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A Perturbation Approach to the $\mathcal{H}^2$ Analysis of Spatially Periodic Systems

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***Abstract*— We consider a class of spatially distributed systems with spatially periodic coefficients. Frequency domain methods are used to convert a linear system belonging to this class into a family of infinite-dimensional LTI ones. The frequency and amplitude of the periodic coefficients are then treated as parameters. A perturbation analysis is then performed on the amplitude of the periodicity to find a computationally efficient method of calculating the $\mathcal{H}^2$-norm. The dependence of the $\mathcal{H}^2$-norm on the amplitude and frequency of the periodic coefficients is demonstrated.**

I. Introduction

The $\mathcal{H}^2$-norm of a linear system is an indicator of the amount of energy amplification by the system. While an eigenvalue analysis determines whether or not a linear system is stable, it lacks information about its transient behavior, a phenomenon that depends on the coupling of the states and non-normal structures. The $\mathcal{H}^2$-norm on the other hand captures such behavior, which makes it a valuable tool in the analysis and synthesis of linear systems [1].

$\mathcal{H}^2$-norm analysis of spatially distributed systems has recently received attention. One application of this has been in the investigation of disturbance amplification and transition to turbulence in channel flows [2] [3] [4] [5], a topic of great practical and theoretical interest. It has also been proposed that certain spatially-periodic structures, called riblets, can lead to a reduction of disturbance amplification in such flows. This serves as one of the main motivations of the present work.

In this paper we study the $\mathcal{H}^2$-norm of spatially-periodic systems that are in the form of a spatially-invariant system with spatially-periodic coefficients. We use a frequency domain representation of periodic operators to convert the spatially-periodic linear system to an infinie-dimensional ODE (i.e. an infinite number of coupled scalar ODEs).

Clearly, to calculate the $\mathcal{H}^2$-norm of such a system, one has to first take a large enough truncation of the system matrices. As a result, finding the $\mathcal{H}^2$-norm can be a very computationally-intensive task. For this reason we pursue a perturbation approach to this problem.

We consider the frequency and amplitude of the spatially-periodic coefficients as parameters. We then take the amplitudes of the periodic coefficients to be small and perform a perturbation analysis of the $\mathcal{H}^2$-norm. Finally, we compare the results for different value of the perturbation frequency.

We show that for certain frequencies of the perturbation, that are related to the parameters of the unperturbed system, the $\mathcal{H}^2$-norm shows significant increase. We refer to this as "parametric resonance". We also give examples where the right choice of the perturbation leads to a decrease of the $\mathcal{H}^2$-norm.

Our presentation is organized as follows: in Section II we review the frequency response representation of periodic operators. Section III introduces the perturbation analysis employed to analyze the $\mathcal{H}^2$-norm of spatially-periodic systems. Sections IV and V apply the methods of Section III to various systems to increase or decrease the $\mathcal{H}^2$-norm. We concluded and suggest directions for future work in Section VI.

II. Frequency Representation of Periodic Systems

A. Periodic Operators

Let $\hat{u}(k_x)$ and $\hat{y}(k_x)$ denote the Fourier transforms[1] of two spatial functions $u(x)$ and $y(x)$ respectively. If u and y are related by a linear operator, then so are their Fourier transforms, and it is in general possible to write the relation between them as

$$\hat{y}(k_x) \ = \int_{-\infty}^{\infty} \hat{G}(k_x, \kappa) \ \hat{u}(\kappa) \ d\kappa, \tag{1}$$

were the function $\hat{G}$ (termed the *kernel function* of the operator) may contain distributions in general.

It is a standard fact that if the operator G is spatially-invariant (i.e. it commutes with all spatial shifts), then its representation in the Fourier domain is a multiplication operator [6], that is, there exists a function $\hat{g}(k_x)$ such that

$$\hat{y}(k_x) \ = \ \hat{g}(k_x) \ \hat{u}(k_x). \tag{2}$$

This means that in (1)

$$\hat{G}(k_x, \kappa) \ = \ \hat{g}(k_x) \ \delta(k_x - \kappa).$$

One way to think about this is that spatially-invariant operators have Fourier kernel functions $\hat{G}(k_x, \kappa)$ that are "diagonal", i.e. they are a function of only $k_x - \kappa$. This can be visualized as an "impulse sheet" along the diagonal $k_x = \kappa$ whose strength is given by the function $\hat{g}(k_x)$.

We now investigate the structure of the kernel function for spatially-periodic operators. Consider a spatially-periodic multiplication operator with period $X = \frac{2\pi}{\Omega}$ of the form

$$y(x) \ = \ e^{j\Omega x} \ u(x).$$

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[1]We use $k_x \in \mathbb{R}$ to denote the spatial-frequency variable, also known as the wave-number.

