathbf{cat}_i \breve{v}^{\text{K}}_{ij}$.

Theorem 8: The relation

$$\breve{w}^{\text{K}}_i = (\mathcal{A}^w_i)^\dagger \mathcal{A}^v_i \breve{v}^{\text{K}}_i + (\mathcal{A}^w_i)^\dagger \zeta_i \quad (42)$$

holds, where for a rectangular matrix Θ with full column rank, the pseudoinverse $\Theta^\dagger$ is defined as $(\Theta^*\Theta)^{-1}\Theta^*$.

Theorem 8 is a key result in the whole theory. This theorem has allowed us to define a new set of interconnection variables for the controller subsystems. We can now construct a new controller with these interconnection signals. From (42) and (39), we get

$$\breve{w}^{\text{K}}_i = (\mathcal{A}^w_i)^\dagger \mathcal{A}^v_i \breve{v}^{\text{K}}_i + (\mathcal{A}^w_i)^\dagger B^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} x_i + (\mathcal{A}^w_i)^\dagger A^{\text{K}}_{\text{ST}_i} x^{\text{K}}_i. \quad (43)$$

Equation (43), together with the equation for the plant interconnection output w_i (obtained from (8)) now begins to resemble the closed-loop interconnection equation (with a new controller):

$$\begin{bmatrix} w_i \\ \breve{w}^{\text{K}}_i \end{bmatrix} = \begin{bmatrix} A_{\text{ST}_i} & 0 \\ \bar{B}^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} & \bar{A}^{\text{K}}_{\text{ST}_i} \end{bmatrix} \begin{bmatrix} x_i \\ x^{\text{K}}_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}^{\text{K}}_{\text{SS}_i} \end{bmatrix} \begin{bmatrix} v_i \\ \breve{v}^{\text{K}}_i \end{bmatrix} + \begin{bmatrix} B^d_{\text{S}_i} \\ 0 \end{bmatrix} d_i, \quad (44)$$

where

$$\bar{A}^{\text{K}}_{\text{ST}_i} := (\mathcal{A}^w_i)^\dagger A^{\text{K}}_{\text{ST}_i},\ \bar{A}^{\text{K}}_{\text{SS}_i} := (\mathcal{A}^w_i)^\dagger \mathcal{A}^v_i,\ \bar{B}^{\text{K}}_{\text{S}_i} := (\mathcal{A}^w_i)^\dagger B^{\text{K}}_{\text{S}_i}. \quad (45)$$

(Compare with (13), (14)).

Now the state equations of the closed loop subsystems are, for each $i \in V$,

$$\dot{x}^{\text{C}}_i = A^{\text{C}}_{\text{TT}_i} x^{\text{C}}_i + A^{\text{C}}_{\text{TS}_i} v^{\text{C}}_i + B^{\text{C}}_{\text{T}_i} d_i. \quad (46)$$

(See (8), (12), (13) and (14)). Since $v^{\text{C}}_i = (v_i\,,\, v^{\text{K}}_i)$, we must express v^{K}_i in terms of $\breve{v}^{\text{K}}_i$. Partition the matrices T^{-1}_{ij} as

$$T^{-1}_{ij} =: S_{ij} = \begin{bmatrix} S^{11}_{ij} & S^{12}_{ij} \\ S^{21}_{ij} & S^{22}_{ij} \end{bmatrix} \quad (47)$$

according to (34). Some algebra later, we obtain

$$v^{\text{K}}_i = (\mathbf{diag}_j S^{21}_{ij}) \bar{B}^{\text{K}}_{\text{S}_i} C^y_{\text{T}_i} x_i + (\mathbf{diag}_j S^{21}_{ij}) \bar{A}^{\text{K}}_{\text{ST}_i} x^{\text{K}}_i + \left[(\mathbf{diag}_j S^{21}_{ij}) \bar{A}^{\text{K}}_{\text{SS}_i} + (\mathbf{diag}_j S^{22}_{ij}) \right] \breve{v}^{\text{K}}_i. \quad (48)$$

The closed loop state equation can now be written in terms of the states and the new interconnection input:

