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Finite-time Control of Linear Time-Varying Systems via Output Feedback

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***Abstract*— This paper deals with various finite-time analysis and design problems for continuous-time time-varying linear systems. We present some necessary and sufficient conditions for finite-time stability and then we turn to the design problem. In this context, we consider both the state feedback and the output feedback problems. For both cases, we end up with some sufficient conditions involving Linear Differential Matrix Inequalities.**

I. Introduction

The concept of finite time control dates back to the Sixties, when the idea of finite time stability (FTS) was introduced in the control literature [6], [3]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval.

It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and vice versa. While LAS deals with the behaviour of a system within a sufficiently long (in principle infinite) time interval, FTS is a more practical concept, useful to study the behaviour of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the state variables do not exceed a given threshold (for example to avoid saturations or the excitation of nonlinear dynamics) during the transients.

FTS in the presence of exogenous inputs leads to the concept of finite-time boundedness (FTB). In other words a system is said to be FTB if, given a bound on the initial condition and a characterization of the set of admissible inputs, the state variables remain below the prescribed limit for all inputs in the set.

FTS and FTB are open loop concepts. The finite-time control problem concerns the design of a linear controller which ensures the FTS or the FTB of the closed loop system. Sufficient conditions for finite-time stabilization in the presence of constant disturbances and zero reference input have been provided in [2] for the state feedback case.

In the recent paper [1] the dynamic output feedback problem has been converted into a LMIs based optimization problem. The drawback of this last approach is that the controller design is performed in two phases; the first step is devoted to design a static state feedback controller, then a state observer which tries to retain the properties guaranteed by the state feedback controller is synthesized. Therefore this approach looks at a subset of the class of admissible dynamic output feedback controllers.

In contrast to the previous literature, the methodology proposed in this paper is based on time-varying Lyapunov functions. Using this approach, we first give some necessary and sufficient conditions for FTS of a time-varying linear system. Then we find a more "tractable", from a computational point of view, sufficient condition, which is expressed in terms of a linear differential matrix inequality with initial and terminal bounds.

Afterwards, we consider the state feedback and the output feedback design problems. Both cases have been solved making use of the proposed approach, ending up with linear differential matrix inequalities.

The paper is divided as follows: in Section II we give some definitions on FTS and FTB and we state the problems that we want to solve. Section III is devoted to the analysis conditions for FTS and FTB, whereas in Section IV we present some sufficient conditions for the design of state feedback and output feedback controllers guaranteeing FTS and/or FTB.

II. Problem Statement

The following definitions deal with various finite-time control problems.

Definition 1 (Finite-time stability (FTS)): Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, and a positive definite symmetric matrix function $\Gamma(t)$ defined over $[0, T]$, the time-varying linear system

$$\dot{x}(t) = A(t)x(t)\,, \quad x(0) = x_0 \tag{1}$$

is said to be finite-time stable (FTS) with respect to $(c_1, c_2, T, \Gamma(t))$, if

$$x_0^T\Gamma(0)x_0 \le c_1 \Rightarrow x(t)^T\Gamma(t)x(t) < c_2 \quad \forall t \in [0, T]\,. \tag{2}$$

◊

Remark 1: Lyapunov Asymptotic Stability (LAS) and FTS are independent concepts: a system which is FTS may be not LAS; conversely a LAS system could be not FTS if, during the transients, its state exceeds the prescribed bounds. △

Definition 2 (Finite-time boundedness (FTB)): Given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix function $\Gamma(t)$ defined over $[0, T]$, and a class of signals $\mathcal{W}$, the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(0) = x_0 \tag{3}$$

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is said to be finite-time bounded with respect to $(c_1, c_2, \mathcal{W}, T, \Gamma(t))$ if

$$x_0^T \Gamma(0) x_0 \le c_1 \Rightarrow x(t)^T \Gamma(t) x(t) < c_2 \quad \forall t \in [0, T] ,$$

for all $w(\cdot) \in \mathcal{W}$. $\diamondsuit$

Remark 2: An important difference between LAS and FTB relies in the fact that, for linear systems, LAS is a structural property of the system, depending only on the system matrix, while FTB clearly depends on the kind and amplitude of the inputs acting on the system. $\triangle$

Note that FTS and FTB refer to open loop systems. The next problem puts together FTS and FTB in the design context; all matrices and vectors are assumed to be of compatible dimensions.