From the standard shift property of the Fourier transform, we have

$$\hat{y}(k_x) \;=\; \hat{u}(k_x - \Omega),$$

i.e. $\hat{y}$ is a shift of $\hat{u}$. Such shifts are represented in (1) by kernel functions of the form

$$\hat{G}(k_x, \kappa) \;=\; \delta(k_x - \kappa - \Omega).$$

This can be visualized as an impulse sheet of constant strength along the subdiagonal $k_x - \kappa = \Omega$.

Now consider multiplication by a general periodic function Γ of period $X = \frac{2\pi}{\Omega}$. Let γ_l be the Fourier series coefficients of Γ, i.e.

$$\Gamma(x) \;=\; \sum_{l=-\infty}^{\infty} \gamma_l \; e^{jl\Omega x}.$$

Using the above, the shift property of the Fourier transform repeatedly, and the linearity of multiplication operators, we conclude that

$$y(x) = \Gamma(x)\; u(x) \quad \Longleftrightarrow \quad \hat{y}(k_x) = \sum_{l=-\infty}^{\infty} \gamma_l \; \hat{u}(k_x - l\Omega),$$

i.e. $\hat{y}$ is the sum of weighted shifts of $\hat{u}$ by integer multiples of Ω. Thus, the kernel function of a periodic pure multiplication operator is of the form

$$\hat{G}(k_x, \kappa) \;=\; \sum_{l=-\infty}^{\infty} \gamma_l \; \delta(k_x - \kappa - l\Omega).$$

This can be visualized as an array of diagonal impulse sheets at $k_x - \kappa \;=\; l\Omega$ with relative strength given by γ_l, the l'th Fourier series coefficient of the function $\Gamma(x)$.

Let us now look at the structure of a general periodic operator. First, the cascade of a pure multiplication by $e^{jl\Omega}$ followed by a spatially-invariant operator with Fourier symbol $\hat{g}(k_x)$ has a kernel function given by

$$\hat{g}(k_x)\; \delta(k_x - \kappa - l\Omega).$$

It is easy to see that sums and cascades of such basic periodic operators produce an operator with a kernel function of the form

$$\hat{G}(k_x, \kappa) \;=\; \sum_{l=-\infty}^{\infty} \hat{g}_l(k_x)\; \delta(k_x - \kappa - l\Omega). \qquad (3)$$

Such a kernel function can be visualized as in Figure 1.

In this paper, we consider spatially-periodic operators with kernel functions of the form (3). These operators are completely specified by the sequence of functions $\{\hat{g}_l(k_x)\}$. It is interesting to observe certain special subclasses.

1) A *spatially-invariant* operator has a kernel function of the form (3) in which $\hat{g}_l \;=\; 0$ if $l \neq 0$ (i.e. it is purely "diagonal").
2) A *periodic pure multiplication* operator has a kernel function of the form (3) in which all the functions $\hat{g}_l$ are constant in their arguments (i.e. it is a "Toeplitz" operator).

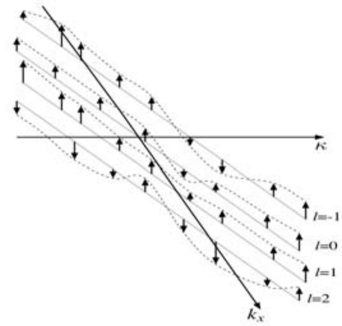


Fig. 1. The kernel function of a general spatially-periodic operator

Let us return to the kernel representation (3). Now, to find $\hat{y}(k_x)$ for a given k_x, one can imagine the action of (3) on $\hat{u}(\kappa)$ as that depicted in Figure 2. Let us write $k_x \in \mathbb{R}$ as

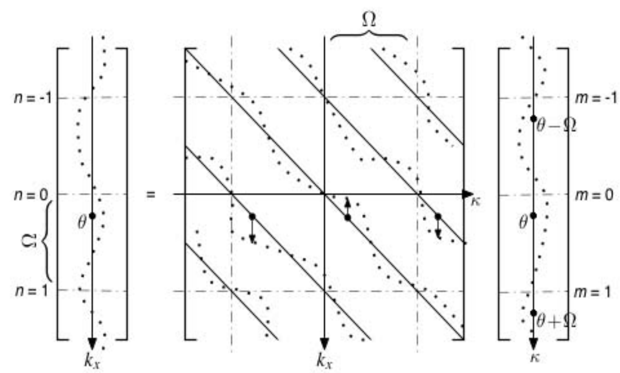


Fig. 2. The action of the kernel function of a periodic operator

$n\Omega + \theta$ for some $n \in \mathbb{Z}$ and $\theta \in [0, \Omega)$. From (1) and (3)

$$\begin{aligned}
\hat{y}(\theta + n\Omega) \;&=\; \int_{-\infty}^{\infty} \hat{G}(\theta + n\Omega, \kappa)\; \hat{u}(\kappa)\; d\kappa, \\
&=\; \sum_{l=-\infty}^{\infty} \hat{g}_l(\theta + n\Omega)\; \hat{u}(\theta + n\Omega - l\Omega) \\
&=\; \sum_{m=-\infty}^{\infty} \hat{g}_{n-m}(\theta + n\Omega)\; \hat{u}(\theta + m\Omega).
\end{aligned}$$