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}^{\text{K}}_i \end{bmatrix} = \begin{bmatrix} A_{\text{TT}_i} + B^u_{\text{T}_i} \bar{D}^{\text{K}}_i C^y_{\text{T}_i} & B^u_{\text{T}_i} \bar{C}^{\text{K}}_{\text{T}_i} \\ \bar{B}^{\text{K}}_{\text{T}_i} C^y_{\text{T}_i} & \bar{A}^{\text{K}}_{\text{TT}_i} \end{bmatrix} \begin{bmatrix} x_i \\ x^{\text{K}}_i \end{bmatrix} + \begin{bmatrix} A_{\text{TS}_i} & B^u_{\text{T}_i} \bar{C}^{\text{K}}_{\text{S}_i} \\ 0 & \bar{A}^{\text{K}}_{\text{TS}_i} \end{bmatrix} \begin{bmatrix} v_i \\ \breve{v}^{\text{K}}_i \end{bmatrix} + B^{\text{C}}_{\text{T}_i} d_i, \quad (49)$$

where the controller matrices $\bar{A}^{\rm K}_{{\rm TS}_i}$, $\bar{C}^{\rm K}_{{\rm S}_i}$, $\bar{B}^{\rm K}_{{\rm T}_i}$, $\bar{A}^{\rm K}_{{\rm TT}_i}$, $\bar{C}^{\rm K}_{{\rm S}_i}$, and $\bar{A}^{\rm K}_{{\rm TS}_i}$ can be computed from the foregoing signal relations. Comparing the closed loop matrices in (49) and (44) with (8), (12), (13) and (14), we see that the matrices $\bar{A}^{\rm K}_{{\rm TT}_i}$, etc, define a valid controller. The first theorem about this controller is that it achieves nominal stability and performance:

Theorem 9: Assume that the plant satisfies the (3) and (4). Suppose the controller subsystems (7) have been computed such that the quadratic form (18) is negative definite for each $i \in V$. Let the controller subsystems be as in Equation 7, with $A^{\rm K}_{{\rm TT}_i}$ replaced by $\bar{A}^{\rm K}_{{\rm TT}_i}$, etc. Let the closed loop subsystems be interconnected according to

$$v_{ij}(t) = w_{ji}(t) \text{ and } v^{\rm K}_{ij}(t) = w^{\rm K}_{ji}(t). \tag{50}$$

The resulting closed loop system is stable and satisfies (17).

The new closed loop system exhibits robust stability and performance to small delays (see Definition 1):

Theorem 10: Assume that the plant satisfies (3), (4) and (5). When the new controller is connected to the plant, the closed loop matrix $\bar{\Xi}_i$, which is just Ξ_i (see (27)) with $A^{\rm K}_{{\rm SS}_i}$ replaced by $\bar{A}^{\rm K}_{{\rm SS}_i}$, satisfies $\bar{\sigma}(\bar{\Xi}_i) < 1$. Consequently, by Theorems 9 and 6, the closed loop exhibits robust stability and performance to small delays.

Remark 3: From the proof of the above theorem (Section V-C), it is seen that $\bar{\sigma}(\bar{A}^{\rm K}_{{\rm SS}_i}) < 1$ whether or not (5) holds. It follows that the closed loop system satisfies the condition $\bar{\sigma}(A^{\rm C}_{{\rm SS}_i}) < 1$, where the closed loop feed-through matrix $A^{\rm C}_{{\rm SS}_i}$, is now given by equation (14) with $A^{\rm K}_{{\rm SS}_i}$ replaced by $\bar{A}^{\rm K}_{{\rm SS}_i}$. Thus from Corollary 7, the closed loop is robustly stable to small delays even if (5) is not satisfied.

