Adaptive Quantized Control for Linear Uncertain Discrete-Time Systems

Tomohisa Hayakawa†, **Hideaki Ishii**‡, **and Koji Tsumura**‡

†CREST, Japan Science and Technology Agency, Saitama, 332-0012, JAPAN

‡Department of Information Physics and Computing, The University of Tokyo, Tokyo, 113-8656, JAPAN

{tomohisa_hayakawa,hideaki_ishii,koji_tsumura}@ipc.i.u-tokyo.ac.jp

Abstract

A direct adaptive control framework for linear uncertain systems with input quantizers is developed. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability; that is, Lyapunov stability of the closed-loop system states and attraction with respect to the plant states. Specifically, the input quantizers are logarithmic and characterized by sector-bound conditions with the conic sector adjusted at each time instant by the adaptive controller in conjunction with the system response. Finally, a numerical example is provided to demonstrate the efficacy of the proposed approach.

1. Introduction

To design control systems whose components are connected by shared networks, it is essential to consider the limitation due to the communication system and to ensure that the systems can operate appropriately within the given bandwidth [1, 2]. An important aspect there is to use quantization schemes that have sufficient precision and, at the same time, require low communication rate. These views have prompted research interests on new quantization methods accounting for characteristics particular to control systems.

One such scheme is presented in [3] for stabilization of a linear discrete-time system where an optimal quantizer is obtained with respect to a certain measure on coarseness of the transmitted information. This quantizer has a unique feature that the quantization levels become finer in the region closer to the origin in a logarithmic way and is hence called the logarithmic quantizer. Moreover, its key parameter is determined solely by the unstable poles of the system. In [4], an alternative proof for the optimal design and more general results are given by viewing such quantizers as sector-bounded nonlinearities. This idea is extended in [5] and applied to the case of uncertain systems with additive bounded uncertainties using H_∞ techniques.

In contrast to fixed-gain robust controllers, adaptive controllers are more appropriate in dealing with uncertain systems with unknown uncertainty bounds. In other words, adaptive controllers can tolerate far greater system uncertainty levels by adjusting feedback gains in response to plant variation to improve system performance.

In this paper, we consider a stabilization problem for uncertain plants over networks via a direct adaptive control approach. The setup is depicted in Figure 1.1. The controller is on the sensor side, and the control input is quantized and coded in the coder to be sent over the channel; we assume that the channel is noiseless, and hence the quantized signal is recovered in the decoder and is applied to the plant.

For a linear time-invariant plant whose parameters are uncertain with unknown bounds, we propose a design method

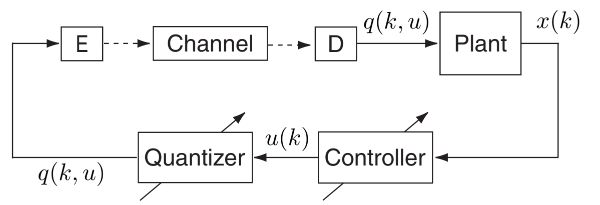


Figure 1.1: Adaptive control scheme with a time-varying quantizer, where E and D represent the encoder and the decoder, respectively.

for an adaptive controller and an input quantizer. The quantizer is time varying, and at each time instant, its parameters are determined and adjusted in response to the update in the controller gain. Following the recent quantization approaches, we aim at maintaining the quantizer as coarse as possible at each moment. In particular, we employ logarithmic quantizers.

In our adaptive control scheme, it turns out that the quantization levels must be fine while the gain is large, and vice versa. In general, this implies that systems with poles that are more unstable would require more information for stabilization. This is in agreement with the implications in [3, 4] as mentioned above. Furthermore, if the system matrices are known, the proposed controller and quantizer reduce to static ones and coincide with the optimal ones given in [3, 4]. Although in the adaptive case, it is difficult to show optimality, we may say that our approach is nonconservative for this reason.

Finally, we emphasize that the proposed adaptive control method is Lyapunov-based and guarantees partial asymptotic stability; that is, Lyapunov stability of the closed-loop system states and attraction with respect to the plant states. (As a result, the adaptive gain states are bounded). Note that most of the adaptive control approaches for discrete-time systems are based on recursive least squares and least mean squares algorithms [6]; there, the primary focus is on state convergence rather than stability. Several notable Lyapunov-based approaches in discrete time are given in [7–11].

The notation used in this paper is fairly standard. Specifically, $\mathbb{I}$ denotes the set of integers, $\mathbb{N}_0$ denotes the set of nonnegative integers, and $(\cdot)^\dagger$ denotes the Moore-Penrose generalized inverse. Furthermore, we write $\lambda_{\max}(M)$ for the maximum eigenvalue of the symmetric matrix M and $\sigma_{\max}(M)$ for the maximum singular value of the matrix M.

2. Adaptive Control for Linear Uncertain Systems with Input Quantizers

In this section we introduce an adaptive feedback control problem for linear uncertain dynamical systems with input quantizers. Specifically, consider the linear uncertain

This research was supported in part by Japan Science and Technology Agency under CREST program.

discrete-time system $\mathcal{G}$ given by

$$x(k+1) = Ax(k) + Bv(k), \quad x(0) = x_0, \quad k \in \mathbb{N}_0, \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $v(k) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. Here we assume that the input vector $v(t)$ takes quantized values that are reconstructed at the decoder. In particular, we assume that $v(k)$ is given by

$$v(k) = q(k, u(k)), \quad (2)$$

where $q(\cdot,\cdot)$ represents the *time-varying logarithmic* quantization function of the form

$$q_i(k, u_i) = \begin{cases} a_i(k)\rho_i^j(k), & \text{if } u_i \in (a_i(k)\rho_i^{j+1}(k), a_i(k)\rho_i^j(k)], \\ -a_i(k)\rho_i^j(k), & \text{if } u_i \in [-a_i(k)\rho_i^j(k), -a_i(k)\rho_i^{j+1}(k)), \\ 0, & \text{if } u_i = 0, \end{cases}$$
$$i = 1, \cdots, m, \quad (3)$$

and $u(k)$ is the control input signal to be quantized at the encoder and is given in the form

$$u(k) = H(k)x(k), \quad (4)$$

where $a_i(k) > 0$, $i = 1, \cdots, m$, $0 < \rho_i(k) < 1$, $i = 1, \cdots, m$, and $q_i(\cdot,\cdot)$ and u_i denote the ith component of $q(\cdot,\cdot)$ and u, respectively. Note that $\rho_i(\cdot)$ determines coarseness of the quantizer $q_i(\cdot,\cdot)$ for each $u_i(\cdot)$, $i = 1, \cdots, m$.