If we define the bi-infinite column vectors $u_\theta := \mathrm{col}\{\cdots, \hat{u}(\theta - \Omega), \hat{u}(\theta), \hat{u}(\theta + \Omega), \cdots\}$, and $y_\theta := \mathrm{col}\{\cdots, \hat{y}(\theta - \Omega), \hat{y}(\theta), \hat{y}(\theta + \Omega), \cdots\}$, then the above equality can be written in matrix form

$$y_\theta = \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & \hat{g}_0(\theta-\Omega) & \hat{g}_{-1}(\theta-\Omega) & \hat{g}_{-2}(\theta-\Omega) & \\ \ddots & \hat{g}_1(\theta) & \hat{g}_0(\theta) & \hat{g}_{-1}(\theta) & \ddots \\ & \hat{g}_2(\theta+\Omega) & \hat{g}_1(\theta+\Omega) & \hat{g}_0(\theta+\Omega) & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix} u_\theta,$$

which we henceforth denote

$$y_\theta \;=\; \mathcal{G}_\theta \; u_\theta.$$

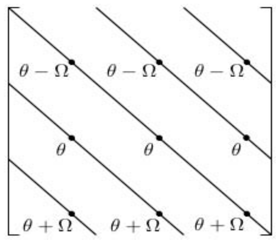


Fig. 3. Interpretation of $\mathcal{G}_\theta$ as "samples" of $\hat{G}$.

As θ varies in $[0, \Omega)$, $\mathcal{G}_\theta$ fully describes the kernel $\hat{G}$.

Remark 1: Another way to interpret the bi-infinite matrix representation introduced above is to think of $\mathcal{G}_\theta$, for every given θ, as a sample of the values of $\hat{G}$ at an array of equally spaced points as shown in Figure 3. As θ changes in $[0, \Omega)$, this "sampling grid" slides diagonally on $\hat{G}$.

In this setting, the special operators discussed before have particularly simple forms.

1) A *spatially-invariant* operator has the diagonal representation

$$\mathcal{G}_\theta = \begin{bmatrix} \ddots & & \\ & \hat{g}(\theta + n\Omega) & \\ & & \ddots \end{bmatrix},$$

2) A *periodic pure multiplication* operator has the (θ-independent) Toeplitz representation

$$\mathcal{G}_\theta = \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & \gamma_0 & \gamma_{-1} & \gamma_{-2} & \\ \ddots & \gamma_1 & \gamma_0 & \gamma_{-1} & \ddots \\ & \gamma_2 & \gamma_1 & \gamma_0 & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix} =: \Gamma.$$

Before ending this section, we should mention that u_θ can be seen as a lifted (in frequency) version of $\hat{u}(k_x)$ [7]. Hence one can define a unitary operator $\mathcal{M}$ such that $u_\theta = \mathcal{M}\hat{u}$, $y_\theta = \mathcal{M}\hat{y}$, and thus $\mathcal{G}_\theta = \mathcal{M}\hat{G}\mathcal{M}^*$. It is easy to show that if $\hat{u} \in L^2(-\infty, \infty)$, then for any given $\theta \in [0, \Omega)$, $u_\theta \in \ell^2$. Clearly $\mathcal{M}$ preserves norms and

$$\begin{aligned} \int_{-\infty}^{\infty} u^*(x)\, u(x)\, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}^*(k_x)\, \hat{u}(k_x)\, dk_x \\ &= \frac{1}{2\pi} \int_0^{\Omega} u_\theta^*\, u_\theta\, d\theta = \frac{1}{2\pi} \int_0^{\Omega} \operatorname{trace}\Big(u_\theta\, u_\theta^*\Big)\, d\theta. \end{aligned}$$

III. Perturbation Analysis of the $\mathcal{H}^2$-Norm

Let us now consider a system of the form

$$\begin{aligned} \partial_t \psi(x,t) &= A_p\, \psi(x,t) \;+\; B\, v(x,t) \\ &= \big(A_0 + B\, \epsilon \Gamma(x)\, C\big)\, \psi(x,t) \;+\; B\, v(x,t), \\ y(t,x) &= C\, \psi(t,x), \end{aligned}$$

where $x \in \mathbb{R}$, and $\psi(x,t)$, for any given (x,t), is a vector in $\mathbb{C}^n$, u is a spatio-temporal input, and y the spatio-temporal output. A_0, B, and C are spatially-invariant operators and $\Gamma(x)$ is a spatially-periodic multiplication operator, all defined on a dense domain $\mathcal{D}(A_p) \subset L^2(-\infty, \infty)$. $\Gamma(x)$ has period $X = \frac{2\pi}{\Omega}$ and zero mean, and ϵ is a *small* real scalar.

We assume that A_p defines an exponentially stable C_0-semigroup on $L^2(-\infty, \infty)$ [8]. Finally, for simplicity, we assume that B and C are constant matrices, and that $\Gamma(x) = 2L\cos(\Omega x)$ for some constant matrix $L \in \mathbb{C}^{n \times n}$.