Remark 4: The assumption $A_{{\rm SS}_i} = 0$ made in this paper can be considerably relaxed if we are only concerned with stability robustness; the method given in this paper goes through without modifications if $\bar{\sigma}(A_{{\rm SS}_i}) < 1$ for each $i \in V$. This can be seen from the proof of Theorem 10 and the fact that, when $A_{{\rm SS}_i} \neq 0$, the closed loop feed-through matrix $A^{\rm C}_{{\rm SS}_i}$ (see (14)), with the new controller in place, becomes

$$A^{\rm C}_{{\rm SS}_i} = \begin{bmatrix} A_{{\rm SS}_i} & 0 \\ 0 & \bar{A}^{\rm K}_{{\rm SS}_i} \end{bmatrix}. \tag{51}$$

From the previous remark, $\bar{\sigma}(\bar{A}^{\rm K}_{{\rm SS}_i}) < 1$ holds; therefore, $\bar{\sigma}(A^{\rm C}_{{\rm SS}_i}) < 1$. Robust stability now follows from Corollary 7.

Thus we see that the assumption of equation (5) is not necessary to guarantee *stability* robustness, which is encouraging, since one cannot really impose this constraint on the plant. Now suppose (5) does not hold, but that (3) and (4) do, and that a controller has been synthesized (as above) such that $\bar{\sigma}(\bar{A}^{\rm K}_{{\rm SS}_i}) < 1$. Thus, by Corollary 7, the system is robustly stable to small delays and exhibits nominal performance. Now the following question arises: is it possible to estimate an upper bound for the $\mathcal{H}^\infty$ norm of the closed loop system from d to z in the presence of small delays? In other words, is it possible to find numbers $\Gamma, \epsilon > 0$ such that

$$\sup_{\|d\|=1} \|z\| < \Gamma \tag{52}$$

whenever the interconnections between subsystems are described by the relations (19) and (20), with $\epsilon_{ijk} < \epsilon$? Theorem 11 below answers this question in the affirmative.

Theorem 11: Suppose $F_i \in \mathbb{R}^{\tilde{m}_i \times \tilde{n}_i}$ (see (6) for the definition of F_i). Let $\bar{\sigma}(F_i) \geq 1$ for at least one $i \in V$, so that robust performance to small delays is not ensured by Theorem 10. Define

$$\Gamma := \max_i \{\tilde{n}_i \bar{\sigma}^2(F_i)\} + \eta, \tag{53}$$

where $\eta > 0$ is any (small) constant. Then there exists $\epsilon > 0$ such that the closed loop system obtained by interconnecting the plant with the new controller is internally stable and satisfies (52) whenever the interconnections between subsystems are described by the relations (19) and (20), with $\epsilon_{ijk} < \epsilon$.

Therefore, if the assumption (5) is not satisfied, it is still possible to bound the *worst-case* $\mathcal{H}_\infty$ gain of the closed loop system in the presence of small delays, in terms of the plant matrices.

By chasing through the sizes of the signals, the following bound can be established on the dimensions of the controller's interconnection signals. The proof is simple and is omitted.

Theorem 12: Suppose the original controller (7) for the plant (1) satisfied the relation

$$\max(\mathbf{dim}(w^{\rm K}_{ij}), \mathbf{dim}(v^{\rm K}_{ij})) \leq \alpha \max(\mathbf{dim}(w_{ij}), \mathbf{dim}(v_{ij})),$$

for some constant α. Then the new controller satisfies the relation

$$\max(\mathbf{dim}(w^{\rm K}_{ij}), \mathbf{dim}(v^{\rm K}_{ij})) \leq 2\alpha \max(\mathbf{dim}(w_{ij}), \mathbf{dim}(v_{ij})).$$

Since we can guarantee a bound $\alpha = 3$ by the LMIs of [11], the new controller's interconnection dimension does not exceed that of the plant (in every channel) by more than a factor of 6.

IV. Conclusions

In this paper, a technique was given to construct a class of structured controllers to stabilize a spatially interconnected system and to render the closed loop $\mathcal{H}_\infty$ norm less than unity in the face of small communication delays. The importance of this problem is self-evident given the spatially distributed nature of the problem considered in this and related work [11], [5]. The results build on those of [11] on distributed control synthesis for large-scale systems and on those of [13] on the stability of feedback loops affected by small delays. The controller construction algorithm has been demonstrated with the help of an example.

