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Identification of continuous-time ARX models using sample cross-covariances

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***Abstract*— The problem of estimating the parameters of a continuous-time ARX (CARX) process from discrete-time data is studied. In the proposed solution, an expression for the cross-covariance function between the input and the output signal of the CARX process is derived. This expression is parameterized by the unknown and searched parameters. The parameters are estimated by fitting the theoretical expression for the cross-covariance function to sample cross-covariances.**

I. Introduction

Mathematical models of continuous-time stochastic systems are of fundamental importance in control design for such systems, [1]–[3]. One of the most common and useful standard model structures is the CARX model

$$A(p)y(t) = B(p)u(t) + e(t), \tag{1}$$

where

$$A(p) = p^n + a_1 p^{n-1} + \ldots + a_n, \tag{2}$$

$$B(p) = b_1 p^{n-1} + \ldots + b_n, \tag{3}$$

$$\mathrm{E}\{e(t)e(s)\} = \sigma_e^2 \delta(t-s) \tag{4}$$

and where p denotes the differentiation operator, $y(t)$ is the output signal, $u(t)$ is the input signal, $e(t)$ is a continuous-time white noise source, and $\delta(\cdot)$ is the Dirac delta function. Note that the continuous-time white noise does not exist *per se*, but that the spectrum

$$\phi(\omega) = \frac{\sigma_e^2}{A(\mathrm{i}\omega)A(-\mathrm{i}\omega)} \tag{5}$$

of the noise part of the output signal $y(t)$ is modeled well, using the description (1)–(4).

The output signal is observed, together with the known input signal, at times $t = h, 2h, 3h, \ldots, Nh$, and the problem of interest in this paper to estimate the parameter vector

$$\boldsymbol{\theta} = \begin{bmatrix} a_1, \ldots, a_n, b_1, \ldots, b_n \end{bmatrix}^T \tag{6}$$

from the discrete-time data. This problem is considered in [4], where the derivatives are approximated by discrete-time differences. This leads to a linear regression and the parameter vector can be obtained by the least squares method. Since the continuous-time parameterization is kept throughout the estimation procedure and there is no need for a discrete-time model description in some intermediate step, it is an example of a *direct* approach. The following advantages hold for the approach in [4] (and for most other direct approaches):

- Good numerical properties for fast sampling, [5], [6].
- Easy applicability for the case of non-uniform sampling, [7].
- Computational efficiency.

However, the choice of derivative approximation is crucial for the approach to give unbiased estimates, [8], [9], see also [10] for an analysis. This is the main disadvantage of direct approaches.

The other class of methods for identifying continuous-time models are the *indirect* approaches. These are characterized by the following steps:

1) Transform the continuous-time model into a discrete-time model, [11], [12].
2) Estimate the parameters in the discrete-time model.
3) Map the estimated discrete-time model parameters onto the searched continuous-time model parameters.

An advantage with the indirect approaches is that a variety of powerful estimation techniques, such as the prediction error method, [13], [14], are available for the estimation in Step 2. However, the main drawback of the indirect approaches is the mapping in Step 3. The problem is that no closed-form expression for the mapping of the zeros exists, [15]–[17]. Another drawback is that the transformation in Step 1 may give rise to numerical problems for small sampling intervals h, since the continuous-time poles tend to $z = 1$ when $h \to 0$, [18], [19]. A comprehensive overview of both indirect and direct approaches is found in [17].

The idea presented in this paper is to first find an expression for the cross-covariance function between the input and output signal of the CARX process. This expression is parameterized by the continuous-time parameters $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$. The parameters are then determined by fitting the parameterized cross-covariance function to a cross-covariance function estimated from discrete-time data. The proposed approach is a direct approach in the sense that the continuous-time parameterization is kept through the whole identification procedure. Techniques similar to the one proposed here are found in the literature for discrete-time systems. For example, suboptimal methods based on sample covariances for estimating the parameters of discrete-time ARMA process parameters, are found in [20], [21] and the references therein.

