2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Output Feedback Stabilization of Nonlinear Feedforward Systems using Arbitrarily Bounded Control

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Abstract— **In this paper, we consider the *output feedback stabilization* problem of a class of nonlinear feedforward (upper triangular) systems, of which the stabilization using an arbitrarily bounded controller has not been addressed previously. This stabilization is accomplished here via nested saturation linear control and dynamic compensator. The novelty in such an approach is that the maximum controller value can be set prior to designing the control, whereas in similar design schemes the maximum controller level is required to be sufficiently small.**

I. Background and Motivation

In this paper we develop a constructive methodology to globally stabilize a nonlinear feedforward system using *output feedback*. The system to be stabilized is of the form

$$\Sigma_1 \begin{cases} \dot{x}_1 = x_2 + f_1(x_3, \ldots, x_n, u) \\ \dot{x}_2 = x_3 + +f_2(x_4, \ldots, x_n, u) \\ \vdots \\ \dot{x}_{n-2} = x_{n-1} + f_{n-2}(x_n, u) \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = u \\ y = x_1 \end{cases}$$

where the nonlinear terms are assumed to be Lipschitz in x and vanish at the origin. Briefly stated, systems of the form of Σ_1 cannot be stabilized by *output feedback* using existing stabilization techniques for feedforward systems while maintaining an arbitrary control bound.

The problem of stabilization of feedforward systems is well known in the nonlinear controls research community and has received considerable attention in recent years. A fairly comprehensive literature review was given in [9]; however, for the sake of completeness, a brief overview of results will be addressed. In particular, state feedback controllers using nested saturation designs [1] have been presented in [3]-[7], [11], while a controller designed using the recursive technique of forwarding was presented in [8]. Forwarding and nested saturation designs were combined in [10] to alleviate restrictive growth conditions. Previous state feedback nested saturation designs such as [3], [4], [6], [7] necessitate the nonlinear term be of higher order, thus requiring the saturation levels to be sufficiently small. This not only severely limits the linear operating range of the controller, but makes the designs very susceptible to additive disturbances and modeling errors. Additionally when the initial state values are relatively large and the controller saturated, the rate of change is a very small linear constant of time inherited from the saturation level, thus substantially increasing correction time. [11] considered locally Lipschitz systems with a switching scheme controller. However, the saturation levels are again assumed to be sufficiently small to accommodate the locally Lipschitz definition, thereby also allowing the designs to be open to disturbance errors leaving the controller unable to stabilize the system. State feedback forwarding schemes [8], on the other hand, are known to be quite complex in their construction and at times necessitate numerical approximations of nonlinear ODE's providing a difficult stabilization path. Such inherent complexities do not appear to lend themselves well to *output feedback stabilization* designs. [9] presented a linearization technique for feedforward systems which makes use of a forwarding-type state feedback controller and eliminates the need to solve a series of nonlinear ODE's, though a new need is encountered in the computation of a series of integrals. As far as the authors know, the issue of *output feedback stabilization* of nonlinear feedforward systems has only been addressed in [12] where an unbounded high-gain scaling controller was developed; in [5], where marginally stable free dynamics were restrictively assumed; and in [2], where the aforementioned forwarding approach is applied, though the form of the final control law and its bound are unknown.

Herein we develop a controller inherently more robust to additive disturbances than existing nested saturation designs since the saturation levels are not restricted to be small, thus greatly expanding the linear operating range of our control design. And since nested saturation is used a bounded controller is guaranteed which is quite straightforward in its construction. The novelty in this approach, as opposed to existing designs, is that the maximum controller value can either be arbitrarily selected or given as a design constraint.

II. Global Output Feedback Stabilization

In this section, it is shown that nonlinear feedforward systems of the form of Σ_1 can be globally asymptotically stabilized using *output feedback*. This is first accomplished

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by designing an observer and proving the error dynamics are asymptotically stable independent of the controller used.

Lemma 2.1: The states of Σ_1 can be asymptotically estimated by output feedback.