Then, as shown in the previous section, the representation of the system in Fourier domain would be the infinite-dimensional system[2]

$$\begin{aligned} \partial_t \psi_\theta(t) &= \big(\mathcal{A}_\theta^{(0)} \;+\; \epsilon\, \mathcal{B}\, \Gamma\, \mathcal{C}\big)\psi_\theta(t) \\ &= \big(\mathcal{A}_\theta^{(0)} \;+\; \epsilon \mathcal{A}^{(1)}\big)\psi_\theta(t), \end{aligned}$$

where

$$\mathcal{A}_\theta^{(0)} := \begin{bmatrix} \ddots & & \\ & A_0(\theta + n\Omega) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{A}^{(1)} := \mathcal{B}\Gamma\mathcal{C} = \begin{bmatrix} \ddots & \ddots & & \\ \ddots & 0 & A_1 & \\ & A_1 & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

and $A_1 := BLC$.

Define

$$\begin{aligned} \mathcal{A}_\theta(\epsilon) &:= \mathcal{A}_\theta^{(0)} + \epsilon \mathcal{A}^{(1)}, \\ \mathcal{P}_\theta(\epsilon) &:= \mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \cdots, \end{aligned}$$

with $\mathcal{P}_\theta^*(\epsilon) = \mathcal{P}_\theta(\epsilon)$. Notice that this implies $\mathcal{P}_\theta^{(m)*} = \mathcal{P}_\theta^{(m)}$ for all $m = 0, 1, 2, \cdots$. We want to find $\mathcal{P}_\theta^{(m)}$ by solving the Lyapunov equation

$$\mathcal{A}_\theta(\epsilon)\mathcal{P}_\theta(\epsilon) + \mathcal{P}_\theta(\epsilon)\mathcal{A}_\theta^*(\epsilon) \equiv -\mathcal{B}\mathcal{B}^*, \tag{4}$$

$$\Downarrow$$

$$\begin{aligned} &(\mathcal{A}_\theta^{(0)} + \epsilon\mathcal{A}^{(1)})(\mathcal{P}_\theta^{(0)} + \epsilon\mathcal{P}_\theta^{(1)} + \epsilon^2\mathcal{P}_\theta^{(2)} + \cdots) + \\ &(\mathcal{P}_\theta^{(0)} + \epsilon\mathcal{P}_\theta^{(1)} + \epsilon^2\mathcal{P}_\theta^{(2)} + \cdots)(\mathcal{A}_\theta^{(0)} + \epsilon\mathcal{A}^{(1)})^* \equiv -\mathcal{B}\mathcal{B}^*. \end{aligned} \tag{5}$$

Our aim is to find $\mathcal{P}_\theta(\epsilon)$ from the above identity and compute the $\mathcal{H}^2$-norm of the system using [9]

$$\|\mathcal{G}\|_2^2 := \frac{1}{2\pi} \int_0^{\Omega} \operatorname{trace}\Big(\mathcal{C}\mathcal{P}_\theta(\epsilon)\mathcal{C}^*\Big) d\theta.$$

It is easy to see from (5) that

$$\mathcal{A}_\theta^{(0)}\mathcal{P}_\theta^{(0)} + \mathcal{P}_\theta^{(0)}\mathcal{A}_\theta^{(0)*} = -\mathcal{B}\mathcal{B}^*, \tag{6}$$

$$\mathcal{A}_\theta^{(0)}\mathcal{P}_\theta^{(1)} + \mathcal{P}_\theta^{(1)}\mathcal{A}_\theta^{(0)*} = -\Big(\mathcal{A}^{(1)}\mathcal{P}_\theta^{(0)} + \mathcal{P}_\theta^{(0)}\mathcal{A}^{(1)*}\Big), \tag{7}$$

$$\mathcal{A}_\theta^{(0)}\mathcal{P}_\theta^{(2)} + \mathcal{P}_\theta^{(2)}\mathcal{A}_\theta^{(0)*} = -\Big(\mathcal{A}^{(1)}\mathcal{P}_\theta^{(1)} + \mathcal{P}_\theta^{(1)}\mathcal{A}^{(1)*}\Big), \tag{8}$$

$$\vdots$$

[2]To avoid clutter, we henceforth drop the "ˆ" on the Fourier symbol of spatially-invariant operators, and omit the θ subscript for operators that are independent of θ.