State space descriptions for the CARX process are given in the next section. Thereafter, the cross-covariance function

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between the input signal $u(t)$ and the output signal $y(t)$ is derived. The estimation is described in Section IV, and sampling of a continuous-time stochastic model is described in Section V. The proposed approach is illustrated by means of numerical studies in Section VI, followed by conclusions and a discussion about future work in the last two sections. In the paper, the notational convention $\mathbf{0}_{(\ell_1|\ell_2)}$ is used for a zero matrix of dimension $(\ell_1|\ell_2)$.

II. State space representations

A state space description of the CARX process (1)–(4) is given in this section. First, an input signal is represented on state space form. This representation is needed for the state space representation of the output signal of the CARX process.

A. The input signal

Assume that the process (1) operates in open loop and that $u(t)$ is given by a continuous-time ARMA (CARMA) process

$$\begin{aligned}&\left(p^{m_1}+\alpha_1 p^{m_1-1}+\ldots+\alpha_{m_1}\right)u(t)\\&\quad=\left(\gamma_0 p^{m_2}+\gamma_1 p^{m_2-1}+\ldots+\gamma_{m_2}\right)v(t),\end{aligned} \tag{7}$$

$$\mathrm{E}\{v(t)v(s)\}=\sigma_v^2\delta(t-s), \tag{8}$$

where

$$m_1\geq m_2+n. \tag{9}$$

Introduce a state space representation for $u(t)$as

$$\dot{\mathbf{z}}(t)=\mathbf{F}_u\mathbf{z}(t)+\mathbf{g}_u v(t), \tag{10}$$

$$u(t)=\mathbf{h}_u^T\mathbf{z}(t). \tag{11}$$

One possible choice of $\mathbf{F}_u$, $\mathbf{g}_u$ and $\mathbf{h}_u$ in (10)–(11) is

$$\underset{(m_1|m_1)}{\mathbf{F}_u}=\begin{bmatrix}-\alpha_1 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1\\ -\alpha_{m_1} & & & \end{bmatrix}, \tag{12}$$

$$\underset{(m_1|1)}{\mathbf{g}_u}=\left[\mathbf{0}_{(1|m_1-m_2-1)},\gamma_0,\ldots,\gamma_{m_2}\right]^T, \tag{13}$$

$$\underset{(1|m_1)}{\mathbf{h}_u}=\left[1,\mathbf{0}_{(1|m_1-1)}\right]^T \tag{14}$$

which gives an observable canonical form.

The input signal must not necessarily be chosen as a CARMA signal. However, the identification approach presented in this paper presupposes that a state space description of the input signal is given. The choice of a CARMA signal is motivated by the fact that by varying $\{\alpha_i\}_{i=1}^{m_1}$ and $\{\gamma_i\}_{i=0}^{m_2}$, this general signal description can describe many different input signals.

B. The output signal

The output signal $y(t)$ of the CARX process of order n, given by (1), can be represented in state space form as

$$\dot{\mathbf{x}}(t)=\mathbf{F}\mathbf{x}(t)+\mathbf{g}_1 v(t)+\mathbf{g}_2 e(t), \tag{15}$$

$$y(t)=\mathbf{h}^T\mathbf{x}(t), \tag{16}$$

where

$$\mathbf{x}(t)=\begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}, \tag{17}$$

$$\mathbf{x}_1(t)=\left[p^{n-1}y(t),\ldots,y(t)\right]^T, \tag{18}$$

$$\mathbf{x}_2(t)=\mathbf{z}(t), \tag{19}$$

and $\mathbf{F}$, $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{h}$ are given by

$$\underset{(n+m_1|n+m_1)}{\mathbf{F}}=\begin{bmatrix}\mathbf{F}_{11} & \mathbf{F}_{12}\\ \mathbf{F}_{21} & \mathbf{F}_{22}\end{bmatrix}, \tag{20}$$

$$\underset{(n|n)}{\mathbf{F}_{11}}=\begin{bmatrix}-a_1 & \cdots & \cdots & -a_n\\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \tag{21}$$