Proof. The observer implemented is of the form

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \frac{1}{L}a_1(x_1-\hat{x}_1) + f_1(\hat{x}_3,\ldots,\hat{x}_n,u) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + \frac{1}{L^2}a_2(x_1-\hat{x}_1) + f_2(\hat{x}_4,\ldots,\hat{x}_n,u) \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + \frac{1}{L^{n-1}}a_{n-1}(x_1-\hat{x}_1) \\
\dot{\hat{x}}_n &= u + \frac{1}{L^n}a_n(x_1-\hat{x}_1) \qquad (1)
\end{aligned}$$

where $L > 1$ is a constant to be determined later and $a_j > 0$, $j = 1,\ldots,n$, are coefficients of the Hurwitz polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Defining $\varepsilon_i = \frac{x_i - \hat{x}_i}{L^{n-i}}$, $i = 1,\ldots,n$, it is straightforward to verify that

$$\dot{\varepsilon} = \frac{1}{L}A\varepsilon + \begin{bmatrix} \frac{1}{L^{n-1}}(f_1-\hat{f}_1) \\ \vdots \\ \frac{1}{L^2}(f_{n-2}-\hat{f}_{n-2}) \\ 0 \\ 0 \end{bmatrix} = \frac{1}{L}A\varepsilon + \psi \qquad (2)$$

where $\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$ and $A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}$.

Since A is a Hurwitz matrix by construction, the existence of a positive definite symmetric matrix P such that $A^TP + PA = -I$ is implicit. The time derivative of the Lyapunov function $V = \varepsilon^T P \varepsilon$ yields

$$\begin{aligned}
\dot{V} &= \frac{1}{L}\varepsilon^T(A^TP + PA)\varepsilon + 2\varepsilon^T P\psi \\
&= -\frac{1}{L}\|\varepsilon\|^2 + 2\varepsilon^T P\psi.
\end{aligned}$$

Examining an arbitrary component of ψ,

$$\left|\frac{1}{L^{n-i}}(f_i(\cdot) - \hat{f}_i(\cdot))\right| \le \frac{1}{L^{n-i}}k_i \left\| \begin{matrix} L^{n-i-2}\varepsilon_{i+2} \\ L^{n-i-3}\varepsilon_{i+3} \\ \vdots \\ L\varepsilon_{n-1} \\ \varepsilon_n \end{matrix} \right\|$$

$$\le \frac{1}{L^{n-i}}k_i \left\|(L^{n-i-2}\varepsilon_{i+2}, L^{n-i-2}\varepsilon_{i+3},\cdots,L^{n-i-2}\varepsilon_n)^T\right\|$$

$$\le k_i\|\varepsilon\|/L^2 \qquad (3)$$

since $L < L^2 < \cdots < L^{n-i-2}$. Taking into account (3), the Lyapunov function under consideration becomes

$$\begin{aligned}
\dot{V} &\le -\frac{1}{L}\|\varepsilon\|^2 + \frac{2}{L^2}\varepsilon^T Pk\|\varepsilon\| \\
&\le -\frac{1}{L}\|\varepsilon\|^2 + \frac{2k}{L^2}\|P\|\|\varepsilon\|^2.
\end{aligned}$$

Choosing $L > 2k\|P\|$, $k = max\{k_i\}$, will guarantee that the Lyapunov function is negative definite and the error dynamics go to zero asymptotically, thereby proving $\hat{x} \to x$ asymptotically. ∎

Definition 2.1: Given a positive constant M_i, a function $\sigma_i : \mathbb{R} \to \mathbb{R}$ is said to be a *simple saturation* if it is a continuous, nondecreasing function with $\sigma_i(s) = s$ for all $|s| < M_i$ and $\sigma_i(s) = M_i sign(s)$ for all $|s| \ge M_i$.

As in [3], a change of coordinates will be utilized, albeit with a simpler form.

Theorem 2.1: Systems of the form Σ_1 are globally stabilizable by output feedback using a nested saturation controller of the form

$$\begin{aligned}
u = -\sigma_n(K_n z_n + \epsilon_n + \lambda_{n-1}\sigma_{n-1}(K_{n-1}z_{n-1} + \epsilon_{n-1} + \\
\cdots + \lambda_1\sigma_1(K_1 z_1 + \epsilon_1))\cdots)
\end{aligned}$$

where $z = T\hat{x}$ is an invertible linear coordinate transformation given by $z_i = \hat{x}_i + \hat{x}_n$ and $z_n = \hat{x}_n$, λ_i and K_j are positive constants, and ϵ_j represents perturbations due to modeling errors, uncertainties, or deviations in the output feedback implementation, $i = 1,\ldots,n-1$, $j = 1,\ldots,n$. Under this construction, the saturation level of the n^{th} function, $\sigma_n(\cdot)$, can be *arbitrarily* set (not necessarily small), thus giving a pre-determined and well-defined maximum control value.