It is important to note that the logarithmic quantizer (3) can be characterized as a class of time-varying sector-bounded memoryless input nonlinearities $\mathcal{Q}$ which is given by

$$\begin{aligned} \mathcal{Q} \triangleq \{ & q : \mathbb{N}_0 \times \mathbb{R}^m \to \mathbb{R}^m : q(\cdot, 0) = 0, \\ & [q(k,u) - M_1(k)u]^{\mathrm{T}}[q(k,u) - M_2(k)u] \le 0, \\ & u \in \mathbb{R}^m, k \in \mathbb{N}_0\}, \end{aligned} \quad (5)$$

where $M_1 \triangleq \mathrm{diag}[M_{1_1}, \cdots, M_{1_m}] > 0$ and $M_2 \triangleq \mathrm{diag}[M_{2_1}, \cdots, M_{2_m}] > 0$ are such that $\rho_i = M_{1_i}/M_{2_i}$, $i = 1, \cdots, m$, and $M_2 - M_1$ is positive definite (Figure 2.2(a)). Note that the sector condition characterizing $\mathcal{Q}$ is implied by the scalar sector conditions

$$M_{1_i}(k)u_i^2 \le q_i(k, u_i)u_i \le M_{2_i}(k)u_i^2, \quad u_i \in \mathbb{R},$$
$$k \in \mathbb{N}_0, \quad i = 1, \cdots, m. \quad (6)$$

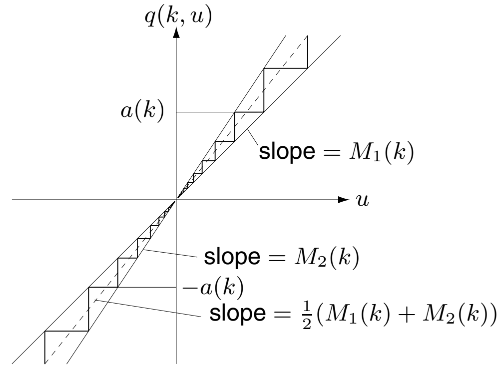
Since $\rho_i(\cdot) = M_{1_i}(\cdot)/M_{2_i}(\cdot)$, $i = 1, \cdots, m$, the coarseness of the quantizer $q_i(\cdot,\cdot)$ is determined by $M_{2_i}(\cdot) - M_{1_i}(\cdot)$ for each $i = 1, \cdots, m$. Even though the time variation of $q(k,\cdot)$ is due solely to the variation of $M_1(k)$ and $M_2(k)$, we write $q(k, u(k))$ instead of $q(M_1(k), M_2(k), u(k))$ for simplicity of exposition.

To design adaptive feedback controllers for (1) we decompose the quantization function $q(\cdot,\cdot)$ into a linear part and a nonlinear part so that

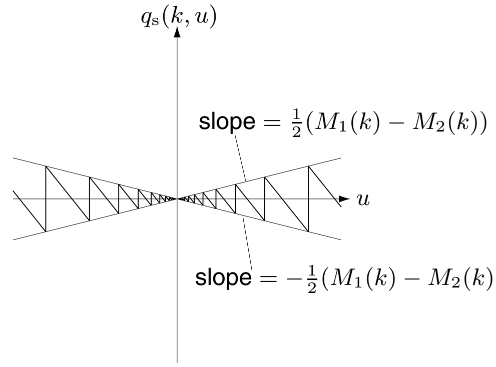
$$q(k, u) = M(k)u + q_{\mathrm{s}}(k, u), \quad (7)$$

where $M(k) \triangleq \frac{1}{2}(M_1(k) + M_2(k))$ (see Figure 2.2(b)). Note that the transformed nonlinearities $q_{\mathrm{s}}(\cdot,\cdot)$ belong to the set $\mathcal{Q}_{\mathrm{s}}$ given by

$$\begin{aligned} \mathcal{Q}_{\mathrm{s}} \triangleq \{ & q_{\mathrm{s}} : \mathbb{N}_0 \times \mathbb{R}^m \to \mathbb{R}^m : q_{\mathrm{s}}(\cdot, 0) = 0, \\ & q_{\mathrm{s}}^{\mathrm{T}}(k,u)q_{\mathrm{s}}(k,u) - \tfrac{1}{4}u^{\mathrm{T}}(M_2(k) - M_1(k))^2 u \le 0, \\ & u \in \mathbb{R}^m, k \in \mathbb{N}_0\}. \end{aligned} \quad (8)$$



(a) Logarithmic quantizer $q(\cdot,\cdot)$



(b) Transformed nonlinearity $q_{\mathrm{s}}(\cdot,\cdot)$

Figure 2.2: Decomposition of a quantization function for $m = 1$

As discussed in the Introduction, we assume that $M_1(\cdot)$ and $M_2(\cdot)$ also take quantized (discrete) values with the aim of using network channels. For the guideline of choosing $M_1(k)$ and $M_2(k)$ at each $k \in \mathbb{N}_0$, see Remark 2.2 below.

