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Natural Observers for Singularly Perturbed Mechanical Systems

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Index Terms— **Vector second order systems, natural observer, sensor dynamics, singular perturbations.**

Abstract— **In this note, we consider vector second order mechanical systems having fast sensor dynamics, and the objective is to obtain accurate estimates of the unmeasurable slow system state variables that are generated by an appropriately designed model-based observer. Using a singular perturbation framework, the proposed observer is designed using the system dynamics that evolves on the slow manifold, and the dynamic behavior of the resulting estimation error is analyzed in the presence of the unmodelled fast sensor dynamics. We show that the observation error generated by neglecting the (unmodeled) fast sensor dynamics is of order $O(\varepsilon)$, where ε is the singular perturbation parameter and a measure of the relative speed/time-constant of the fast (sensor) and the slow component (vector second-order system) of the overall system dynamics. The performance of the proposed natural observer is evaluated in an example of a two-degree of freedom mechanical system.**

I. Introduction

The customary approach to address a state estimation problem for a vector second-order system is to first place it in an equivalent first-order form in state space, and then design a standard Luenburger observer or a Kalman/Wiener filter [3]. Besides the obvious adverse consequence of the above first-order formulation that inevitably leads to an increased computational load associated with the calculation of the observer gains [3], one of the main disadvantages of the above approach is that any state observer design that is implemented yields an estimated state that might not carry the requisite information about the physical state of the original system; for example, the components of the estimated state might not be the estimates of the components of the true physical state [3]. This was convincingly demonstrated in [1], wherein it was shown that for a mechanical system, the derivative of the position estimate was not equal to the velocity estimate. Notice that in the aforementioned approaches the sensor dynamics is not integrated into their respective observer design framework, and thus its impact on the viability and performance of the proposed observer remains unaddressed [8]. However, one could in principle consider the overall instrumented system dynamics that is comprised of the original vector second order system along with the fast sensor dynamics, and realize the design of the natural observer through the restriction of the system dynamics on the slow manifold, followed by a rigorous analysis on the impact of the ignored/unmodeled fast sensor dynamics (otherwise known as "parasitics") on the convergence properties of the proposed natural observer [5], [6], [8]. Please notice that the aforementioned problem has been studied for first-order systems [5], [8], however, to the best of our knowledge, it has not been formulated and addressed as a natural observer design problem for vector second order systems in the presence of unmodelled fast sensor dynamics. It should be emphasized, that in light of the above considerations, the need for the development of a *natural* observer design method for vector second order systems which can effectively address both the computational requirements and the physical interpretation of the estimator states is warranted.

The contribution of the present research work is twofold: first, a natural observer for a class of vector second order systems in the absence of sensor dynamics is introduced. It is shown that the proposed *natural* observer identifies the states of the plant by ensuring that the derivative of the position estimate *is* the estimate of the velocity, a case not always true when a system is placed in first order form. Secondly, the proposed natural observer is extended to a class of second order systems with fast unmodelled sensor dynamics. A natural observer is designed on the basis of the system dynamics evolving on the slow manifold, and the effect of the fast component (sensor dynamics) of the system dynamics on the estimation error dynamics is analyzed. In particular, it is shown, that in the proposed method the observer error generated by neglecting the fast sensor dynamics is of order $O(\varepsilon)$, where ε is the singular perturbation parameter and a measure of the relative speed/time-constant of the fast and the slow component of the overall system dynamics.

II. Problem Formulation

We consider a vector second-order dynamical system accompanied by sensor dynamics and represented in a standard singular perturbation state-space form

$$\begin{aligned} &M\ddot{\zeta}(t)+D\dot{\zeta}(t)+K\zeta(t)+N_1w_1(t)+N_2w_2(t)=Bu(t),\\ &\varepsilon_1\dot{w}_1(t)=M_1\zeta(t)+M_2w_1(t),\\ &\varepsilon_2\dot{w}_2(t)=M_3\dot{\zeta}(t)+M_4w_2(t),\\ &y_p(t)=C_pw_1(t),\\ &y_v(t)=C_vw_2(t), \end{aligned} \tag{1}$$

where $\zeta\in\mathbb{R}^n$ is the vector of spatial (position) coordinates, $\dot{\zeta}\in\mathbb{R}^n$ represents the associated velocity vector, $w_1\in\mathbb{R}^m$ is the vector of states associated with the sensor w_1-dynamics for the position measurements, $w_2\in\mathbb{R}^p$ is the vector of