Now since $\mathcal{A}_\theta^{(0)}$ and $\mathcal{B}\mathcal{B}^*$ are block-diagonal in (6), so is $\mathcal{P}_\theta^{(0)}$. In (7), the right hand side operator has the structure of being nonzero only on the first upper and lower block-subdiagonals, and hence $\mathcal{P}_\theta^{(1)}$ inherits the same structure (since $\mathcal{A}_\theta^{(0)}$ is block-diagonal). In the same manner, one can show that $\mathcal{P}_\theta^{(2)}$ is only nonzero on the main block-diagonal and the second upper and lower block-subdiagonals, and so on for other $\mathcal{P}_\theta^{(m)}$. We have

$$\mathcal{P}_\theta^{(0)} = \begin{bmatrix} \ddots & & \\ & P_0(\theta+n\Omega) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{P}_\theta^{(1)} = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & 0 & P_1(\theta+n\Omega) & & \\ & P_1(\theta+n\Omega) & 0 & \ddots & \\ & & \ddots & \ddots & \end{bmatrix},$$

$$\mathcal{P}_\theta^{(2)} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & \ddots & 0 & \widetilde{P}_2(\theta+n\Omega) & & \\ \ddots & 0 & P_2(\theta+n\Omega) & 0 & \ddots & \\ & \widetilde{P}_2(\theta+n\Omega) & 0 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \end{bmatrix},$$
$$\vdots$$

It is important to realize that, not only is $\mathcal{P}_\theta^{(m)}$ not a "full" operator, it has at most m nonzero block-subdiagonals. Also, all $\mathcal{P}_\theta^{(m)}$ for odd m are trace-free operators.

Now returning to (6)-(8), $\mathcal{P}_\theta^{(0)}$, $\mathcal{P}_\theta^{(1)}$ and $\mathcal{P}_\theta^{(2)}$ are found by equating, element by element, the bi-infinite matrices on both sides of these equations. For example, (6) leads to

$$A_0(\theta+n\Omega)P_0(\theta+n\Omega) + P_0(\theta+n\Omega)A_0^*(\theta+n\Omega) = -BB^*$$

for every $n \in \mathbb{Z}$, and $\theta \in [0,\Omega)$. But notice that as n runs over all integers and θ changes in $[0,\Omega)$, $k_x = \theta + n\Omega$ runs over all reals, and one can rewrite the above equation as

$$A_0(k_x)P_0(k_x) + P_0(k_x)A_0^*(k_x) = -BB^*.$$

Applying the same procedure to (6)-(8), one arrives at

$$\begin{aligned} &A_0(k_x)P_0(k_x) + P_0(k_x)A_0^*(k_x) = -BB^*, \\ &A_0(k_x)P_1(k_x) + P_1(k_x)A_0^*(k_x+\Omega) = \\ &\qquad -\Big(A_1P_0(k_x+\Omega) + P_0(k_x)A_1^*\Big), \\ &A_0(k_x)P_2(k_x) + P_2(k_x)A_0^*(k_x) = \\ &\qquad -\Big(A_1P_1(k_x-\Omega) + P_1(k_x-\Omega)A_1^* + \\ &\qquad\qquad A_1P_1(k_x) + P_1(k_x)A_1^*\Big), \\ &A_0(k_x)\widetilde{P}_2(k_x) + \widetilde{P}_2(k_x)A_0^*(k_x+2\Omega) = \\ &\qquad -\Big(A_1P_1(k_x+\Omega) + P_1(k_x)A_1^*\Big), \end{aligned}$$

and so on for all nonzero diagonals of $\mathcal{P}_\theta^{(m)}$, $m = 3, 4, \cdots$. Notice that from the above equations, one first finds $P_0(\,\cdot\,)$ from the first equation, then $P_1(\,\cdot\,)$ from the second equation, and so on. This "decoupling" of the subdiagonals would not have been possible had we not employed a perturbation approach and had attempted to solve (4) directly.

Returning to the calculation of the $\mathcal{H}^2$-norm, let us first separate the block-diagonal part of $\mathcal{P}_\theta^{(2)}$ by rewriting it as $\mathcal{P}_\theta^{(2)} = \overline{\mathcal{P}}_\theta^{(2)} + \widetilde{\mathcal{P}}_\theta^{(2)}$, where

$$\overline{\mathcal{P}}_\theta^{(2)} := \begin{bmatrix} \ddots & & \\ & P_2(\theta+n\Omega) & \\ & & \ddots \end{bmatrix},$$

and $\widetilde{\mathcal{P}}_\theta^{(2)}$ contains the rest of $\mathcal{P}_\theta^{(2)}$. Also, recall that

$$\mathrm{trace}\Big(\mathcal{P}_\theta^{(2m+1)}\Big) = 0, \qquad m = 0, 1, 2, \cdots.$$