Now we state the main theorem of this paper. Our objective is to design an adaptive controller in the form of (4) and a quantization rule for $u(k)$ to reduce bit rates to be sent over the communication channel. The following result provides a control architecture that ensures stability of the closed-loop system in the case where the system matrix A is unknown but the input matrix B is known. The case where B is also unknown is addressed in Corollary 2.1. For the statement of Theorem 2.1 and Corollary 2.1 define $\mathcal{A} \triangleq \{\tilde{A} \in \mathbb{R}^{n\times n} : \tilde{A} = A + BK_{\mathrm{g1}},\ K_{\mathrm{g1}} \in \mathbb{R}^{m\times n}\}$.

Theorem 2.1. Consider the linear uncertain system $\mathcal{G}$ given by (1) where $A \in \mathbb{R}^{n\times n}$ is an unknown matrix, $B \in \mathbb{R}^{n\times m}$ is such that $\mathrm{rank}\, B = m$, and the pair (A, B) is stabilizable. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution of the Riccati equation

$$P = \tilde{A}^{\mathrm{T}}P\tilde{A} + R - \tilde{A}^{\mathrm{T}}PB(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\tilde{A}, \quad (9)$$

with $P \ge I_n$, where $\tilde{A} \in \mathcal{A}$ and $R \in \mathbb{R}^{n\times n}$ is positive definite. Furthermore, let $A_{\mathrm{s}} \triangleq \tilde{A} + BK_{\mathrm{g2}}$, where $K_{\mathrm{g2}} \triangleq -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\tilde{A}$, and let $Q \in \mathbb{R}^{m\times m}$ and $\varepsilon \in \mathbb{R}$ be such that $Q > 0$ and $\varepsilon > 0$ satisfy

$$\tilde{R} \triangleq \tfrac{1}{\varepsilon}(I_m - Q) - B^{\mathrm{T}}PB \ge 0. \quad (10)$$

Then the adaptive feedback control law

$$u(k)=M^{-1}(k)K(k)x(k), \tag{11}$$

where $K(k)\in\mathbb{R}^{m\times n}$, $M_1(k)$ and $M_2(k)$ satisfy

$$R-\tfrac{1}{4\varepsilon}K^{\mathrm T}(k)(M_2(k)-M_1(k))^2M^{-2}(k)K(k)\geq\gamma I_n>0, \tag{12}$$

at each time $k\in\mathbb{N}_0$, and $\gamma\in\mathbb{R}$ is an arbitrarily small constant, with the quantizer (2) and the update law

$$\begin{aligned}K(k+1) &= K(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}QB^{\dagger}[x(k+1)\\ &\quad -A_{\mathrm s}x(k)]x^{\mathrm T}(k),\quad K(0)=K_0, \end{aligned}\tag{13}$$

guarantees that the solution $(x(k),K(k))\equiv(0,K_{\mathrm g})$, where $K_{\mathrm g}\triangleq-(B^{\mathrm T}PB)^{-1}B^{\mathrm T}PA$, of the closed-loop system given by (1), (11), and (13) is Lyapunov stable and $x(k)\to0$ as $k\to\infty$ for all $x_0\in\mathbb{R}^n$.

Proof. First, note that

$$\begin{aligned}A_{\mathrm s}^{\mathrm T}PBB^{\mathrm T}PA_{\mathrm s} &= (\tilde A+BK_{\mathrm{g2}})^{\mathrm T}PBB^{\mathrm T}P(\tilde A+BK_{\mathrm{g2}})\\ &= (\tilde A-B(B^{\mathrm T}PB)^{-1}B^{\mathrm T}P\tilde A)^{\mathrm T}PBB^{\mathrm T}P\\ &\quad\cdot(\tilde A-B(B^{\mathrm T}PB)^{-1}B^{\mathrm T}P\tilde A)\\ &= 0,\end{aligned}\tag{14}$$

and hence, since $A_{\mathrm s}^{\mathrm T}PBB^{\mathrm T}PA_{\mathrm s}$ is nonnegative definite, $A_{\mathrm s}^{\mathrm T}PB=0$. Furthermore, note that from (9)

$$P=A_{\mathrm s}^{\mathrm T}PA_{\mathrm s}+R. \tag{15}$$

Next, define $\tilde K(k)\triangleq K(k)-K_{\mathrm g}$, $\tilde u(k)\triangleq\tilde K(k)x(k)$, and let K_{g1} be such that $\tilde A=A+BK_{\mathrm{g1}}$. Note that

$$\begin{aligned}K_{\mathrm g} &= -(B^{\mathrm T}PB)^{-1}B^{\mathrm T}PA\\ &= K_{\mathrm{g1}}-(B^{\mathrm T}PB)^{-1}B^{\mathrm T}P(A+BK_{\mathrm{g1}})\\ &= K_{\mathrm{g1}}+K_{\mathrm{g2}}.\end{aligned}\tag{16}$$

Furthermore, with $u(k)$ given by (11) it follows from (7) that

$$\begin{aligned}x(k+1) &= A_{\mathrm s}x(k)+B\tilde K(k)x(k)+Bq_{\mathrm s}(k,u(k))\\ &= A_{\mathrm s}x(k)+B\tilde u(k)+Bq_{\mathrm s}(k,u(k)),\\ &\qquad x(0)=x_0,\quad k\in\mathbb{N}_0.\end{aligned}\tag{17}$$

In addition, note that by subtracting $K_{\mathrm g}$ from both sides of (13) and using (17) it follows that

$$\begin{aligned}\tilde K(k+1) &= \tilde K(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}QB^{\dagger}[B\tilde K(k)x(k)\\ &\quad +Bq_{\mathrm s}(k,u(k))]x^{\mathrm T}(k)\\ &= \tilde K(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Q\tilde K(k)x(k)x^{\mathrm T}(k)\\ &\quad -\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Qq_{\mathrm s}(k,u(k))x^{\mathrm T}(k).\end{aligned}\tag{18}$$