$$\underset{(n|m_1)}{\mathbf{F}_{12}}=\begin{bmatrix}b_1 & \cdots & b_n\\ & \mathbf{0}_{(n-1|n)} & \end{bmatrix}\begin{bmatrix}\mathbf{h}_u^T\mathbf{F}_u^{n-1}\\ \vdots\\ \mathbf{h}_u^T\mathbf{F}_u\\ \mathbf{h}_u^T\end{bmatrix}, \tag{22}$$

$$\mathbf{F}_{21}=\mathbf{0}_{(m_1|n)}, \tag{23}$$

$$\underset{(m_1|m_1)}{\mathbf{F}_{22}}=\mathbf{F}_u, \tag{24}$$

$$\underset{(n+m_1|1)}{\mathbf{g}_1}=\left[\mathbf{0}_{(1|n)},\mathbf{g}_u^T\right]^T, \tag{25}$$

$$\underset{(n+m_1|1)}{\mathbf{g}_2}=\left[1,\mathbf{0}_{(1|n+m_1-1)}\right]^T, \tag{26}$$

$$\underset{(n+m_1|1)}{\mathbf{h}}=\left[\mathbf{0}_{(1|n-1)},1,\mathbf{0}_{(1|m_1)}\right]^T. \tag{27}$$

It should be emphasized that the state space representations chosen here are of course not the only possible choices. The state space forms are used in the next section for finding an expression for the cross-covariance function between the input and output signal.

III. The cross-covariance function

The covariance matrix $\mathbf{R_x}$ of the state vector $\mathbf{x}(t)$ is given from a continuous-time Lyapunov equation, [12],

$$\mathbf{F}\mathbf{R_x}+\mathbf{R_x}\mathbf{F}^T+\begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2\end{bmatrix}\begin{bmatrix}\sigma_v^2 & 0\\ 0 & \sigma_e^2\end{bmatrix}\begin{bmatrix}\mathbf{g}_1^T\\ \mathbf{g}_2^T\end{bmatrix}=0. \tag{28}$$

The covariance function $\boldsymbol{\mathcal{R}}_{\mathbf{x}}(\tau)$ of $\mathbf{x}(t)$ is evaluated, using the solution to (15), as

$$\begin{aligned}\boldsymbol{\mathcal{R}}_{\mathbf{x}}(\tau)&=\mathrm{E}\{\mathbf{x}(t+\tau)\mathbf{x}^T(t)\}\\&=\mathrm{E}\{e^{\mathbf{F}\tau}\mathbf{x}(t)\\&\quad+\int_t^{t+\tau}e^{\mathbf{F}(t+\tau-s)}\begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2\end{bmatrix}\begin{bmatrix}v(s)\\ e(s)\end{bmatrix}\mathrm{d}s\}\mathbf{x}^T(t)\\&=e^{\mathbf{F}\tau}\mathbf{R_x}.\end{aligned} \tag{29}$$

This gives, together with (16) and

$$u(t) = \boldsymbol{\kappa}^T \mathbf{x}(t), \tag{30}$$

where

$$\boldsymbol{\kappa} = \left[\mathbf{0}_{(1|n)}, \mathbf{h}_u^T\right]^T, \tag{31}$$

that the cross-covariance function $r_{yu}(\tau)$ between $y(t)$ and $u(t)$ can be expressed as

$$r_{yu}(\tau, \boldsymbol{\theta}) = \mathrm{E}\{y(t+\tau)u^T(t)\} = \mathbf{h}^T \boldsymbol{\mathcal{R}}_{\mathbf{x}}(\tau)\boldsymbol{\kappa}. \tag{32}$$

Here, the dependency on the parameter vector $\boldsymbol{\theta}$ in (6) is emphasized. The theoretical expression (32) for the cross-covariance function is used in the next section, together with sample cross-covariances, for estimating the parameter vector $\boldsymbol{\theta}$.