V. Appendix

A. Proof of Theorem 8

In view of the discussion before Theorem 8, it is sufficient to prove that $\mathcal{A}^w_i$ has a left inverse.

Proof: Define $\Phi_i := (x^{\rm C}_i)^* X_{{\rm T}_i} \dot{x}^{\rm C}_i + (\dot{x}^{\rm C}_i)^* X_{{\rm T}_i} x^{\rm C}_i$. Note that while the state $x^{\rm C}_i$ is really a signal in $\mathcal{L}_{2e}$, we are treating it as a vector (in $\mathbb{R}^{n^{\rm C}_i}$) here. We shall continue to do so for all other signals without further comment. Also, if $\beta(f)$ is a quadratic form in a vector f, we use the notation $\beta(f) < 0$ to convey the fact that β is negative definite (though the strict inequality holds only if $f \neq 0$).

Recall the meaning of the negative definiteness of the quadratic form (18): vectors $x^{\rm C}_i$, $v^{\rm C}_i$ and d_i are independent variables and $\dot{x}^{\rm C}_i$, $w^{\rm C}_i$ and z_i must be computed from the closed loop equations (8) and (12) - (15). Now, if we set $x_i = x^{\rm K}_i = v_{ij} = d_i = 0$, we see from (8) and (12) - (15) that $w_{ij} = z_i = 0$; so (18) becomes

$$\sum_{j=1}^{L} \begin{bmatrix} w^{\rm K}_{ij} \\ v^{\rm K}_{ij} \end{bmatrix}^* U^{\rm K}_{ij} \begin{bmatrix} w^{\rm K}_{ij} \\ v^{\rm K}_{ij} \end{bmatrix} < 0, \tag{54}$$

where $U^{\rm K}_{ij}$ was defined in (31). From (34), (32) and (33), we have

$$\sum_{j=1}^{L} |\check{w}^{\rm K}_{ij}|^2 - \sum_{j=1}^{L} |\check{v}^{\rm K}_{ij}|^2 < 0 \implies |\check{w}^{\rm K}_i|^2 - |\check{v}^{\rm K}_i|^2 < 0. \tag{55}$$

From equation (41), noting that $\zeta_i = 0$ (this is because we have set $x_i = x^{\rm K}_i = 0$), we have

$$\mathcal{A}^w_i \check{w}^{\rm K}_i = \mathcal{A}^v_i \check{v}^{\rm K}_i. \tag{56}$$

From (55) and (56), we have that, for all ω and ν satisfying $\mathcal{A}^w_i \omega - \mathcal{A}^v_i \nu = 0$, the inequality $|\omega|^2 - |\nu|^2 < 0$ holds provided $|\omega|^2 + |\nu|^2 \neq 0$. Now *suppose* $\mathcal{A}^w_i \omega = 0$ for some $\omega \neq 0$. Then by choosing $\nu = 0$, we can satisfy $\mathcal{A}^w_i \omega - \mathcal{A}^v_i \nu = 0$, but on the other hand, $|\omega|^2 - |\nu|^2 = |\omega|^2 > 0$, a contradiction. Hence $\mathcal{A}^w_i \omega = 0$ implies that $\omega = 0$. Thus $\mathcal{A}^w_i$ has full column rank; in other words it has a left inverse. ■

B. Proof of Theorem 9

From (18), (34), (32) and (47), we have

$$\Phi_i + \sum_{j=1}^{L} \begin{bmatrix} \check{w}^{\rm C}_{ij} \\ \check{v}^{\rm C}_{ij} \end{bmatrix}^* \begin{bmatrix} \check{X}_{ij} & \check{Y}_{ij} \\ \check{Y}^*_{ij} & \check{Z}_{ij} \end{bmatrix} \begin{bmatrix} \check{w}^{\rm C}_{ij} \\ \check{v}^{\rm C}_{ij} \end{bmatrix} + z^*_i z_i - (1-\kappa) d^*_i d_i < 0, \tag{57}$$

where $\check{w}^{\rm C}_{ij} := (w_{ij}\,,\, \check{w}^{\rm K}_{ij})$, $\check{v}^{\rm C}_{ij} := (v_{ij}\,,\, \check{v}^{\rm K}_{ij})$ and