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states associated with the sensor w_2-dynamics for the velocity measurements, $\varepsilon_1, \varepsilon_2$ are the perturbation parameters that represent a measure of the relative speed/time-constant of the sensor dynamics (the fast component of the overall system dynamics (1)) and the slow system dynamics (the slow component of the overall instrumented system dynamics (1)), and $y_p \in \mathbb{R}^{q_1}, y_v \in \mathbb{R}^{q_2}$ are the available sensor measurements for the position and velocity respectively; *i.e.* y_p, y_v are the output signals of the sensing devices. In the above description of the system dynamics, the $n \times n$ matrices $M = M^T > 0,\ D = D^T \geq 0,\ K = K^T > 0$ are the mass, damping and stiffness matrices respectively. The distribution matrix for the input signal is denoted by B and the r-dimensional input signal is given by u. Furthermore, $M_1, M_2, M_3, M_4, N_1, N_2, C_p, C_v$ are constant matrices of appropriate dimensions with M_2, M_4 being non-singular $m \times m$ and $p \times p$ matrices. Without loss of generality, it is assumed that the origin $(\zeta, \dot{\zeta}, w_1, w_2) = (0,0,0,0)$ is an equilibrium point of (1).

When one sets the singular perturbation parameters equal to zero: $\varepsilon_1 = \varepsilon_2 = 0$, one then obtains the reduced-order system dynamics $(\overline{\zeta}, \overline{\zeta})$ on the slow manifold [6]

$$w_1 = \left(-M_2^{-1} M_1\right) \overline{\zeta}, \quad w_2 = \left(-M_4^{-1} M_3\right) \dot{\overline{\zeta}}, \tag{2}$$

given by

$$\begin{aligned} M\ddot{\overline{\zeta}}(t) + D_0\dot{\overline{\zeta}}(t) + K_0\overline{\zeta}(t) &= Bu, \\ \overline{y}_p(t) &= C_{p,0}\overline{\zeta}(t), \\ \overline{y}_v(t) &= C_{v,0}\dot{\overline{\zeta}}(t), \end{aligned} \tag{3}$$

where

$$\begin{aligned} D_0 &= D - N_2 M_4^{-1} M_3, \qquad C_{p,0} = C_p\Big(-M_2^{-1} M_1\Big), \\ K_0 &= K - N_1 M_2^{-1} M_1, \qquad C_{v,0} = C_v\Big(-M_4^{-1} M_3\Big). \end{aligned} \tag{4}$$

Using only the above reduced-order system (3) that represents the restriction of the original system dynamics on the slow manifold (2), an appropriate natural observer could, in principle, be designed, and thus lead to estimates $(\dot{\widehat{\zeta}}, \widehat{\zeta})$ of both the full velocity $\dot{\zeta}$ as well as the position ζ vectors. Please notice, that ignoring the fast sensor dynamics will induce an observer error, and therefore, a rigorous analysis of its effect on the convergence properties of the proposed *natural observer* should be carefully performed. The latter represents the main objective of the present study.

Applying the natural observer design method presented in [2], [3] to the reduced-order system (3) we have

$$\begin{aligned} M\ddot{\widehat{\zeta}}(t) + D_0\dot{\widehat{\zeta}}(t) + K_0\widehat{\zeta}(t) &= Bu(t) \\ &+ L_p(y_p(t) - \widehat{y}_p(t)) + L_v(y_v(t) - \widehat{y}_v(t)) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \widehat{y}_p(t) &= C_p\Big(-M_2^{-1} M_1\Big)\widehat{\zeta}(t) = C_{p,0}\widehat{\zeta}(t), \\ \widehat{y}_v(t) &= C_v\Big(-M_4^{-1} M_3\Big)\dot{\widehat{\zeta}}(t) = C_{v,0}\dot{\widehat{\zeta}}(t), \end{aligned} \tag{6}$$

and L_p, L_v are constant gains that assign the desirable eigenspectrum to the matrices $\widetilde{K} \triangleq K_0 + L_p C_{p,0}$ and $\widetilde{D} \triangleq D_0 + L_v C_{v,0}$ respectively, under the proper detectability conditions imposed on the matrix pairs $\{C_{p,0}, K_0\}$, $\{C_{v,0}, D_0\}$, see [9].

Remark 2.1: The pairs $\{C_{p,0}, K_0\}$ and $\{C_{v,0}, D_0\}$ are detectable, in the sense that one can assign the spectrum of both $\widetilde{K} \triangleq K_0 + L_p C_{p,0}$ and $\widetilde{D} \triangleq D_0 + L_v C_{v,0}$ such that the quadratic pencil (quadratic matrix polynomial [7])

$$\mathcal{P}_o(\lambda) = \lambda^2 M + \lambda\,(D_0 + L_v C_{v,0}) + (K_0 + L_p C_{p,0})$$

has the desired eigenstructure.

One can easily show that in the absence of sensor dynamics (*i.e.* $\varepsilon_1 = \varepsilon_2 = 0$), the natural observer (5) induces the following linear error dynamics for the reduced-order system (3):

$$M\ddot{e}_s(t) + \widetilde{D}\dot{e}_s(t) + \widetilde{K}e_s(t) = 0, \tag{7}$$

where $e_s(t) \triangleq \overline{\zeta}(t) - \widehat{\zeta}(t)$, and the dynamic modes are assignable through the L_p, L_v gains.