IV. Estimation

Assume that the output signal $y(t)$ of a stationary CARX process is observed, together with the known input signal $u(t)$, at the sampling instants $t = h, \dots, Nh$. A natural estimator of the cross-covariance function $r_{yu}(\tau)$ in (32) is then

$$\hat{r}_{yu}(\tau) = \frac{1}{N-\tau} \sum_{t=1}^{N-\tau} (y(t+\tau) - \hat{m}_y)(u(t) - \hat{m}_u), \tag{33}$$

where t and τ, $\tau \geq 0$, are multiple integers of the sampling interval h, and

$$\hat{m}_y = \frac{1}{N}\sum_{t=1}^{N} y(t), \qquad \hat{m}_u = \frac{1}{N}\sum_{t=1}^{N} u(t). \tag{34}$$

Assume that $\hat{r}_{yu}(\tau)$ and $r_{yu}(\tau, \boldsymbol{\theta})$ are available for $\tau = 0, \dots, \tau_{\text{max}}$, and define the loss function

$$V(\boldsymbol{\theta}) = \sum_{\tau=0}^{\tau_{\text{max}}} \big(\hat{r}_{yu}(\tau) - r_{yu}(\tau, \boldsymbol{\theta})\big)^2 \tag{35}$$

from which an estimate $\hat{\boldsymbol{\theta}}$ is obtained as

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} V(\boldsymbol{\theta}). \tag{36}$$

Instead of defining a criterion function based on the cross-covariances, an alternative is to consider

$$J(\boldsymbol{\theta}) = \sum_{\tau=0}^{\tau_{\text{max}}} \big(\hat{r}_{y}(\tau) - r_{y}(\tau, \boldsymbol{\theta})\big)^2, \tag{37}$$

i.e., a criterion function based on covariances of $y(t)$, from which an estimate $\hat{\boldsymbol{\theta}}$ is obtained as

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}). \tag{38}$$

Here,

$$r_{y}(\tau, \boldsymbol{\theta}) = \mathrm{E}\{y(t+\tau)y^T(t)\} = \mathbf{h}^T \boldsymbol{\mathcal{R}}_{\mathbf{x}}(\tau)\mathbf{h}, \tag{39}$$

and

$$\hat{r}_{y}(\tau) = \frac{1}{N-\tau} \sum_{t=1}^{N-\tau} (y(t+\tau) - \hat{m}_y)(y(t) - \hat{m}_y), \tag{40}$$

where t and τ, $\tau \geq 0$, are multiple integers of the sampling interval h. However, it can be argued that the loss function (35) may be a better choice for estimation problems, such as this, where an input signal is present. The reason is that the cross-correlation between a known signal $u(t)$ and a measured signal $y(t)$ is estimated in $\hat{r}_{yu}(\tau)$, which is later used in (35). This is in contrast to (37), where the estimated correlation function $\hat{r}_y(\tau)$ between different samples of a measured signal $y(t)$ is used. An interpretation of this is that the information about the known input signal is better utilized in (35) than in (37). Of course, for systems where no input signal is present, the only alternative is to consider loss functions similar to (37). This is done for discrete-time stochastic processes in [20] and [21], and for continuous-time stochastic signals in [22].

V. Sampling

The generation of discrete-time data from a CARX process is described in this section. The results presented here are used when generating data for the numerical study in Section VI.

Let the sampling instants be $t = h, \dots, Nh$. Integration of the process (15) over one sampling period gives

$$\begin{aligned} \mathbf{x}(kh+h) &= e^{\mathbf{F}h}\mathbf{x}(kh) \\ &\quad + \int_{kh}^{kh+h} e^{\mathbf{F}(kh+h-s)} \begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2\end{bmatrix} \begin{bmatrix} v(s) \\ e(s)\end{bmatrix} \mathrm{d}s \\ &\triangleq \mathbf{F}_d \mathbf{x}(kh) + \mathbf{w}(kh) \end{aligned} \tag{41}$$

with k being an integer. The random variable $\mathbf{w}(kh)$ is discrete-time white noise with covariance matrix

$$\begin{aligned} \mathbf{C}_w &= \mathrm{E}\{\mathbf{w}(kh)\mathbf{w}^T(kh)\} = \ldots = \\ &= \int_0^h e^{\mathbf{F}s} \begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2\end{bmatrix} \begin{bmatrix}\sigma_v^2 & 0 \\ 0 & \sigma_e^2\end{bmatrix} \begin{bmatrix}\mathbf{g}_1^T \\ \mathbf{g}_2^T\end{bmatrix} e^{\mathbf{F}^T s}\mathrm{d}s. \end{aligned} \tag{42}$$