$$\check{X}_{ij} := \begin{bmatrix} X^{\rm G}_{ij} & \star \\ (S^{11}_{ij})^* (X^{\rm GK}_{ij})^* + (S^{21}_{ij})^* (Y^{\rm GK}_{ij})^* & I \end{bmatrix}, \tag{58}$$

$$\check{Y}_{ij} := \begin{bmatrix} Y^{\rm G}_{ij} & Y^{\rm GK}_{ij} S^{22}_{ij} + X^{\rm GK}_{ij} S^{12}_{ij} \\ (S^{11}_{ij})^* Y^{\rm KG}_{ij} + (S^{21}_{ij})^* (Z^{\rm GK}_{ij})^* & 0 \end{bmatrix}, \tag{59}$$

$$\check{Z}_{ij} := \begin{bmatrix} Z^{\text{G}}_{ij} & \star \\ (S^{22}_{ij})^*(Z^{\text{GK}}_{ij})^* + (S^{12}_{ij})^* Y^{\text{KG}}_{ij} & -I \end{bmatrix}, \qquad (60)$$

where $\star$ denotes terms induced by symmetry. Recall that $\mathbf{dim}(w^{\text{K}}_{ij}) = m^{\text{K},+}_{ij}$ and $\mathbf{dim}(v^{\text{K}}_{ij}) = m^{\text{K},-}_{ij}$; let $\mathbf{dim}(\check{w}^{\text{K}}_{ij}) := \check{m}^{\text{K},+}_{ij}$ and $\mathbf{dim}(\check{v}^{\text{K}}_{ij}) := \check{m}^{\text{K},-}_{ij}$. From (16), we have $X^{\text{K}}_{ij} = -Z^{\text{K}}_{ji}$ and $(Y^{\text{K}}_{ij})^* = -Y^{\text{K}}_{ji}$. These can be written as

$$\begin{bmatrix} X^{\text{K}}_{ij} & Y^{\text{K}}_{ij} \\ (Y^{\text{K}}_{ij})^* & Z^{\text{K}}_{ij} \end{bmatrix} = \begin{bmatrix} 0 & I_{m^{\text{K},-}_{ji}} \\ I_{m^{\text{K},+}_{ji}} & 0 \end{bmatrix} \left(- \begin{bmatrix} X^{\text{K}}_{ji} & Y^{\text{K}}_{ji} \\ (Y^{\text{K}}_{ji})^* & Z^{\text{K}}_{ji} \end{bmatrix} \right) \times \begin{bmatrix} 0 & I_{m^{\text{K},+}_{ji}} \\ I_{m^{\text{K},-}_{ji}} & 0 \end{bmatrix} \qquad (61)$$

$$\implies \begin{bmatrix} X^{\text{K}}_{ij} & Y^{\text{K}}_{ij} \\ (Y^{\text{K}}_{ij})^* & Z^{\text{K}}_{ij} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I_{m^{\text{K},-}_{ji}} \\ I_{m^{\text{K},+}_{ji}} & 0 \end{bmatrix} \left(- \begin{bmatrix} X^{\text{K}}_{ji} & Y^{\text{K}}_{ji} \\ (Y^{\text{K}}_{ji})^* & Z^{\text{K}}_{ji} \end{bmatrix}^{-1} \right) \times \begin{bmatrix} 0 & I_{m^{\text{K},+}_{ji}} \\ I_{m^{\text{K},-}_{ji}} & 0 \end{bmatrix}. \qquad (62)$$