Now one can write the following

$$\begin{aligned} \|\mathcal{G}\|_2^2 &= \frac{1}{2\pi}\int_0^\Omega \mathrm{trace}\Big(\mathcal{C}\mathcal{P}_\theta(\epsilon)\mathcal{C}^*\Big)d\theta \\ &= \frac{1}{2\pi}\int_0^\Omega \mathrm{trace}\Big(\mathcal{C}\mathcal{P}_\theta^{(0)}\mathcal{C}^* + \epsilon^2\mathcal{C}\mathcal{P}_\theta^{(2)}\mathcal{C}^*\Big)d\theta + O(\epsilon^4) \\ &= \frac{1}{2\pi}\int_0^\Omega \mathrm{trace}\Big(\mathcal{C}\mathcal{P}_\theta^{(0)}\mathcal{C}^* + \epsilon^2\mathcal{C}\overline{\mathcal{P}}_\theta^{(2)}\mathcal{C}^*\Big)d\theta + O(\epsilon^4), \end{aligned}$$

where the last equation follows from the fact that $\mathrm{trace}\Big(\mathcal{C}\widetilde{\mathcal{P}}_\theta^{(2)}\mathcal{C}^*\Big) = 0$. Next, using $\overline{\mathcal{P}}_\theta^{(0)} = \mathcal{M}P_0(k_x)\mathcal{M}^*$, where $\mathcal{M}$ is the unitary (lifting) operator defined earlier, and

$$\begin{aligned} \mathcal{C}\mathcal{P}_\theta^{(0)}\mathcal{C}^* &= (\mathcal{M}C\mathcal{M}^*)(\mathcal{M}P_0(k_x)\mathcal{M}^*)(\mathcal{M}C^*\mathcal{M}^*) \\ &= \mathcal{M}CP_0(k_x)C^*\mathcal{M}^*, \end{aligned}$$

we have

$$\int_0^\Omega \mathrm{trace}\Big(\mathcal{M}CP_0(k_x)C^*\mathcal{M}^*\Big)d\theta = \int_{-\infty}^\infty \mathrm{trace}\Big(CP_0(k_x)C^*\Big)dk_x,$$

with the same procedure applied to $\mathrm{trace}\Big(\mathcal{C}\mathcal{P}_\theta^{(2)}\mathcal{C}^*\Big)$. Thus

$$\begin{aligned} \|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{trace}\Big(&CP_0(k_x)C^* + \\ &\epsilon^2 CP_2(k_x)C^*\Big)dk_x + O(\epsilon^4). \end{aligned}$$

IV. An Example of "Parametric Resonance"

As an application of the preceding discussion, we investigate the occurrence of parametric resonance for a class of spatially-periodic systems. Let us take

$$A_0(k_x) = a_0(k_x) = -(k_x^2 - k_0^2)^2 - c,$$

for some $0 \neq k_0 \in \mathbb{R}$, $c > 0$, and assume that $B = C = 1$, $L = 1$ ($\Gamma(x) = 2\cos(\Omega x)$), and hence $A_1 = 1$.

For scalar systems, the functions $P_0(k_x) = p_0(k_x)$ and $P_2(k_x) = p_2(k_x)$ in the previous section simplify to[3]

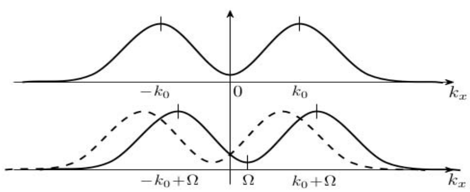
$$\begin{aligned} p_0(k_x) &= \frac{-1}{2a_0(k_x)}, \\ p_2(k_x) &= \frac{1}{\big(a_0(k_x)\big)^2}\left(\frac{-1}{2a_0(k_x-\Omega)} + \frac{-1}{2a_0(k_x+\Omega)}\right) \qquad (9) \\ &= 4\big(p_0(k_x)\big)^2\big(p_0(k_x-\Omega) + p_0(k_x+\Omega)\big), \end{aligned}$$

[3]To find $p_2(k_x)$ here, one needs to first find $p_1(k_x)$, but we have omitted the details for brevity.

and it is our aim to find the $\mathcal{H}^2$-norm

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\big(p_0(k_x) + \epsilon^2 p_2(k_x)\big)dk_x + O(\epsilon^4). \quad (10)$$

From (9), $p_0(k_x) = \frac{1}{2}\frac{1}{(k_x^2-k_0^2)^2+c}$. The first plot below shows $p_0(k_x)$, while the second shows $p_0(k_x - \Omega)$ and $p_0(k_x + \Omega)$ (dashed), for a given value of $\Omega \neq 0$. As Ω increases $p_0(k_x - \Omega)$ slides to the right and $p_0(k_x + \Omega)$ to the left. From (9) it is clear that to find $p_2(\cdot)$ for any



Ω, one would sum the two functions in the second plot and multiply the result by the square of the first plot. The interesting question now is, for what value(s) of $\Omega \in (0,\infty)$ would the $\mathcal{H}^2$-norm in (10) be maximized.

One can easily see that as $\Omega \to 0$, the peaks of $p_0(\,\cdot - \Omega)$ and $p_0(\,\cdot + \Omega)$ merge toward that of $\big(p_0(\,\cdot\,)\big)^2$, namely to the peaks at $k_x = \pm k_0$, thus $\int_{-\infty}^{\infty} p_2(k_x)dk_x$ grows, and hence $\|G\|_2$ grows. (Remember that $p_0(k_x)$ is independent of Ω, and thus $\int_{-\infty}^{\infty} p_0(k_x)dk_x$ is constant.) This is intuitively clear; as $\Omega \to 0$, the perturbation is tending toward a constant function, $\Gamma(x) = 2\cos(\Omega x) \to 2$, which shifts the spectrum of $A_0 = a_0$ toward the RHP and hence increases the $\mathcal{H}^2$-norm.