Lemma 2.1: Consider the vector second order system on the slow manifold (3). The proposed natural observer (5) guarantees the convergence of the state position and state velocity errors to zero

$$\lim_{t\to\infty} \|e_s(t)\| = 0, \text{ and } \lim_{t\to\infty} \|\dot{e}_s(t)\| = 0.$$

The convergence is exponential $(\|e_s(t)\|^2 + \|\dot{e}_s(t)\|^2) \leq e^{-\alpha(t-t_0)}(\|e_s(t_0\|^2 + \|\dot{e}_s(t_0\|^2)$, with the rate α depending on the matrices $M, \widetilde{K}, K_1, \widetilde{D}, D_1$ and the parameter γ. Furthermore, it ensures that the derivative of the estimated position vector is the estimate of the actual velocity vector.

Proof of Lemma 2.1: The stability proof utilizes the *parameter dependent* Lyapunov function [3]

$$V = \frac{\gamma}{2}\dot{e}_s^T M\dot{e}_s + \frac{\gamma}{2}e_s^T\widetilde{K}e_s + e_s^T M\dot{e}_s + \frac{1}{2}e_s^T\widetilde{D}e_s$$

where the gain γ is chosen as

$$\gamma \geq \max\left\{\frac{\lambda_{max}(M)}{\lambda_{min}(M)}, \frac{\lambda_{max}(M) - \lambda_{min}(\widetilde{D})}{\lambda_{min}(\widetilde{K})}, \frac{\lambda_{max}(M) - \lambda_{min}(\widetilde{K})}{\lambda_{min}(\widetilde{D})}\right\}. \tag{8}$$

The above guarantees that

$$\alpha_1\left(\|e_s\|^2 + \|\dot{e}_s\|^2\right) \leq V \leq \alpha_2\left(\|e_s\|^2 + \|\dot{e}_s\|^2\right).$$

where α_1, α_2 depend on $\gamma, \widetilde{K}, M, \widetilde{D}$. When the derivative of the Lyapunov function is evaluated along the trajectories of the estimation error on the slow manifold, it results in

$$\begin{aligned} \dot{V} &= -\gamma\dot{e}_s^T\widetilde{D}\dot{e}_s - e_s^T\widetilde{K}e_s + \dot{e}_s^T M\dot{e}_s \\ &\leq -\alpha_3\left(\|e_s\|^2 + \|\dot{e}_s\|^2\right) \end{aligned}$$

for some $\alpha_3 > 0$. Since $\dot{V} \leq -\alpha_4 V$, $\alpha_4 > 0$, one obtains the desired convergence. Following [3], one has $\frac{d}{d}(\widehat{\zeta}) = \dot{\widehat{\zeta}}$ for all $t \geq t_0$, a case not always true when the system (1) is placed in a first order form ($2n$-dimensional system), and then design a state observer. □

However, in the presence of the sensor dynamics, the behavior of the estimation error dynamics is no longer the one described by (7), and an observer error will be inevitably encountered. In this case, the estimation error ($e(t) \triangleq \zeta(t) - \widehat{\zeta}(t)$) dynamics becomes

$$\begin{aligned} M\ddot{e} &= -\left(\widetilde{D} + (L_v C_v + N_2) M_4^{-1} M_3\right)\dot{\zeta} + \widetilde{D}\dot{\widehat{\zeta}} \\ &\quad - \left(\widetilde{K} + (L_p C_p + N_1) M_2^{-1} M_1\right)\zeta + \widetilde{K}\widehat{\zeta} \\ &\quad - (L_p C_p + N_1) w_1 - (L_v C_v + N_2) w_2. \end{aligned}$$

This implies that

$$\begin{aligned} M\ddot{e} + \widetilde{D}\dot{e} + \widetilde{K}e = &-(L_v C_v + N_2)(M_4^{-1} M_3 \dot{\zeta} + w_2) \\ &-(L_p C_p + N_1)(M_2^{-1} M_1 \zeta + w_1). \end{aligned} \tag{9}$$

As expected, by considering the above expression for the estimation error dynamics (9), it can be concluded that, even in the absence of an observer initialization error, there is an inevitable observer error due to the presence of the ignored/unmodeled sensor dynamics in the design of the proposed natural observer (5). Within the framework of singular perturbation theory however, it will be shown that the natural observer error induced is of order $O(\varepsilon_1), O(\varepsilon_2)$.