Thus $(U^{\text{K}}_{ij})^{-1}$ and $-(U^{\text{K}}_{ji})^{-1}$ have the same inertia (see (31)). From (32) and (47), the identity

$$(U^{\text{K}}_{ij})^{-1} = S_{ij} R_{ij} S^*_{ij} \qquad (63)$$

holds. Since the inertia of $(U^{\text{K}}_{ij})^{-1}$ is captured by R_{ij}, we can in fact compute S_{ij} via (32), (47) for $i < j$ and get S_{ij} for $i > j$ from

$$\begin{bmatrix} S^{11}_{ij} & S^{12}_{ij} \\ S^{21}_{ij} & S^{22}_{ij} \end{bmatrix} = \begin{bmatrix} 0 & I_{m^{\text{K},+}_{ji}} \\ I_{m^{\text{K},-}_{ji}} & 0 \end{bmatrix} \begin{bmatrix} S^{11}_{ji} & S^{12}_{ji} \\ S^{21}_{ji} & S^{22}_{ji} \end{bmatrix} \begin{bmatrix} 0 & I_{\check{m}^{\text{K},+}_{ji}} \\ I_{\check{m}^{\text{K},-}_{ji}} & 0 \end{bmatrix}. \qquad (64)$$

Assume that this has been done prior to controller construction. Thus

$$\begin{bmatrix} S^{11}_{ji} & S^{12}_{ji} \\ S^{21}_{ji} & S^{22}_{ji} \end{bmatrix} = \begin{bmatrix} S^{22}_{ji} & S^{21}_{ji} \\ S^{12}_{ji} & S^{11}_{ji} \end{bmatrix} \qquad (65)$$

Substituting equations (16), (65) into (58), (59) and (60) gives

$$\check{X}_{ij} = -\check{Z}_{ji}, \check{Y}^*_{ij} = -\check{Y}_{ji}. \qquad (66)$$

The theorem now follows from (57), (66) the stability criterion given in Theorem 1.

C. Proof of Theorem 10

From (45), the controller matrix $\bar{A}^{\text{K}}_{\text{SS}_i}$ is nothing but $(\mathcal{A}^w_i)^\dagger \mathcal{A}^v_i$. We claim that $\bar{\sigma}(\bar{A}^{\text{K}}_{\text{SS}_i}) < 1$. We know that for all ω, ν such that $\mathcal{A}^w_i \omega = \mathcal{A}^v_i \nu$, the inequality $|\omega|^2 < |\nu|^2$ holds provided ω and ν are not both zero (see the proof of Theorem 8). Thus for all $\nu \neq 0$, it holds that $|(\mathcal{A}^w_i)^\dagger \mathcal{A}^v_i \nu|^2 < \nu^2$, or that $\bar{\sigma}(\bar{A}^{\text{K}}_{\text{SS}_i}) < 1$. From (5) and the structure of the matrix Ξ_i (see Remark 2 and (28) for example), the theorem follows.

D. Proof of Theorem 11

Note that, since $\bar{\sigma}(F_i) \geq 1$ for at least one $i \in V$, the identity $\Gamma > 1$ holds. Define a new closed loop system by scaling the outputs z_i and the inputs d_i in (8) as follows:

$$z'_i := \frac{z_i}{\sqrt{\Gamma}}, \; d'_i := d_i \sqrt{\Gamma}, \qquad (67)$$

where z'_i and d'_i are the new output and input respectively of subsystem i.

In terms of the new output and input, the matrix F_i in (6) is transformed to

$$F'_i := \begin{bmatrix} 0 & \frac{B^d_{\text{S}_i}}{\sqrt{\Gamma}} \\ \frac{C^z_{\text{S}_i}}{\sqrt{\Gamma}} & \frac{D^{zd}_i}{\Gamma} \end{bmatrix}. \qquad (68)$$

We claim that $\bar{\sigma}(F'_i) < 1$. Assume that $F'_i \neq 0$. Recall that the Frobenius norm of any matrix A, denoted by $\|A\|_F$, is given by the Euclidean norm of the vector of all elements of A. Since $\Gamma > 1$,

$$\|F'_i\|_F \leq \frac{\|F_i\|_F}{\sqrt{\Gamma}} \qquad (69)$$

$$\implies \|F'_i\|_F < \frac{\|F_i\|_F}{\bar{\sigma}(F_i)\sqrt{\tilde{n}_i}}. \qquad (70)$$