To show Lyapunov stability of the closed-loop system (17) and (18), consider the Lyapunov function candidate given by

$$V(x,K)=\ln(1+x^{\mathrm T}Px)+\tfrac{1}{\varepsilon}\mathrm{tr}(K-K_{\mathrm g})^{\mathrm T}Q^{-1}(K-K_{\mathrm g}). \tag{19}$$

Note that $V(0,K_{\mathrm g})=0$ and, since P and Q are positive definite and $\varepsilon>0$, $V(x,K)>0$ for all $(x,K)\neq(0,K_{\mathrm g})$. Furthermore, $V(x,K)$ is radially unbounded. Now, let $x(k)$ denote the solution of the closed-loop system (17). Then, using (15), (18), and the fact that $A_{\mathrm s}^{\mathrm T}PB=0$, the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}&\Delta V(x(k),K(k))\\ &\quad\triangleq V(x(k+1),K(k+1))-V(x(k),K(k))\\ &\quad= \ln\Big(1+(A_{\mathrm s}x(k)+B\tilde u(k)+Bq_{\mathrm s}(k,u(k)))^{\mathrm T}\\ &\qquad\cdot P(A_{\mathrm s}x(k)+B\tilde u(k)+Bq_{\mathrm s}(k,u(k)))\Big)\\ &\qquad+\tfrac{1}{\varepsilon}\mathrm{tr}\Big(\tilde K(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Q\tilde K(k)x(k)x^{\mathrm T}(k)\\ &\qquad-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Qq_{\mathrm s}(k,u(k))x^{\mathrm T}(k)\Big)^{\mathrm T}Q^{-1}\\ &\qquad\cdot\Big(\tilde K(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Q\tilde K(k)x(k)x^{\mathrm T}(k)\\ &\qquad-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Qq_{\mathrm s}(k,u(k))x^{\mathrm T}(k)\Big)\\ &\qquad-\ln(1+x^{\mathrm T}(k)Px(k))-\tfrac{1}{\varepsilon}\mathrm{tr}\tilde K^{\mathrm T}(k)Q^{-1}\tilde K(k)\\ &\quad= \ln\Big(1+[1+x^{\mathrm T}(k)Px(k)]^{-1}\Big[x^{\mathrm T}(t)A_{\mathrm s}^{\mathrm T}PA_{\mathrm s}x(k)\\ &\qquad+\tilde u^{\mathrm T}(k)B^{\mathrm T}PB\tilde u(k)+2\tilde u^{\mathrm T}(k)B^{\mathrm T}PBq_{\mathrm s}(k,u(k))\\ &\qquad+q_{\mathrm s}^{\mathrm T}(k,u(k))B^{\mathrm T}PBq_{\mathrm s}(k,u(k))-x^{\mathrm T}(k)Px(k)\Big]\Big)\\ &\qquad+\tfrac{1}{\varepsilon}\mathrm{tr}\,\tilde K^{\mathrm T}(k)Q^{-1}\tilde K(k)+\tfrac{1}{\varepsilon(1+x^{\mathrm T}(k)Px(k))^2}\\ &\qquad\cdot\mathrm{tr}\,x(k)q_{\mathrm s}^{\mathrm T}(k,u(k))Qq_{\mathrm s}(k,u(k))x^{\mathrm T}(k)\\ &\qquad+\tfrac{1}{\varepsilon(1+x^{\mathrm T}(k)Px(k))^2}\mathrm{tr}[x(k)x^{\mathrm T}(k)\tilde K^{\mathrm T}(k)Q\\ &\qquad\cdot\tilde K(k)x(k)x^{\mathrm T}(k)]\\ &\qquad-\tfrac{2}{\varepsilon(1+x^{\mathrm T}(k)Px(k))}\mathrm{tr}\,\tilde K^{\mathrm T}(k)\tilde K(k)x(k)x^{\mathrm T}(k)\\ &\qquad-\tfrac{2}{\varepsilon(1+x^{\mathrm T}(k)Px(k))}\mathrm{tr}\,\tilde K^{\mathrm T}(k)q_{\mathrm s}(k,u(k))x^{\mathrm T}(k)\\ &\qquad+\tfrac{2}{\varepsilon(1+x^{\mathrm T}(k)Px(k))^2}\mathrm{tr}[x(k)x^{\mathrm T}(k)\tilde K^{\mathrm T}(k)\\ &\qquad\cdot Qq_{\mathrm s}(k,u(k))x^{\mathrm T}(k)-\tfrac{1}{\varepsilon}\mathrm{tr}\tilde K^{\mathrm T}(k)Q^{-1}\tilde K(k)]\\ &\quad\leq [1+x^{\mathrm T}(k)Px(k)]^{-1}\Big[-x^{\mathrm T}(k)Rx(k)\\ &\qquad+q_{\mathrm s}^{\mathrm T}(k,u(k))(B^{\mathrm T}PB+\tfrac{1}{\varepsilon}Q)q_{\mathrm s}(k,u(k))\\ &\qquad-\tilde u^{\mathrm T}(k)[\tfrac{1}{\varepsilon}(2I_m-Q)-B^{\mathrm T}PB]\tilde u(k)\\ &\qquad-2q_{\mathrm s}^{\mathrm T}(k)[\tfrac{1}{\varepsilon}(I_m-Q)-B^{\mathrm T}PB]\tilde u(k)\Big],\quad k\in\mathbb{N}_0,\end{aligned}\tag{20}$$

where in (20) we used $\ln a-\ln b=\ln\frac{a}{b}$ and $\ln(1+c)\leq c$ for $a,b>0$ and $c\geq-1$, respectively, and $\frac{x^{\mathrm T}x}{1+x^{\mathrm T}Px}<1$ since $P\geq I_n$. Now, using (12) and the fact that $q_{\mathrm s}(\cdot,\cdot)$ belongs to $\mathcal{Q}_{\mathrm s}$ given by (8), it further follows from (10) and (20) that

$$\begin{aligned}&\Delta V(x(k),K(k))\\ &\quad\leq [1+x^{\mathrm T}(k)Px(k)]^{-1}\Big[-x^{\mathrm T}(k)Rx(k)\\ &\qquad+q_{\mathrm s}^{\mathrm T}(k,u(k))(B^{\mathrm T}PB+\tfrac{1}{\varepsilon}Q)q_{\mathrm s}(k,u(k))\\ &\qquad-\tfrac{1}{\varepsilon}\tilde u^{\mathrm T}(k)\tilde u(k)\end{aligned}$$