Remark 2.2: When the natural observer is not implemented, one will arrive at a situation in which the derivative of the first component of the estimated state vector *is not* equal to the second component of the estimated vector. This is further complicated by the presence of sensor dynamics. To demonstrate this, consider (1) in a first order form

$$\frac{d}{dt}\begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}\begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -M^{-1}N_1 & -M^{-1}N_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix} u$$

$$\begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & M_3 \end{bmatrix}\begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} M_2 & 0 \\ 0 & M_4 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} y_p \\ y_v \end{bmatrix} = \begin{bmatrix} C_p & 0 \\ 0 & C_v \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

A standard first order Luenberger observer for the above plant, with observer states (z_1, z_2) takes the form

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K_0 & -M^{-1}D_0 \end{bmatrix}\begin{bmatrix} z_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix} u + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} C_p w_1 - C_{p,0} z_1 \\ C_v w_2 - C_{v,0} z_2 \end{bmatrix}$$

$$\widehat{y}_1 = C_{p,0} z_1,$$

$$\widehat{y}_2 = C_{v,0} z_2.$$

Notice that the first component is given by

$$\begin{aligned} \frac{d}{dt} z_1 &= z_2 + L_{11}(C_p w_1 - C_{p,0} z_1) + L_{12}(C_v w_2 - C_{v,0} z_2) \\ &= z_2 + L_{11} C_p \left(w_1 + M_2^{-1} M_1 z_1\right) \\ &\quad + L_{12} C_v \left(w_2 + M_4^{-1} M_3 z_2\right). \end{aligned}$$

It is easily seen that the derivative of the first component z_1 of the estimated state vector (z_1, z_2) is not equal to its second component z_2. In the subsequent analysis, one may show that *eventually* the derivative of z_1 will converge to z_2, but that of course provides an "artificial" relation between estimated position and velocity vectors.

III. MAIN RESULTS

It is quite standard within a singular perturbation context and sensor modelling framework to assume that matrices M_2, M_4 appearing in the sensor dynamic equations are Hurwitz [5], [8]. An immediate consequence is that the mismatch: $||w_1(t) - \overline{w}_1(t)||$, $||w_2(t) - \overline{w_2}(t)||$ between the sensors' true states w_1, w_2 and the states $\overline{w}_1, \overline{w}_2$ associated with the reduced-order dynamics (2) (the quasi-steady-state approximants): $\overline{w}_1(t) = -M_2^{-1} M_1 \overline{\zeta}(t), \overline{w}_2(t) = -M_4^{-1} M_3 \dot{\overline{\zeta}}(t)$, asymptotically decay with time-constants of the order of $O(\varepsilon_1), O(\varepsilon_2)$ respectively [5], [6]. The following technical lemma is necessary:

Lemma 3.1: [5], [6] If matrices M_2, M_4 are Hurwitz, then

$$\begin{aligned} w_1(t) = &\exp\left(\tfrac{M_2(t-t_0)}{\varepsilon_1}\right)\left(w_1(t_0) + M_2^{-1} M_1 \zeta(t_0)\right) \\ &- M_2^{-1} M_1 \zeta(t) + O(\varepsilon_1) \end{aligned} \tag{10}$$

and

$$\begin{aligned} w_2(t) = &\exp\left(\tfrac{M_4(t-t_0)}{\varepsilon_2}\right)\left(w_2(t_0) + M_4^{-1} M_3 \dot{\zeta}(t_0)\right) \\ &- M_4^{-1} M_3 \dot{\zeta}(t) + O(\varepsilon_2) \end{aligned} \tag{11}$$

where t_0 is the initial time instant.

We now present the main result.

Theorem 3.1: For the original system (1) and in the presence of unmodelled sensor dynamics, the estimation error $e(t) \triangleq \zeta(t) - \widehat{\zeta}(t)$ that is induced by the natural observer (5) satisfies the following inequality:

$$||e(t)|| \leq k e^{-\lambda(t-t_0)} + E(t, \varepsilon_1, \varepsilon_2) \tag{12}$$

where k, λ are positive constants and the observation error term $E(t, \varepsilon_1, \varepsilon_2)$ is of order $O(\varepsilon_1), O(\varepsilon_2)$ with respect to the singular perturbation parameters $\varepsilon_1, \varepsilon_2$.

Proof of Theorem 3.1: Using the results of Lemma 3.1 (*i.e.* equations (10) and (11)), equation (9) becomes:

$$\begin{aligned} M\ddot{e}(t) + \widetilde{D}\dot{e}(t) + \widetilde{K}e(t) = &-(L_p C_p + N_1)\times \\ &\left[\exp\left(\frac{M_2(t-t_0)}{\varepsilon_1}\right)\left(w_1(t_0) + M_2^{-1} M_1 \zeta(t_0)\right) + O(\varepsilon_1)\right] \\ &-(L_v C_v + N_2)\times \\ &\left[\exp\left(\frac{M_4(t-t_0)}{\varepsilon_2}\right)\left(w_2(t_0) + M_4^{-1} M_3 \dot{\zeta}(t_0)\right) + O(\varepsilon_2)\right]. \end{aligned}$$