Since $\| \bullet \|_F$ and $\bar{\sigma}(\bullet)$ are equivalent norms satisfying

$$\bar{\sigma}(A) \leq \|A\|_F \leq \bar{\sigma}(A)\sqrt{n} \qquad (71)$$

for any $A \in \mathbb{R}^{m \times n}$(see for example [9], page 56), one obtains

$$\|F'_i\|_F < 1 \qquad (72)$$

$$\implies \bar{\sigma}(F'_i) < 1, \qquad (73)$$

which proves the claim. Now the nominal system mapping $d' := \mathbf{cat}_i d'_i$ to $z' := \mathbf{cat}_i z'_i$ (is stable and) has $\mathcal{H}_\infty$ norm less than unity, since

$$\frac{\|z'\|}{\|d'\|} = \frac{1}{\Gamma} \frac{\|z\|}{\|d\|} \leq \frac{\sqrt{1-\kappa}}{\Gamma} \qquad (74)$$

whenever $d \neq 0$; the last inequality is because the nominal system satisfies the performance equation (17) for some $\kappa > 0$. Since $\bar{\sigma}(\bar{A}^{\text{K}}_{\text{SS}_i}) < 1$ by hypothesis, Theorem 10 and the form of equation (28) imply that the system mapping d' to z' exhibits robust performance to small time delays; thus

$$\sup_{\|d'\|=1} \|z'\| < 1 \qquad (75)$$

in the presence of small delays. From the first equality of (74), the theorem follows.

References

[1] P. Apkarian, P. Gahinet, and G. Becker. Self-scheduled $\mathcal{H}_\infty$ control of linear parameter-varying systems: A design example. *Automatica*, 31(9):1251–1261, 1995.

[2] B. Bamieh, F. Paganini, and M. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7):1091–1118, 2002.

[3] V. Blondel and J. N. Tsitsiklis. NP-hardness of some linear control design problems. *SIAM Journal on Control and Optimization*, 35(5):2218–2127, November 1997.

[4] R. S. Chandra, C. Langbort, and R. D'Andrea. Distributed control design for spatially interconnected systems with robustness to small communication delays. *Mathematical Theory of Networks and Systems*, 2004. Available at http://www.mtns2004.be/database/papersubmission/upload/454.pdf.

[5] R. D'Andrea and G. E. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, 48(9):1478–1495, 2003.

[6] C. Desoer and M. Vidyasagar. *Feedback systems: input-output properties*. Academic Press, 1975.

[7] G. E. Dullerud and R. D'Andrea. Distributed control of heterogeneous systems. *IEEE Transactions on Automatic Control*, 2004. To appear.

[8] G. E. Dullerud and F. Paganini. *A Course in Robust Control Theory*. Springer-Verlag, 2000.

[9] G.H. Golub and C.F. Van Loan. *Matrix Computations*. The Jonhs Hopkins University Press, 1996.

[10] D. Gorinevsky and G. Stein. Structured uncertainty analysis of robust stability for multidimensional array systems. *IEEE Transactions on Automatic Control*, 48(8):1557–1568, 2003.

[11] C. Langbort, R. S. Chandra, and R. D'Andrea. Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control*, 49(9):1502, 2004.

[12] C. Langbort and R. D'Andrea. Distributed control of heterogeneous systems interconnected over arbitrary graphs. In *IEEE Conference on Decision and Control*, pages 2835–2840, 2003.

[13] G. Meinsma, M. Fu, and T. Iwasaki. Robustness of the stability of feedback systems with respect to small time delays. *Systems and Control Letters*, 36:131–134, 1999.

[14] P.J. Moylan and D.J. Hill. Stability criteria for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):143–149, 1999.

[15] A. Packard and J. C. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[16] C. Scherer. LPV control and full block multipliers. *Automatica*, 37:361–375, 2001.

[17] O. Toker and H. Ozbay. On the complexity of purely complex mu computation and related problems in multidimensional systems. *IEEE Transactions on Automatic Control*, 43(3):409–414, 1998.

[18] J. C. Willems. *The analysis of feedback systems*. MIT Press, 1971.

[19] J.C. Willems. Dissipative dynamical systems I & II. *Archive for Rational Mechanics and Analysis*, 45:321–393, 1972.

[20] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1996.