But we are more interested in nonzero frequencies Ω that exhibit a local increase in the $\mathcal{H}^2$-norm. Now notice that a different alignment of the peaks can also occur which leads to another local maximum of the $\mathcal{H}^2$-norm as a function of Ω. More specifically, this happens when the peak of $p_0(\,\cdot - \Omega)$ at $k_x = -k_0 + \Omega$ becomes aligned with the peak of $\big(p_0(\,\cdot\,)\big)^2$ at $k_x = k_0$, and, simultaneously, the peak of $p_0(\,\cdot + \Omega)$ at $k_x = k_0 - \Omega$ becomes aligned with the peak of $\big(p_0(\,\cdot\,)\big)^2$ at $k_x = -k_0$. Clearly this occurs when

$$-k_0 + \Omega_{res} = k_0 \quad \Longrightarrow \quad \Omega_{res} = 2k_0.$$

This agrees exactly with the result obtained in [10], where parametric resonance was shown to occur in the perturbation of the spectrum, for periodic perturbations whose frequency satisfies the relation $\Omega = 2k_0$.

V. Reduction of $\mathcal{H}^2$-Norm

In this section we continue with some examples that demonstrate the affect of spatially-periodic perturbations on the $\mathcal{H}^2$-norm of spatially distributed systems, and show that by appropriately choosing the frequency of the perturbation one can either induce parametric resonance or reduce the $\mathcal{H}^2$-norm of the system.

Example 1: Let us now perform a numerical analysis of the scalar system of the previous section. Take $A_0(k_x) = a_0(k_x) = -(k_x^2-1)^2 - 0.1$, $B = C = 1$. Clearly $A_1 = L$, and we allow L to be either a purely real or a purely imaginary scalar.[4]

The plots in Figure 4 show numerical calculations performed in MATLAB. Notice that for $A_1 \in \mathbb{R}$ the results are in complete agreement with those shown for the system in the previous section, namely that the $\mathcal{H}^2$-norm has local increases at $\Omega \to 0$ and the resonant frequency $\Omega_{res} = 2k_0 = 2$.

It is also seen that for $A_1 \in j\mathbb{R}$, one can actually *reduce* the $\mathcal{H}^2$-norm at certain frequencies of the perturbation. Interestingly, this reduction becomes negligible at the resonant frequency $\Omega_{res} = 2k_0 = 2$.

Remark 2: Again this is in agreement with the results previously reported in [10], where it is illustrated that purely-imaginary (in general *skew-symmetric*) periodic feedback acts as a form of damping and pushes the system modes farther into the LHP, whereas purely real (in general *self-adjoint*) periodic feedback will move the closed-loop modes toward the RHP.

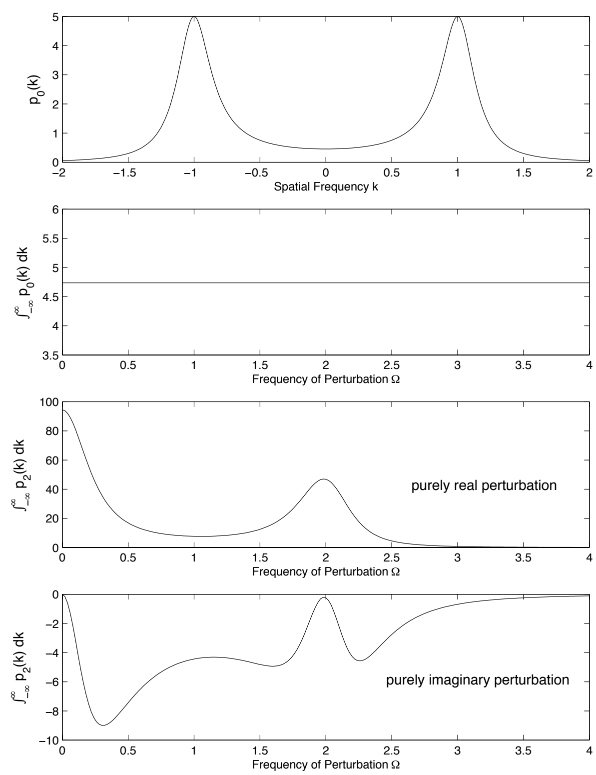


Fig. 4. Graphs of Example 1

Example 2: The following system is motivated by channel flow problems. Take $B = C = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $L = \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$,

$$A_0(k_x) = \begin{bmatrix} -\frac{1}{R}(k_x^2 + c) & 0 \\ jk_x & -\frac{1}{R}(k_x^2+c) \end{bmatrix}.$$

[4]The physical interpretation of such an imaginary perturbation will be discussed in the next section.