The model (41) is therefore a standard discrete-time stochastic state space model and it has the same covariance function, measured at multiples of the sampling interval, as the original continuous-time process (15). It also holds that the discrete-time spectral density tends to the continuous-time spectral density as the sampling interval tends to zero. More material on sampling of stochastic processes can be found in [11] and [12].

VI. Numerical studies

Data are generated after instantaneous and exact sampling, as described in Section V, of a second order CARX process

$$\left(p^2 + a_1 p + a_2\right) y(t) = \left(b_1 p + b_2\right) u(t) + e(t), \tag{43}$$

where $e(t)$ is a continuous-time white noise source with $\sigma_e^2 = 1$, and the input signal $u(t)$ is given by a continuous-time AR(3) process

$$\left(p^3 + \alpha_1 p^2 + \alpha_2 p + \alpha_3\right) u(t) = \gamma_0 v(t) \tag{44}$$

with $v(t)$ being a continuous-time white noise source with $\sigma_v^2 = 1$. All parameters, except γ_0, are chosen equal to 2. The parameter γ_0 is used to obtain different values of the signal-to-noise ratio (SNR), as defined below. The parameters in (43) are estimated MC= 100 times in four different Monte Carlo studies. In these studies, the influence of the following four different quantities on the quality of the estimates, are studied in Examples 1–4, respectively:

1) The sampling interval h.
2) The SNR, defined as

$$\text{SNR} = \frac{\text{E}\{y_u^2(t)\}}{\text{E}\{y_e^2(t)\}}, \tag{45}$$

where $y_u(t)$ and $y_e(t)$ are defined in

$$y(t) = \frac{B(p)}{A(p)}u(t) + \frac{1}{A(p)}e(t) \triangleq y_u(t) + y_e(t), \tag{46}$$

where the two terms in (46) are independent.

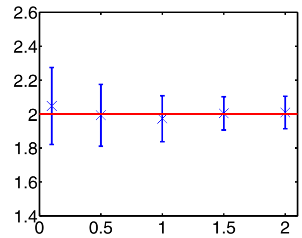
3) The maximum value τ_{max} for which the cross-covariances are considered in the loss function (35).
4) The total measurement time Nh.

Example 1: The influence of h. The sampling interval h is varied in the interval $[0.1, 2]$, with $N = 20000$, SNR $= 25$ dB, and $\tau_{\text{max}} = 50$. The mean values and the empirical standard deviations for the estimates of a_1 and b_1 are shown in Fig. 1. The estimates of a_2 and b_2 have similar properties as the estimates of a_1 and b_1, and are therefore not shown. From the simulations, it is evident that h must be chosen in a rather narrow interval, if parameter estimates of low variance are to be obtained. This is in agreement with the results in [23], which show that the Cramér-Rao lower bound (CRB) is changing drastically with h. For this particular system, h should ideally be chosen somewhere in the interval $[1, 2]$.

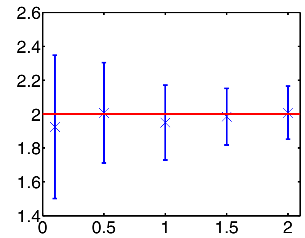
To illustrate the accuracy in the estimation of the cross-covariance function between $y(t)$ and $u(t)$, the MC= 100 estimates from the Monte Carlo study are plotted together with the true cross-covariance function, for $h = 0.5$, in Fig. 2. It is seen from the graph that the cross-covariance function is estimated with high accuracy.