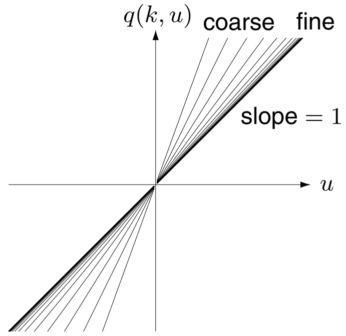


Figure 2.3: An example of sector bounds for the time-varying logarithmic quantizor ($M_{2_i}(k) \in \{1 + \hat{a}\mu_i^j : j \in \mathbb{I}\}$, $M_{1_i}(k) \equiv 1$)

$$
\begin{aligned}
& - \left[\tilde{u}^{\mathrm{T}}(k),\, q_{\mathrm{s}}^{\mathrm{T}}(k, u(k))\right] \begin{bmatrix} \tilde{R} & \tilde{R} \\ \tilde{R} & \tilde{R} \end{bmatrix} \begin{bmatrix} \tilde{u}(k) \\ q_{\mathrm{s}}(k, u(k)) \end{bmatrix} \\
& + q_{\mathrm{s}}^{\mathrm{T}}(k, u(k))[\tfrac{1}{\varepsilon}(I_m - Q) - B^{\mathrm{T}}PB]q_{\mathrm{s}}(k, u(k))\Big] \\
\leq\ & [1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
& + \tfrac{1}{\varepsilon}q_{\mathrm{s}}^{\mathrm{T}}(k, u(k))q_{\mathrm{s}}(k, u(k))\Big] \\
\leq\ & -[1 + x^{\mathrm{T}}(k)Px(k)]^{-1}x^{\mathrm{T}}(k)[R - \tfrac{1}{4\varepsilon}K^{\mathrm{T}}(k) \\
& \cdot (M_2(k) - M_1(k))^2 M^{-2}(k)K(k)]x(k) \\
\leq\ & -\gamma[1 + x^{\mathrm{T}}(k)Px(k)]^{-1}x^{\mathrm{T}}(k)x(k) \\
\leq\ & 0, \quad k \in \mathbb{N}_0. \qquad (21)
\end{aligned}
$$

This proves that the solution $(x(k), K(k)) \equiv (0, K_{\mathrm{g}})$ to (17) and (18) is Lyapunov stable. Furthermore, it follows from (the discrete-time version of) Theorem 4.4 of [12] that $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

Remark 2.1. The conditions in Theorem 2.1 imply partial asymptotic stability; that is, the solution $(x(k), K(k)) \equiv (0, K_{\mathrm{g}})$ of the overall closed-loop system is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$. Hence, it follows from (13) that $K(k+1) - K(k) \to 0$ as $k \to \infty$. ♢

Remark 2.2. Note that the choice of $M_1(k)$ and $M_2(k)$ is arbitrary so long as (12) holds for a given ε that satisfies (10). To construct a coarse quantizer, we obviously need to take $M_1(k)$ and $M_2(k)$ such that $M_2(k) - M_1(k)$ is as large as possible *at each time instant*. Furthermore, it follows from (10) that the smaller the maximum eigenvalue of Q is, the larger ε can be and hence, by (12), $M_2(k) - M_1(k)$ can be taken to be large. There are several simple ways to determine $M_1(k)$ and $M_2(k)$. For example, let $M_1(k) \equiv I_m$ and $M_{2_i}(k)$, $i = 1, \cdots, m$, be given by $M_{2_i}(k) \in \{1 + \hat{a}\mu_i^j : j \in \mathbb{I}\}$, where $\hat{a} > 0$ and $\mu_i > 0$, $i = 1, \cdots, m$. This implies that the smaller $M_{2_i}(k) - M_{1_i}(k)$ needs to be, the closer $M_{2_i}(k)$ becomes to $M_{1_i}(k)$ in a logarithmic manner for each $i = 1, \cdots, m$ (see Figure 2.3). (Note that it is realistic in practice to impose an upper bound for $M_2(\cdot) - M_1(\cdot)$ even while $K(\cdot)$ stays close to the zero matrix.) Alternatively, another simple way to determine $M_1(\cdot)$ and $M_2(\cdot)$ is to set $M_{1_i}(k) \in \{1 - \hat{a}\mu_i^j : j \in \mathbb{I}\}$ and $M_{2_i}(k) \in \{1 + \hat{a}\mu_i^j : j \in \mathbb{I}\}$ so that $M(k) = I_m$, $k \in \mathbb{N}_0$, and $M_{2_i}(k) - M_{1_i}(k) = 2\hat{a}\mu_i^j < 2$, $k \in \mathbb{N}_0$, $i = 1, \cdots, m$, since $M_1(k) > 0$ for all $k \in \mathbb{N}_0$. In either case above, there *always* exist $M_1(\cdot)$ and $M_2(\cdot)$ such that (12) is satisfied since $M_2(\cdot) - M_1(\cdot)$ can be arbitrarily small. ♢

Remark 2.3. In the case of single input systems ($m = 1$) with unstable system matrices A, Theorem 2.1 has a close connection with the results given in [3]. In fact, if we have the perfect knowledge of the system dynamics, then the sector condition used in Theorem 2.1 for the largest possible conic sector reduces to the results in Theorem 2.1 of [3].