Consider now the homogeneous differential equation:

$$M\ddot{e}(t) + \widetilde{D}\dot{e}(t) + \widetilde{K}e(t) = 0. \tag{13}$$

The Lyapunov stability analysis performed in [2] and also summarized in Lemma 2.1, led to the existence and explicit construction of a *parameter-dependent* Lyapunov function $V(e,\dot{e})$ such that:

$$\frac{dV}{dt} \leq -\alpha V \tag{14}$$

with α being a positive constant that depends on the Lyapunov gain γ in (9) [3]. Furthermore:

$$c_2||x^h||^2 \leq V(x^h) \leq c_1||x^h||^2 \tag{15}$$

where $x^h = [e\,|\,\dot{e}]$ is the solution to the homogeneous differential equation (13) (when it is equivalently viewed as a system of first-order differential equations in e and $\dot{e}$), and c_1, c_2 positive constants given by

$$c_2 = \min\left(\frac{\gamma\lambda_{min}(\widetilde{K})-\lambda_{max}(M)+\lambda_{min}(\widetilde{D})}{2}, \frac{\gamma\lambda_{min}(M)-\lambda_{max}(M)}{2}\right),$$

$$c_1 = \max\left(\frac{\gamma\lambda_{max}(\widetilde{K})+\lambda_{max}(M)+\lambda_{max}(\widetilde{D})}{2}, \frac{(\gamma+1)\lambda_{max}(M)}{2}\right).$$

Using (14) and the comparison theorem [6], one obtains:

$$V(x^h) \leq V(x_0)\, e^{-\alpha(t-t_0)} \tag{16}$$

where $x_0 = x^h(t_0)$. Hence:

$$||x^h|| \leq \sqrt{\frac{V(x_0)}{c_2}}\, e^{\left(-\frac{\alpha}{2}(t-t_0)\right)} \leq k\|x_0\|\, e^{-\lambda(t-t_0)} \tag{17}$$

with $k = \sqrt{\frac{c_1}{c_2}}$ and $\lambda = \frac{\alpha}{2}$. The solution $x = [e\,|\,\dot{e}]$ to the non-homogeneous differential equation (9) (when itself is viewed as a system of first-order differential equations in e and $\dot{e}$) can be represented as follows

$$x(t) = x^h(t) + \int_0^t e^{\widetilde{A}(t-\tau)}\left[f(\tau)+O(\varepsilon_1)+O(\varepsilon_2)\right]d\tau \tag{18}$$

where

$$\widetilde{A} = \begin{bmatrix} 0 & I \\ -M^{-1}\widetilde{K} & -M^{-1}\widetilde{D} \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ \Pi(t) \end{bmatrix}, \tag{19}$$

with

$$\begin{aligned} \Pi(t) \;=\; & -M^{-1}(L_pC_p+N_1)\exp\left(\frac{M_2(t-t_0)}{\varepsilon_1}\right)\times \\ & \left(w_1(t_0)+M_2^{-1}M_1\zeta(t_0)\right) \\ & -M^{-1}(L_vC_v+N_2)\exp\left(\frac{M_4(t-t_0)}{\varepsilon_2}\right)\times \\ & \left(w_2(t_0)+M_4^{-1}M_3\dot{\zeta}(t_0)\right), \end{aligned}$$

$$g_1 = \begin{bmatrix} 0 \\ -M^{-1}(L_pC_p+N_1) \end{bmatrix},\; g_2 = \begin{bmatrix} 0 \\ -M^{-1}(L_vC_v+N_2) \end{bmatrix},$$

and $x^h = \exp(\widetilde{A}(t-t_0))x_0$. In light of (17), one obtains

$$\|x^h(t)\| = \|\exp(\widetilde{A}(t-t_0))x_0\| \leq k\,\|x_0\|\,e^{-\lambda(t-t_0)}. \tag{20}$$

Therefore, since x_0 is arbitrary

$$\|\exp(\widetilde{A}(t-t_0))\| \leq k\,e^{-\lambda(t-t_0)} \tag{21}$$

From equation (18) the following inequality can be established

$$\begin{aligned} \|x(t)\| \leq\; & k\,e^{-\lambda(t-t_0)} \\ & +k\|M^{-1}(L_pC_p+N_1)\|\,\|w_1(t_0)+M_2^{-1}M_1\zeta(t_0)\|\times \\ & \int_0^t e^{-\lambda(t-\tau)}\left\|\exp\left(\frac{M_2(\tau-t_0)}{\varepsilon_1}\right)\right\|d\tau \\ & +k\|M^{-1}(L_vC_v+N_2)\|\,\|w_2(t_0)+M_4^{-1}M_3\dot{\zeta}(t_0)\|\times \\ & \int_0^t e^{-\lambda(t-\tau)}\left\|\exp\left(\frac{M_4(\tau-t_0)}{\varepsilon_2}\right)\right\|d\tau \\ & +O(\varepsilon_1)+O(\varepsilon_2). \end{aligned}$$