Proof. Under the linear coordinate transformation $z = T\hat{x}$ the new system is given as

$$\Sigma_2 \begin{cases}
\dot{z}_1 = z_2 - z_n + f_1(x_3,\ldots,x_n,u) + u \\
\dot{z}_2 = z_3 - z_n + f_2(x_4,\ldots,x_n,u) + u \\
\vdots \\
\dot{z}_{n-2} = z_{n-1} - z_n + f_{n-2}(x_n,u) + u \\
\dot{z}_{n-1} = z_n + u \\
\dot{z}_n = u
\end{cases}$$

Before proceeding some preliminaries need to be addressed. We must first note that since the nonlinear term is Lipschitz, then it is also bounded by a linear growth in x and therefore by a linear growth in z. It is simple to verify that since $|f_i| \le k_i\|\bar{x}_{i+2}\|$, where $\bar{x}_{i+2} = [x_{i+2} \ \cdots \ x_n]^T$, then $|f_i| \le c_{f_i}(|x_{i+2}| + \cdots + |x_n|)$, and $|f_i| \le c_{f_i}(|z_{i+2}| + \cdots + |z_{n-1}|) + \gamma|z_n|)$, for some positive constants c_{f_i}, γ. In designing the gain values we first let $p_1 = 1$, $p_2 = 3$, and $p_i = max\{c_{f_{i-2}},\ldots,c_{f_1}\} + 2$, $i = 3,\ldots,n$. Define $K_n = p_n$ and

$$K_i = p_i/(\lambda_{n-1}\cdots\lambda_i), \qquad i = 1,\ldots,n-1. \qquad (4)$$

Choose $\lambda_{n-1} = 2p_n$, let $\lambda_{n-2} = 2p_{n-1}$ and

$$\lambda_i = \frac{2p_{j+1}}{\lambda_{i+1}}, \qquad i = 1, \ldots, n-3. \tag{5}$$

Note that the gains are selected in such a way that at any certain level, $\lambda_{n-1}\lambda_i = \lambda_{n-1}(2p_{i+1}) > 2p_{i+1} > p_{i+1} \geq \lambda_{n-1}K_{i+1}$.

For all saturation levels, select

$$M_{i-1} < \frac{1}{4\lambda_{i-1}} M_i, \qquad i = 1, \ldots, n \tag{6}$$

and suppose $|\epsilon_i| < \frac{\lambda_1}{4n} M_1 \ \forall i = 1, \ldots, n$.

The proof is iterative in its approach and is accomplished by first considering the nth state $z_n = x_n$. Define $\Omega_n = \{z_n : K_n|z_n| \leq \frac{1}{2}M_n\}$, where M_n is the controller maximum value. As can be seen from the state equation $\dot{z}_n = u$, the evolution of z_n depends solely on the controller. Suppose $z_n \notin \Omega_n$ and note that $|\epsilon_n| < \frac{1}{16}M_{n-1}$. In the instance when $K_n z_n > \frac{1}{2}M_n$, $K_n z_n + \epsilon_n + \lambda_{n-1}\sigma_{n-1}(\cdot) > \frac{1}{2}M_n - \frac{1}{16}M_{n-1} - \lambda_{n-1}M_{n-1} = b_n > 0$, since by definition $|\sigma_{n-1}(\cdot)| \leq M_{n-1}$ and from (6) $M_{n-1} < \frac{1}{4\lambda_{n-1}}M_n$. Therefore, $\dot{z}_n = -\sigma_n(K_n z_n + \epsilon_n + \lambda_{n-1}\sigma_{n-1}(\cdot)) \leq -\sigma_n(b_n)$ which shows that the trajectory of z_n is a strictly decreasing function of time in this range, giving that $z_n \in \Omega_n$ in finite time.

For the case when $K_n z_n < -\frac{1}{2}M_n$, $K_n z_n + \epsilon_n + \lambda_{n-1}\sigma_{n-1}(\cdot) < -\frac{1}{2}M_n + \frac{1}{16}M_{n-1} + \lambda_{n-1}M_{n-1} = -b_n < 0$. So, $\dot{z}_n = -\sigma_n(K_n z_n + \epsilon_n + \lambda_{n-1}\sigma_{n-1}(\cdot)) \geq \sigma(b_n)$ which shows that the trajectory of z_n is a strictly increasing function of time in this range, giving that $z_n \in \Omega_n$ in finite time. Thus, after some time T_1, $z_n \in \Omega_n$ and stays thereafter. The state equations for z_n and z_{n-1} are now

$$\dot{z}_n = -K_n z_n - \epsilon_n - \lambda_{n-1}\sigma_{n-1}(\cdot) \tag{7}$$

$$\dot{z}_{n-1} = -(K_n - 1)z_n - \epsilon_n - \lambda_{n-1}\sigma_{n-1}(\cdot) \tag{8}$$

where $\sigma_{n-1}(\cdot) = \sigma_{n-1}(K_{n-1}z_{n-1} + \epsilon_{n-1} + \lambda_{n-2}\sigma_{n-2}(\cdot))$. Define $\Omega_{n-1} = \{z_{n-1} : K_{n-1}|z_{n-1}| \leq \frac{1}{2}M_{n-1}\}$ and suppose $z_{n-1} \notin \Omega_{n-1}$. Note $|\epsilon_{n-1}| < \frac{1}{16}M_{n-2}$.