To see this, suppose we have the explicit knowledge of the system matrices A and B so that we do not have to take adaptive strategy. In particular, let

$$K(k) \equiv K_{\mathrm{g}} = -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA, \qquad (22)$$

where P is the solution of the Riccati equation (9) with $\tilde{A} = A$ (i.e., $K_{\mathrm{g1}} = 0$). In this case, the update law (13) is superfluous by letting $Q = 0$ and hence it follows from (10) that the upper bound of ε is given by

$$\varepsilon \leq 1/(B^{\mathrm{T}}PB). \qquad (23)$$

Furthermore, take $M_1 \equiv 1-\delta$ and $M_2 \equiv 1+\delta$, where $\delta > 0$, so that $M \equiv 1$. Then it follows from (12) and (22) that

$$4\delta^2(B^{\mathrm{T}}PB)^{-2}R^{-1/2}A^{\mathrm{T}}PBB^{\mathrm{T}}PAR^{-1/2} < 4\varepsilon I_n, \quad (24)$$

which, with (23), further implies that

$$\delta^2(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PAR^{-1}A^{\mathrm{T}}PB < 1. \qquad (25)$$

Therefore, the upper bound $\delta_{\max}$ of δ is given by

$$\delta_{\max} = \sqrt{\frac{B^{\mathrm{T}}PB}{B^{\mathrm{T}}PAR^{-1}A^{\mathrm{T}}PB}}. \qquad (26)$$

This is precisely the result given in [3] that characterizes the coarsest possible quantizer for the given matrices A, B, and R. In particular, Elia & Mitter showed in [3] that properly choosing R in (26) further leads to the coarsest possible quantizer which is determined solely by the unstable poles of A. ♢

Remark 2.4. In Theorem 2.1 we assume that P is the solution to (9) which constitutes the optimal gain K_{g2} for the pair $(\tilde{A}, B)$ with the quadratic cost function to be minimized [13] given by

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x^{\mathrm{T}}(k)Rx(k). \qquad (27)$$

This construction yields the condition (12) that results in the identical sector bound for the case of static (non-adaptive) feedback control given in the literature [3] (see also Remark 2.3 for details). In fact, as far as stability is concerned, the matrix P can be replaced by the solution of the Lyapunov equation (15) with A_{s} being an arbitrary Schur (asymptotically stable) matrix that is constructed in the form of $A + BK_{\mathrm{g}}$, where $K_{\mathrm{g}} \in \mathbb{R}^{m\times n}$. In this case, closed-loop stability can be shown in a similar way to the proof of Theorem 2.1 with a new condition (instead of (12)) which permits a possibly finer quantizer. ♢

It is important to note that the adaptive control law (11) and (13) does *not* require explicit knowledge of the system matrix A nor the gain matrix $K_{\rm g}$ $(= K_{\rm g1} + K_{\rm g2})$ even though Theorem 2.1 requires that the pair (A, B) be stabilizable so that there exists a stabilizing solution to the Riccati equation (9). Furthermore, if (1) is in controllable canonical form (with asymptotically stable zero dynamics) [14], then we can *always* construct matrices $A_{\rm s}$ and P without requiring knowledge of the system dynamics.

To elucidate the above discussion assume that the linear uncertain system $\mathcal{G}$ is generated by the difference model

$$\begin{aligned} z_i(k+\tau_i) + a_{i,\tau_i-1}z_i(k+(\tau_i-1)) + \cdots + a_{i,0}z_i(k) \\ = \textstyle\sum_{j=1}^{m} B_{{\rm s}(i,j)}u_j(k), \quad k \in \mathbb{N}_0, \quad i = 1, \cdots, m, \end{aligned} \quad (28)$$

where $\tau_i \in \mathbb{N}_0$ denotes the time delay (or relative degree) with respect to the output z_i. Here, we assume that the square matrix $B_{\rm s}$ composed of the entries $B_{{\rm s}(i,j)}$, $i, j = 1, \cdots, m$, is such that $\det B_{\rm s} \neq 0$. Furthermore, since (28) is in a form where it does not possess internal dynamics, it follows that $\tau_1 + \cdots + \tau_m$ is the dimension of the system (28). The case where (28) possesses asymptotically stable zero dynamics can be analogously handled as shown in [11].

Next, define $x_i(k) \triangleq [\, z_i(k), \cdots, z_i(k+\tau_i-2)]^{\rm T}$, $i = 1, \cdots, m$, $x_{m+1}(k) \triangleq [\, z_1(k+\tau_1-1), \cdots, z_m(k+\tau_m-1)]^{\rm T}$, and $x(k) \triangleq [\, x_1^{\rm T}(k), \cdots, x_{m+1}^{\rm T}(k)]^{\rm T}$ so that (28) can be described by (1) with

$$A = \left[\begin{array}{c} A_0 \\ \Theta \end{array}\right], \quad B = \left[\begin{array}{c} 0_{(n-m)\times m} \\ B_{\rm s} \end{array}\right], \quad (29)$$

where $A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [14], $\Theta \in \mathbb{R}^{m\times n}$ is a matrix of uncertain constant parameters, and $B_{\rm s} \in \mathbb{R}^{m\times m}$. Next, to apply Theorem 2.1 to the uncertain system (1), let $K_{\rm g1} \in \mathbb{R}^{m\times s}$ be given by

$$K_{\rm g1} = B_{\rm s}^{-1}[\Theta_{\rm n1} - \Theta], \quad (30)$$

where $\Theta_{\rm n1} \in \mathbb{R}^{m\times n}$ is an arbitrary matrix so that $\tilde{A} = A + BK_{\rm g1}$ is a known matrix (not necessarily stable). Now, since stabilizability is invariant under feedback, the pair $(\tilde{A}, B)$ is also stabilizable and hence there exists a stabilizing solution to the Riccati equation (10) so that $A_{\rm s}$ can be computed and used in the update law (13). Specifically, if the positive-definite matrix R is diagonal, the resulting positive-definite solution P to (9) is also diagonal and $K_{\rm g2}$ is calculated to be $K_{\rm g2} = B_{\rm s}^{-1}\Theta_{\rm n}$. In this case, it follows that $A_{\rm s} = \left[\begin{array}{c} A_0 \\ 0_{m\times n} \end{array}\right]$ and hence the update law (13) is simplified as

$$\begin{aligned} K(k+1) \;=\; K(k) - \tfrac{1}{1+x^{\rm T}(k)Px(k)}QB^{\dagger}x(k+1)x^{\rm T}(k), \\ K(0) = K_0, \end{aligned} \quad (31)$$

since $B^{\dagger}A_{\rm s} = 0$.