Since both matrices M_2, M_4 are Hurwitz, there exist positive constants k_1, a_1, k_2, a_2 such that [5], [6]

$$\left\|\exp\left(\frac{M_2(\tau-t_0)}{\varepsilon_1}\right)\right\| \leq k_1\,e^{\left(-\frac{a_1(\tau-t_0)}{\varepsilon_1}\right)}, \tag{22}$$

and

$$\left\|\exp\left(\frac{M_4(\tau-t_0)}{\varepsilon_2}\right)\right\| \leq k_2\,e^{\left(-\frac{a_2(\tau-t_0)}{\varepsilon_2}\right)}. \tag{23}$$

As a result

$$\begin{aligned} |e(t)| \leq\; & \left(|e(t)|^2+|\dot{e}(t)|^2\right)^{\frac{1}{2}} = \|x(t)\| \leq ke^{-\lambda(t-t_0)} \\ & +\frac{\varepsilon_1 kk_1}{a_1-\varepsilon_1\lambda}\left\|M^{-1}(L_pC_p+N_1)\right\|\times \\ & \left\|w_1(t_0)+M_2^{-1}M_1\zeta(t_0)\right\|\left(e^{-\lambda(t-t_0)}-e^{-\frac{a_1(t-t_0)}{\varepsilon_1}}\right) \\ & +\frac{\varepsilon_2 kk_2}{a_2-\varepsilon_2\lambda}\left\|M^{-1}(L_vC_v+N_2)\right\|\times \\ & \left\|w_2(t_0)+M_4^{-1}M_3\dot{\zeta}(t_0)\right\|\left(e^{-\lambda(t-t_0)}-e^{-\frac{a_2(t-t_0)}{\varepsilon_2}}\right). \end{aligned}$$

One can easily verify that the observer error term

$$\begin{aligned} E(t,\varepsilon_1,\varepsilon_2) \triangleq\; & \frac{\varepsilon_1 kk_1}{a_1-\varepsilon_1\lambda}\left\|M^{-1}(L_pC_p+N_1)\right\|\times \\ & \left\|w_1(t_0)+M_2^{-1}M_1\zeta(t_0)\right\|\left(e^{-\lambda(t-t_0)}-e^{-\frac{a_1(t-t_0)}{\varepsilon_1}}\right) \\ & +\frac{\varepsilon_2 kk_2}{a_2-\varepsilon_2\lambda}\left\|M^{-1}(L_vC_v+N_2)\right\|\times \\ & \left\|w_2(t_0)+M_4^{-1}M_3\dot{\zeta}(t_0)\right\|\left(e^{-\lambda(t-t_0)}-e^{-\frac{a_2(t-t_0}{\varepsilon_2}}\right), \end{aligned}$$

is of order $O(\varepsilon_1)$, $O(\varepsilon_2)$, and the proof is complete. □

Remark 3.1: Continuing with the analysis on an observer based on a first order formulation, define $e_1 \triangleq \zeta - z_1$ and $e_2 \triangleq \dot{\zeta} - z_2$. One then has

$$\begin{aligned} \dot{e}_1 =\; & e_2 - L_{11}C_p\left(w_1+M_2^{-1}M_1z_1\right) \\ & -L_{12}C_v\left(w_2+M_4^{-1}M_3z_2\right) \end{aligned}$$

$$\begin{aligned} M\dot{e}_2+\widetilde{D}e_2+\widetilde{K}e_1 =\; & -\left(L_{21}C_p+N_1\right)\left(w_1+M_2^{-1}M_1z_1\right) \\ & -\left(L_{22}C_v+N_2\right)\left(w_2+M_4^{-1}M_3z_2\right). \end{aligned} \tag{24}$$

The above set of equations will in fact coincide with the proposed natural observer when (i) $\mathcal{R}(C_p) \subset L_{11}$ and $\mathcal{R}(C_v) \subset L_{12}$ or when (ii) $L_{11} = 0 = L_{12}$. Since the observer design based on a first order formulation cannot guarantee this, one then concludes that the derivative of z_1 will not be equal to z_2 initially. Even when sensor dynamics are not present (which would imply that: $y_p \to \widehat{y}_1$ and $y_v \to \widehat{y}_2$), the above condition will only be satisfied for large values of t. When the sensor dynamics are considered, then one can never guarantee that the equation $\dot{z}_1 = z_2$ will be satisfied since use of Lemma 3.1 gives

$$\begin{aligned}\dot{e}_1 = e_2 - L_{11}C_p \exp\left(\frac{M_2(t-t_0)}{\varepsilon_1}\right) \times \\ \left[w_1(t_0) + M_2^{-1}M_1\zeta(t_0)\right] + O(\varepsilon_1) \\ -L_{12}C_v \exp\left(\frac{M_4(t-t_0)}{\varepsilon_1}\right) \times \\ \left[w_2(t_0) + M_4^{-1}M_3\dot{\zeta}(t_0)\right] + O(\varepsilon_2).\end{aligned} \tag{25}$$