Example 2: The influence of the SNR. The SNR is varied from -1 dB to 35 dB, by changing γ_0 in (44). Here, $N = 20000$, $h = 1.5$, and $\tau_{\text{max}} = 50$. The mean values and the empirical standard deviations for the estimates of a_1 and b_1 are shown in Fig. 3. The quality of the estimates are not affected much by the SNR. One possible explanation is that the input signal is persistently exciting of high order. As a consequence, the output signal contains a lot of valuable information about the system to be identified, even for lower SNRs.

Example 3: The influence of τ_{max}. The maximum value τ_{max} is varied in the interval $[3, 25]$, with $h = 0.5$, $N = 20000$, and SNR $= 25$ dB. The mean values and the empirical standard deviations for the estimates of a_1 and b_1 are shown in Fig. 4. The best results, both in terms of low bias and low variance, are obtained for $\tau_{\text{max}} = 10$. If τ_{max} is chosen small, the parameter estimates are based on few, but accurately estimated cross-covariances. However, a larger number of estimated cross-covariances might be needed to gather enough information about the system to be identified. But the estimates of the cross-covariance for large τ:s are less accurate than for small τ:s, since they are based on shorter data sets. Therefore, this extra information might not be beneficial for the quality of the parameter estimates.



(a) a_1



(b) b_1

Fig. 1. The mean values and the empirical standard deviations for the estimates of (a) a_1, and (b) b_1, as functions of h. Here, MC= 100, $N = 20000$, SNR $= 25$ dB, and $\tau_{\text{max}} = 50$. The true parameter values are indicated with solid, horizontal lines.

It is also important to stress that the choice of τ_{max} is strongly connected to the choices of N and h.

Example 4: The influence of Nh. In a practical scenario, the total measurement time Nh can be limited. It is therefore relevant to study not only the individual effects of N or h separately, but also how N and h should be chosen for a given value of Nh. Here, $Nh = 2000$, h is varied in the interval $[0.1, 1.5]$, and N is chosen so that $Nh = 2000$. Moreover, SNR= 25 dB, and $\tau_{\text{max}} = 20$. The mean values and the empirical standard deviations for the estimates of a_1 and b_1 are shown in Fig. 5. The unambiguous conclusion from the simulation is that the shortest sampling interval in the range studied here should be chosen. An obvious motivation for this choice is that it gives the largest data set for the estimation of the cross-covariance function.

VII. Conclusions

The problem of estimating the parameters in a CARX model from discrete-time data was studied. Here, the problem was solved by using the cross-covariance function between the input and output signal. More exactly, an expression for the cross-covariance function was first derived. This expression, which is parameterized by the searched parameters, was then compared with sample cross-covariances in a criterion function. The parameter estimates were finally obtained by minimizing the criterion function.

The proposed solution keeps the continuous-time parameterization throughout the identification process. However, in contrast to most direct approaches for identification

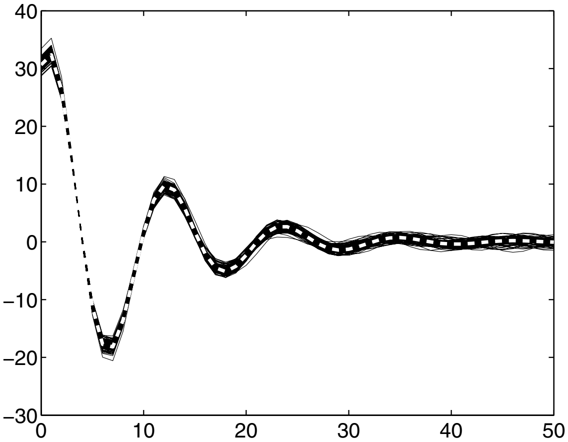
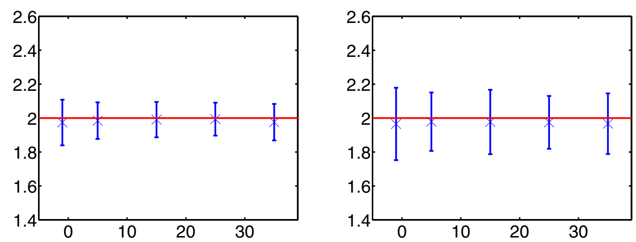
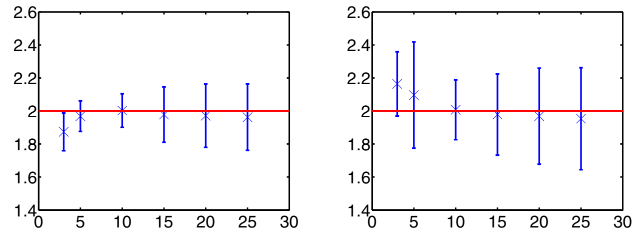