Next, we consider the case where A and B are *both* uncertain. Specifically, we assume that the system matrices A and B are given in the form of (29) and B is such that $B_{\rm s}$, with $\sigma_{\max}(B_{\rm s}) \leq \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix but an upper bound α of the maximum singular value of $B_{\rm s}$ and the sign definiteness of $B_{\rm s}$ are known; that is, $B_{\rm s} > 0$ or $B_{\rm s} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\rm T}$ for $B_{\rm s} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\rm T}$ for $B_{\rm s} < 0$.

Corollary 2.1. Consider the linear system $\mathcal{G}$ given by (1) with A and B given by (29), where $B_{\rm s}$, with $\sigma_{\max}(B_{\rm s}) < \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix and the sign definiteness of $B_{\rm s}$ is known. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution of the Riccati equation

$$P = \tilde{A}^{\rm T}P\tilde{A} + R - \tilde{A}^{\rm T}PB_0(B_0^{\rm T}PB_0)^{-1}B_0^{\rm T}P\tilde{A}, \quad (32)$$

with $P \geq I_n$, where $\tilde{A} \in \mathcal{A}$ and $R \in \mathbb{R}^{n\times n}$ is positive definite. Furthermore, let $A_{\rm s} \triangleq A + B_0K_{\rm g2}$, where $K_{\rm g2} \triangleq -(B_0^{\rm T}PB_0)^{-1}B_0P\tilde{A}$, and let $\varepsilon, \tilde{\gamma} \in \mathbb{R}$ be such that $\varepsilon > 0$ and $\tilde{\gamma} > 1$ satisfy

$$\tilde{R} \triangleq \tfrac{1}{\varepsilon}(1 - \tfrac{1}{\tilde{\gamma}})I_m - \alpha^2 B_0^{\rm T}PB_0 \geq 0. \quad (33)$$

Then the adaptive feedback control law

$$u(k) = M^{-1}(k)K(k)x(k), \quad (34)$$

where $K(k) \in \mathbb{R}^{m\times n}$ and $M_1(k)$ and $M_2(k)$ satisfy (12) at each $k \in \mathbb{N}_0$, with the quantizer (3) and the update law

$$\begin{aligned} K(k+1) \;=\; K(k) - \tfrac{\alpha^{-1}\tilde{\gamma}^{-1}}{1+x^{\rm T}(k)Px(k)}B_0^{\rm T}[x(k+1) \\ - A_{\rm s}x(k)]x^{\rm T}(k), \quad K(0) = K_0, \end{aligned} \quad (35)$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\rm g})$, where $K_{\rm g} \in \mathbb{R}^{m\times n}$, of the closed-loop system given by (1), (34), and (35) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. The result is a direct consequence of Theorem 2.1. Specifically, since Q in (13) is an arbitrary positive-definite matrix with $\lambda_{\max}(Q) < 1$, it can be replaced by $\alpha^{-1}\gamma^{-1}|B_{\rm s}| = \alpha^{-1}\gamma^{-1}(B_{\rm s}^2)^{\frac{1}{2}} \leq \gamma^{-1}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, the proof follows from the Schur decomposition. □

3. Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the proposed discrete-time adaptive control framework in the face of input quantization. Specifically, consider the linear uncertain system given by

$$\begin{aligned} z(k+2) + a_1 z(k+1) + a_0 z(k) = bv(k), \\ z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathbb{N}_0, \end{aligned} \quad (36)$$

where $a_0, a_1, b \in \mathbb{R}$ are unknown constants, $z(k) \in \mathbb{R}$, and $v(k) \in \mathbb{R}$ is to be quantized. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (36) can be written in state space form (1) with $x = [x_1, x_2]^{\rm T}$, $A = \left[\begin{array}{cc} 0 & 1 \\ -a_0 & -a_1 \end{array}\right]$, and $B = [0, b]^{\rm T}$. Here, we further assume that $\operatorname{sgn} b$ is known and $|b| < \alpha = 1$. Next, let $K_{\rm g1} = \frac{1}{b}\left[\theta_{{\rm n}_1} + a_0, \theta_{{\rm n}_2} + a_1\right]$, where $\theta_{{\rm n}_1}$, $\theta_{{\rm n}_2}$ are arbitrary scalars, so that $\tilde{A} = \left[\begin{array}{cc} 0 & 1 \\ \theta_{{\rm n}_1} & \theta_{{\rm n}_2} \end{array}\right]$. Now, it follows from Corollary 2.1 that the adaptive feedback controller (34) along with the quantizer (2) and the update law (35) guarantees that $x(k) \to 0$ as $k \to \infty$. Specifically,