When $K_{n-1}z_{n-1} > \frac{1}{2}M_{n-1}$, $K_{n-1}z_{n-1} + \epsilon_{n-1} + \lambda_{n-2}\sigma_{n-2}(\cdot) > \frac{1}{2}M_{n-1} - \frac{1}{16}M_{n-2} - \lambda_{n-2}M_{n-2} = b_{n-1} > 0$, since by definition $|\sigma_{n-2}(\cdot)| \leq M_{n-2}$ and from (6) $M_{n-2} < \frac{1}{4\lambda_{n-2}}M_{n-1}$. Therefore, $\dot{z}_n = -K_n z_n - \epsilon_n - \lambda_{n-1}\sigma_{n-1}(\cdot)) \leq -K_n z_n - \epsilon_n - \lambda_{n-1}\sigma_{n-1}(b_{n-1})$ which shows that after some time τ_1, z_n is approximately less than or equal to $\frac{-\epsilon_n}{K_n} - \frac{\lambda_{n-1}}{K_n}\sigma_{n-1}(b_{n-1})$. The trajectory for z_{n-1} is now governed by $\dot{z}_{n-1} \leq \frac{-\epsilon_n}{K_n} + (K_n - 1)\frac{1}{K_n}\sigma_{n-1}(b_{n-1}) - \lambda_{n-1}\sigma_{n-1}(b_{n-1}) = -c_{n-1}$ for some positive constant c_{n-1}, since ϵ_n can be shown to satisfy $|\epsilon_n| < |b_{n-1}|$. As before, this implies that $z_{n-1} \in \Omega_{n-1}$ in finite time.

A similar argument can be made for the instance when $K_{n-1}z_{n-1} < -\frac{1}{2}M_{n-1}$, thus proving that after some time $T_2 > T_1 + \tau_1$, $z_{n-1} \in \Omega_{n-1}$ and remains thereafter. The controller equation is now

$$u = -p_n z_n - \epsilon_n - p_{n-1}z_{n-1} - \lambda_{n-1}\epsilon_{n-1} - \lambda_{n-1}\lambda_{n-2}\sigma_{n-2}(\cdot) \tag{9}$$

As with z_n, after some time τ_2, z_{n-1} can be approximated by $\frac{-\epsilon_n}{\lambda_{n-1}K_{n-1}} - \frac{\epsilon_{n-1}}{K_{n-1}} - \frac{\lambda_{n-2}}{K_{n-1}}\sigma_{n-2}(\cdot)$, while z_n can now be approximated by $\frac{\lambda_{n-1}\epsilon_{n-1}}{K_n} \in \Omega_n$.

Examining the state equation for z_{n-2} yields

$$\begin{aligned} \dot{z}_{n-2} &= z_{n-1} - z_n + f_{n-2} + u \\ &\leq z_{n-1} - z_n + c_{f_{n-2}}|z_n| + u \\ &= -(p_{n-1} - 1)z_{n-1} - c_n z_n - \lambda_{n-1}\lambda_{n-2}\sigma_{n-2}(\cdot) \\ &\qquad - \epsilon_n - \lambda_{n-1}\epsilon_{n-1} \end{aligned}$$

where c_n is a positive constant. Substituting in the approximations for z_n and z_{n-1} gives $z_{n-2} = -c_{\epsilon_n}\epsilon_n - c_{\epsilon_{n-1}}\epsilon_{n-1} - c_{n-2}\sigma_{n-2}(\cdot)$ for constants c_{ϵ_n} and $c_{\epsilon_{n-1}}$ and a positive constant c_{n-2} and the previous arguments can be recycled. $n - 3$ repetitions of this process yields a linear controller with all states proven to be globally stable, i.e. in Ω_i. ∎

Remark 2.1: In the case when the modeled perturbations ϵ_i are vanishing, then all the states of Σ_1 are proven to be globally asymptotically stable.

Remark 2.2: If state feedback stabilization is the control goal for systems of the form of Σ_1, then the dynamic equation for x_i can be allowed to contain nonlinear terms with x_{i+1} in them under the restriction that the term must either vanish at $u = 0$ or at $(x_{i+1}, 0, \ldots, 0)$. For output feedback, however, this term cannot be present.

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