Fig. 2. The MC= 100 estimates of the cross-covariance function between $y(t)$ and $u(t)$ (solid lines), and the true cross-covariance function (dashed line), for $N = 20000$, $h = 0.5$, SNR = 25 dB, as functions of τ.



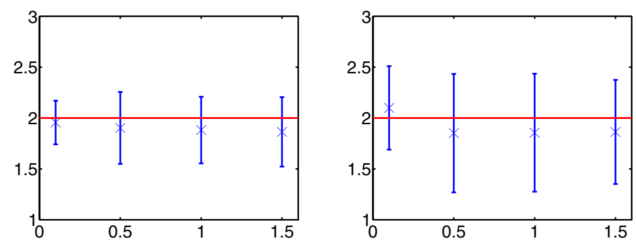
(a) a_1 (b) b_1

Fig. 3. The mean values and the empirical standard deviations for the estimates of (a) a_1, and (b) b_1, as functions of the SNR. Here, MC= 100, $N = 20000$, $h = 1.5$, and $\tau_{\text{max}} = 50$. The true parameter values are indicated with solid, horizontal lines.



(a) a_1 (b) b_1

Fig. 4. The mean values and the empirical standard deviations for the estimates of (a) a_1, and (b) b_1, as functions of τ_{max}. Here, MC= 100, $N = 20000$, $h = 0.5$, and SNR= 25 dB. The true parameter values are indicated with solid, horizontal lines.



(a) a_1 (b) b_1

Fig. 5. The mean values and the empirical standard deviations for the estimates of (a) a_1, and (b) b_1, as functions of h, when Nh is fixed as $Nh \equiv 2000$. Here, MC= 100, SNR= 25 dB, and $\tau_{\text{max}} = 20$. The true parameter values are indicated with solid, horizontal lines.

of continuous-time models, there is no need to construct derivatives of signals from discrete-time data. Instead, the information in the discrete-time data is taken in form of robustly estimated sample cross-covariances.

VIII. FUTURE WORK

The derivation of an expression for the covariance matrix of the estimated parameters is planned for future work. Such an expression can be valid either for a large number of data N or for a high SNR. It would be interesting to compare such an expression with the CRB, which is available in [23]. The expression could also be compared with the covariance matrix derived in [24]. There, the estimates given by a direct approach were the differentiation operator in (1) is replaced by an approximation, e.g., the delta operator, is analyzed.

Another issue for the future is to extend the technique proposed in this paper for recursive identification purposes. A third issue is to extend the technique to multiple-input-multiple-output (MIMO) CARX models.

REFERENCES

[1] H. J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd ed. New York: Springer-Verlag, 2001.

[2] V. Arnăutu and P. Neittaanmäki, *Optimal Control from Theory to Computer Programs*. Dordrecht: Kluwer Academic Publishers, 2003.

[3] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer-Verlag, 1999.

[4] T. Söderström, H. Fan, B. Carlsson, and S. Bigi, "Least squares parameter estimation of continuous-time ARX models from discrete-time data," *IEEE Trans. on Automatic Control*, vol. 42, pp. 659–673, May 1997.

[5] G. C. Goodwin, R. H. Middleton, and H. V. Poor, "High-speed digital signal processing and control," *IEEE Proc.*, vol. 80, no. 2, pp. 240–259, 1992.