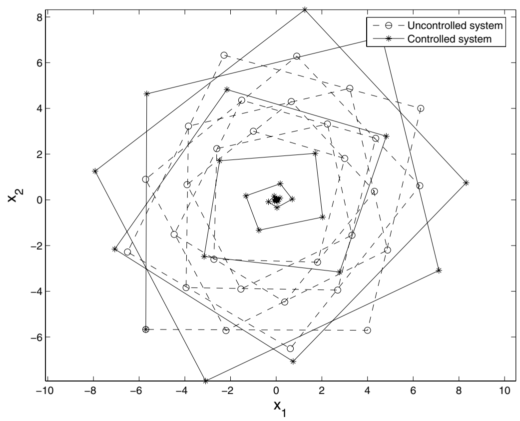


Figure 3.4: Phase portrait of controlled and uncontrolled system

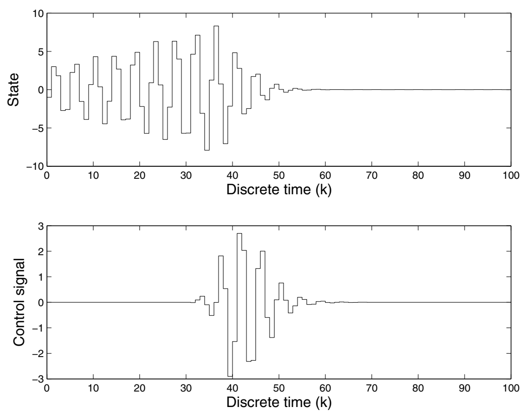


Figure 3.5: State trajectory and control signal versus time

here we choose $R = I_2$ so that P satisfying (32) is given by $P = \mathrm{diag}[1, 2]$ $(> I_2)$ (irrespective of θ_{n_1} and θ_{n_2} since R is diagonal). With $a_0 = 1.06$, $a_1 = -0.25$, $b = 0.4$, $\alpha = 1$, $\gamma = 5$, $M_1(k) \equiv 1$, $M_2(k) \in \{1+3 \cdot 1.3^j, j \in \mathbb{I}\}$, and initial conditions $x(0) = [-1, 3]^{\mathrm{T}}$ and $K(0) = [0, 0]$, Figure 3.4 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $k = 30$. Figure 3.5 shows the state trajectory versus time and the control signal versus time. Finally, Figure 3.6 shows the adaptive gain history and the profile of $M_2(k)$. It can be seen from Figure 3.6 that $M_2(k)$ remains the original value of 10 for several time steps after the controller is switched on. This implies that the required communication bit rates for control are low while the values of the adaptive gains are small.

4. Conclusion

A discrete-time direct adaptive control framework for adaptive stabilization of multivariable linear uncertain dynamical systems with input logarithmic quantizers was developed. The proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, overall closed-loop stability and attraction with respect to the plant states. Furthermore, in the case where the system is represented in controllable canonical form, the adaptive controllers can be simplified without knowledge of the system dynamics. Our control approach was not conservative in the sense that the required quantization fineness for non-

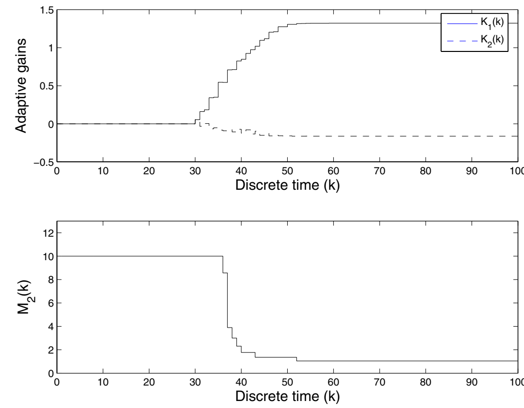


Figure 3.6: Adaptive gain history and profile of $M_2(k)$

uncertain linear systems coincides with the results presented in [3] which provides the coarsest quantizer. Future research will involve extending the discrete-time adaptive control results to the case where the number of quantization levels is finite in the neighborhood of the equilibrium point. Finally, output quantization extensions will also be considered.

References

[1] L. Bushnell (Guest Editor), "Special Section: Networks and Control," *Contr. Syst. Mag.*, vol. 21, no. 1, pp. 22–99, 2001.

[2] H. Ishii and B. A. Francis, *Limited Data Rate in Control Systems with Networks*, vol. 275 of *Lect. Notes Contr. Info. Sci.* Berlin: Springer, 2002.

[3] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Contr.*, vol. 46, no. 9, pp. 1384–1400, 2001.

[4] M. Fu and L. Xie, "On control of linear systems using quantized feedback," in *Proc. Amer. Contr. Conf.*, (Anchorage, AK), pp. 4567–4572, June 2003.

[5] M. Fu, "Robust stabilization of linear uncertain systems via quantized feedback," in *Proc. IEEE Conf. Dec. Contr.*, (Maui, HI), pp. 199–203, December 2003.

[6] G. C. Goodwin and K. S. Sin, *Adaptive filtering prediction and control*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[7] R. Johansson, "Global Lyapunov stability and exponential convergence of direct adaptive control," *Int. J. Contr.*, vol. 50, no. 3, pp. 859–869, 1989.

[8] P.-C. Yeh and P. V. Kokotović, "Adaptive control of a class of nonlinear discrete-time systems," *Int. J. Contr.*, vol. 62, pp. 303–324, 1995.

[9] M. R. Rokui and K. Khorasani, "An indirect adaptive control for fully feedback linearizable discrete-time non-linear systems," *Int. J. Adapt. Control Signal Process.*, vol. 11, pp. 665–680, 1997.

[10] R. Venugopal, V. G. Rao, and D. S. Bernstein, "Lyapunov-based backward-horizon adaptive stabilization," *Int. J. Adapt. Control Signal Process.*, vol. 17, no. 1, pp. 67–84, 2003.

[11] T. Hayakawa, W. M. Haddad, and A. Leonessa, "A Lyapunov-based adaptive control framework for discrete-time nonlinear systems with exogenous disturbances," *Int. J. Contr.*, vol. 77, pp. 250–263, 2004.

[12] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2 ed., 1996.

[13] K. J. Åström and B. Wittenmark, *Computer-Controlled Systems*. Englewood Cliffs, NJ: Prentice Hall, 1990.

[14] C.-T. Chen, *Linear System Theory and Design*. New York: Holt, Rinehart, and Winston, 1984.