IV. EXAMPLE

We consider a two-degree of freedom (2DOF) mechanical translational system given by

$$\begin{aligned}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} d & -d \\ -d & d \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} w_1 \\ + \begin{bmatrix} n_{21} \\ n_{22} \end{bmatrix} w_2 = \begin{bmatrix} 0 \\ b \end{bmatrix} u(t),\end{aligned}$$

$$\begin{aligned}&\varepsilon_1 \dot{w}_1 = m_{11}x_1 + m_{12}x_2 + M_2 w_1, \\ &\varepsilon_2 \dot{w}_2 = m_{21}\dot{x}_1 + m_{22}\dot{x}_2 + M_4 w_2, \\ &y_p = C_p w_1, \qquad y_v = C_v w_2,\end{aligned}$$

which models an automobile suspension in which the simulation parameters are given in Table 1. It is assumed that only the position and velocity of the first mass can be measured and hence

$$w_1(t) = \frac{1}{\frac{\varepsilon_1}{2}s+1}[x_1(t)], \;\; w_2(t) = \frac{1}{\frac{\varepsilon_2}{3}s+1}[\dot{x}_1(t)].$$

In the above, we adopted the notation in [4] for a filter operator in the Laplace s-domain $w_i(t) = G_i(s)[x_i(t)]$, meaning $W_i(s) = G_i(s)X_i(s)$ or $w_i(t) = \int_0^t g(t-\tau)x_i(\tau)\,d\tau$. Alternatively, one has

$$\begin{aligned}w_1(t) = e^{-\frac{2}{\varepsilon_1}t}w_1(0) + \int_0^t e^{-\frac{2}{\varepsilon_1}(t-\tau)}x_1(\tau)\,d\tau, \\ w_2(t) = e^{-\frac{3}{\varepsilon_2}t}w_2(0) + \int_0^t e^{-\frac{3}{\varepsilon_2}(t-\tau)}\dot{x}_1(\tau)\,d\tau.\end{aligned}$$

For low values of $\varepsilon_i \ll 1$, both sensor dynamics model an all-pass filter ($w_1 \approx x_1, w_2 \approx \dot{x}_1$) which becomes a low-pass filter for moderately nonzero values of ε_i. For the current set of simulations, the initial conditions were chosen as follows: $x_1(0) = 3$, $x_2(0) = -2$, $\dot{x}_1(0) = -4$, $\dot{x}_2(0) = 1$, $w_1(0) = 0 = w_2(0)$ and $\widehat{x}_1(0) = \widehat{x}_2(0) = \dot{\widehat{x}}_1(0) = \dot{\widehat{x}}_2(0) = 0$, with the input signal given by $u(t) = 0.1\sin(2t)$.

m_1	m_2	d	k_1	k_2
5	1	5	1.5	3
n_{11}	n_{12}	n_{21}	n_{22}	m_{11}
10^{-3}	0	2×10^{-3}	0	2
m_{12}	m_{21}	m_{22}	M_2	M_4
0	3	0	-2	-3
b	C_p	C_v	L_p	L_v
1	1	1	$25C_{p,0}^T$	$30C_{v,0}^T$

TABLE I
NUMERICAL VALUES OF SIMULATION PARAMETERS.

Figure 1 depicts the energy norm (sum of kinetic and potential energy) of the observation error when $\varepsilon_i = 0$ (*i.e.* no sensor dynamics) and when $\varepsilon_1 = \varepsilon_2 = 0.8$. It is observed that in both cases the error norm converges to zero exponentially. The system outputs (position and velocity) for $\varepsilon_1 = \varepsilon_2 = 0$ are depicted in Figure 2 and in Figure 3 for $\varepsilon_1 = \varepsilon_2 = 0.8$. When $\varepsilon_i = 0$ (*i.e.* no sensor dynamics), the outputs $C_{p,0}\zeta$ and $C_{v,0}\dot{\zeta}$ coincide with sensor outputs y_p and y_v, respectively. At the same time, the outputs of the (ideal) natural observer converge to the outputs of the plant. When unmodelled sensor dynamics is present, one observes that there is a small $O(\varepsilon)$ mismatch between the system outputs $(C_{p,0}\zeta, C_{v,0}\dot{\zeta})$ and the sensor outputs (y_p, y_v). The outputs of the natural observer $(C_{p,0}\widehat{\zeta}, C_{v,0}\dot{\widehat{\zeta}})$ converge to the sensor outputs (y_p, y_v), as predicted by the theory.