The graphs in Figure 5 show simulation results for $R = 6$, $c = 1$. Clearly, the $\mathcal{H}^2$-norm can be decreased by the the application of periodic perturbations with frequency around $\Omega \approx 0.7$ and also for high frequencies $\Omega \gtrsim 3.5$.

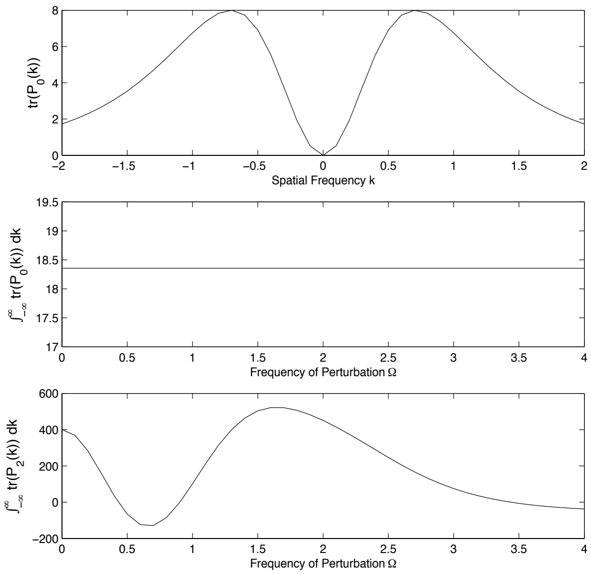


Fig. 5. Graphs of Example 2

A. Interpretation of Imaginary States

It was shown above that one can decrease the $\mathcal{H}^2$-norm of certain systems by choosing the perturbation amplitude A_1 to be purely imaginary (or, in general, skew-symmetric). This would yield a perturbed system that can in general have states with nonzero imaginary parts. One could then ask the physical interpretation of such a system.

For any operator A_p and function ψ one can write

$$\begin{aligned} A_p &= A_r + jA_i, \qquad A_r, A_i \in \mathbb{R}^{n\times n}, \\ \psi &= \psi_r + j\psi_i, \qquad \psi_r, \psi_i \in \mathbb{R}^n. \end{aligned}$$

Then the system equations can be written as

$$\partial_t(\psi_r + j\psi_i) = (A_r + jA_i)(\psi_r + j\psi_i)$$
$$\Updownarrow$$
$$\left\{ \begin{array}{l} \partial_t \psi_r = A_r\psi_r - A_i\psi_i \\ \partial_t \psi_i = A_r\psi_i + A_i\psi_r \end{array} \right.$$
$$\Updownarrow$$
$$\partial_t \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} = \begin{bmatrix} A_r & -A_i \\ A_i & A_r \end{bmatrix} \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix}.$$

Clearly the state dimension is twice that of the original system with imaginary coefficients, but now $\left[\begin{smallmatrix} \psi_r \\ \psi_i \end{smallmatrix}\right] \in \mathbb{R}^{2n}$.

Let us give a simple example. Assume the heat equation, with $A_1 = j\epsilon$, $\epsilon \in \mathbb{R}$, i.e. $A_p = \partial_x^2 - c + j\epsilon\cos(\Omega x)$. Then $A_r = A_0 = \partial_x^2 - c$, $A_i = A_1 = j\epsilon\cos(\Omega x)$, and thus

$$\begin{aligned} \partial_t \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} &= \begin{bmatrix} \partial_x^2 - c & -\epsilon\cos(\Omega x) \\ \epsilon\cos(\Omega x) & \partial_x^2 - c \end{bmatrix} \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} \\ &= \left(\begin{bmatrix} \partial_x^2 - c & 0 \\ 0 & \partial_x^2 - c \end{bmatrix} + \epsilon\cos(\Omega x) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} \end{aligned}$$

which describes two identical systems coupled through the periodic perturbation.

VI. Conclusions and Future Work

We use perturbation analysis to find a computationally efficient way of calculating the $\mathcal{H}^2$-norm of spatially-periodic systems. We show that for certain classes of systems, the periodicity can be chosen so as to increase the $\mathcal{H}^2$-norm and induce parametric resonance. An application of this would be in mixing problems. It is also shown that the $\mathcal{H}^2$-norm can be made to decrease for an appropriate choice of the frequency of the perturbation.

Our approach can also be used in systems with many spatial directions. For example, consider the PDE

$$\psi_t = \psi_{yy} + \psi_{xx} + c\psi + \epsilon\cos(\Omega x)\psi$$

with $y \in [-1, 1]$ and $x \in R$. To put this system into the developed framework one would only have to perform a discrete approximation of the operator ∂_y^2.

Future research in this direction would include an exact characterization of the frequencies for which the $\mathcal{H}^2$-norm is most reduced. Also, one could consider more general periodic coefficients, namely ones with higher order perturbation terms and multiple harmonics of the basic frequency. The perturbation methods presented here could also be generalized to bi-infinite Sylvester equations, which arise in many fluids problems.

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