[6] G. Li and M. Gevers, "Data filtering, reparametrization, and the numerical accuracy of parameter estimators," in *Proc. 31st IEEE Conf. Decision and Control*, vol. 4, Tucson, AZ, December 16–18 1992, pp. 3692–3697.

[7] E. K. Larsson and T. Söderström, "Identification of continuous-time AR processes from unevenly sampled data," *Automatica*, vol. 38, no. 4, pp. 709–718, 2002.

[8] T. Söderström, H. Fan, B. Carlsson, and M. Mossberg, "Some approaches on how to use the delta operator when identifying continuous-time processes," in *Proc. 36th IEEE Conf. Decision and Control*, vol. 1, San Diego, CA, December 10–12 1997, pp. 890–895.

[9] H. Fan, T. Söderström, M. Mossberg, B. Carlsson, and Y. Zou, "Estimation of continuous-time AR process parameters from discrete-time data," *IEEE Trans. on Signal Processing*, vol. 47, no. 5, pp. 1232–1244, May 1999.

[10] T. Söderström and M. Mossberg, "Performance evaluation of methods for identifying continuous-time autoregressive processes," *Automatica*, vol. 36, no. 1, pp. 53–59, January 2000.

[11] K. J. Åström, *Introduction to Stochastic Control Theory*. New York: Academic Press, 1970.

[12] T. Söderström, *Discrete-Time Stochastic Systems*, 2nd ed. London: Springer-Verlag, 2002.

[13] L. Ljung, *System Identification*, 2nd ed. Upper Saddle River, NJ, USA: Prentice–Hall, 1999.

[14] T. Söderström and P. Stoica, *System Identification*. Hemel Hempstead, U.K.: Prentice Hall, 1989.
[15] B. Wahlberg, "Limit results for sampled systems," *Int. J. Control*, vol. 48, pp. 1267–1283, March 1988.
[16] ——, "The effects of rapid sampling in system identification," *Automatica*, vol. 26, no. 1, pp. 167–170, 1990.
[17] E. K. Larsson, "Identification of stochastic continuous-time systems: Algorithms, irregular sampling and Cramér-Rao bounds," Ph.D. dissertation, Uppsala University, February 2004.
[18] E. K. Larsson and M. Mossberg, "On possibilities for estimating continuous-time ARMA parameters," in *Proc. 13th IFAC Symp. System Identification (SYSID)*, Rotterdam, The Netherlands, August 27–29 2003, pp. 641–646, invited session.
[19] E. K. Larsson, M. Mossberg, and T. Söderström, "Practical aspects of continuous-time ARMA system identification," Dept. of Electrical Engineering, Karlstad Univ., Karlstad, Sweden, Tech. Rep. EE-2004-01, 2004.
[20] S. P. Bruzzone and M. Kaveh, "Information tradeoffs in using the sample autocorrelation function in ARMA parameter estimation," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 32, no. 4, pp. 701–715, August 1984.
[21] B. Porat and B. Friedlander, "Bounds on the accuracy of Gaussian ARMA parameter estimation methods based on sample covariances," *IEEE Trans. on Automatic Control*, vol. 31, no. 6, pp. 579–582, June 1986.
[22] M. Mossberg, "Parameter estimation in continuous-time stochastic signals using covariance functions," in *Proc. 23rd IASTED Int. Conf. on Modelling, Identification and Control*, Grindelwald, Switzerland, February 23–25 2004, pp. 187–192.
[23] E. K. Larsson, M. Mossberg, and T. Söderström, "The Cramér-Rao lower bound for estimation of continuous-time ARX parameters from irregularly sampled data," in *Proc. 16th IFAC World Congress*, Prague, Czech Republic, July 4–8 2005, invited session.
[24] M. Mossberg and T. Söderström, "System identification by approximating the differentiation operator—accuracy analysis and results," in *Proc. European Control Conf.*, Karlsruhe, August 31–September 3 1999, invited session.