V. CONCLUDING REMARKS

A new natural observer design method for vector second order systems in the presence of two time-scale multiplicity was presented. In particular, vector second order mechanical systems were considered along with fast unmodelled sensor dynamics, and the primary objective was to use an appropriately designed natural observer in order to obtain accurate estimates of the unmeasurable slow system state variables. Within a singular perturbation framework, the proposed natural observer was designed on the basis of the system dynamics that evolves on the slow manifold, and the dynamic behavior of the estimation error that induces was analyzed in the presence of the unmodelled fast sensor dynamics. Specifically, it was shown, that the observation error generated by neglecting the unmodelled fast sensor dynamics is of order $O(\varepsilon)$, where ε is the singular perturbation parameter and a measure of the relative speed/time-constant of the fast (sensor) and the slow component (vector second-order system) of the overall instrumented system dynamics.

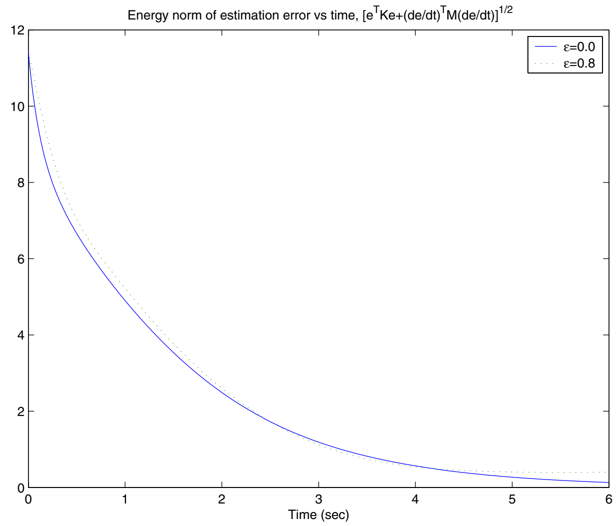

Fig. 1. Energy norm, $\left(e^T(t)Ke(t)+\dot{e}^T(t)M\dot{e}(t)\right)^{\frac{1}{2}}$.

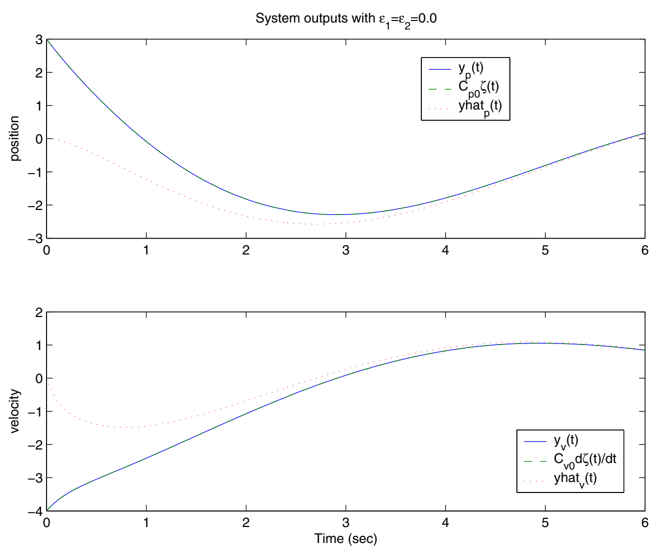

Fig. 2. Position and velocity outputs for $\varepsilon_1 = \varepsilon_2 = 0.0$.

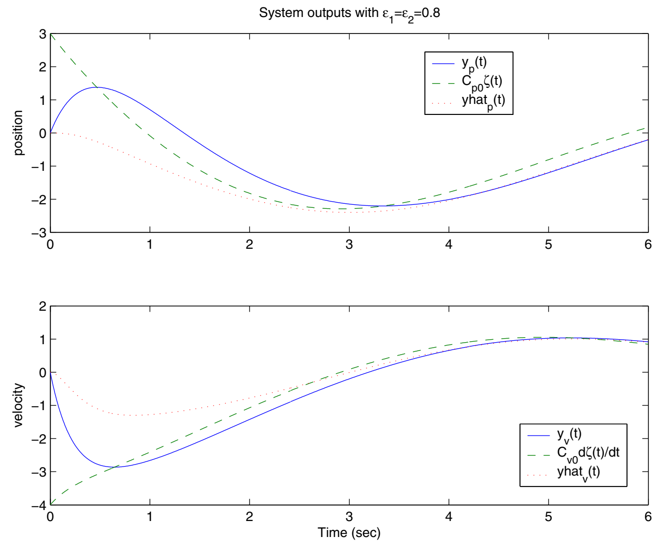

Fig. 3. Position and velocity outputs for $\varepsilon_1 = \varepsilon_2 = 0.8$.

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