Problem 1 (Finite-Time Control via State Feedback): Consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t) , \quad x(0) = x_0 . \quad (4)$$

where $u(t)$ is the control input and $w(t)$ is the disturbance. Then, given three positive scalars c_1, c_2, T, with $c_1 < c_2$, a positive definite symmetric matrix function $\Gamma(t)$ defined over $[0, T]$, and the class of signals $\mathcal{W}$ find a state feedback controller in the form

$$u(t) = K(t)x(t) , \quad (5)$$

such that the closed loop system obtained by the connection of (4) and (5), namely

$$\dot{x}(t) = (A(t) + B(t)K(t))\, x(t) + G(t)w(t) ,\ x(0) = x_0 , \quad (6)$$

is FTB with respect to $(c_1, c_2, \mathcal{W}, T, \Gamma(t))$. $\diamondsuit$

Problem 2 (Finite-Time Control via Output Feedback): Consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t) , \quad x(0) = x_0 \quad (7a)$$

$$y(t) = C(t)x(t) + H(t)w(t) \quad (7b)$$

where $u(t)$ is the control input, $w(t)$ is the disturbance and $y(t)$ is the output. Then, given three positive scalars c_1, c_2, T, with $c_1 < c_2$, two positive definite symmetric matrix functions $\Gamma(t), \Gamma_K(t)$ defined over $[0, T]$, and a class of signals $\mathcal{W}$, find a dynamic output feedback controller in the form

$$\dot{x}_c(t) = A_K(t)x_c(t) + B_K(t)y(t) \quad (8a)$$

$$u(t) = C_K(t)x_c(t) + D_K(t)y(t) , \quad (8b)$$

where $x_c(t)$ has the same dimension of $x(t)$, such that the closed loop system obtained by the connection of (7) and (8) is FTB with respect to $(c_1, c_2, \mathcal{W}, T, \text{blockdiag}(\Gamma(t), \Gamma_K(t)))$. $\diamondsuit$

Obviously, by letting $w(t) = 0$ in Problems 1 and 2, we obtain the corresponding finite-time stabilization problems.

III. MAIN RESULTS: ANALYSIS

A. Finite-Time Stability

The following Theorem gives two necessary and sufficient conditions for FTS of system (1) and one sufficient condition.

Theorem 1: The following statements are equivalent

$i)$ System (1) is FTS wrt $(c_1, c_2, T, \Gamma(t))$.

$ii)$ For all $t \in [0, T]$

$$\Phi(t,0)^T \Gamma(t) \Phi(t,0) < \frac{c_2}{c_1} \Gamma(0) ,$$

where $\Phi(t, 0)$ is the state transition matrix.

$iii)$ For all $t \in [0, T]$ the differential Lyapunov inequality with terminal and initial conditions

$$\dot{P}(\tau) + A(\tau)^T P(\tau) + P(\tau)A(\tau) < 0 , \quad \tau \in]0, t] \quad (9a)$$

$$P(t) \ge \Gamma(t) \quad (9b)$$

$$P(0) < \frac{c_2}{c_1} \Gamma(0) , \quad (9c)$$

admits a piecewise continuously differentiable symmetric solution $P(\cdot)$.

Moreover the following condition is sufficient for FTS.

$iv)$ The differential Lyapunov inequality

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) < 0 \quad (10a)$$

$$P(t) \ge \Gamma(t) , \quad \forall t \in [0, T] \quad (10b)$$

$$P(0) < \frac{c_2}{c_1} \Gamma(0) , \quad (10c)$$

admits a piecewise continuously differentiable symmetric solution $P(\cdot)$.

Proof: $\boxed{ii) \Longrightarrow i)}$ Let $x_0^T \Gamma(0) x_0 \le c_1$. Then

$$x(t)^T \Gamma(t) x(t) = x_0^T \Phi(t,0)^T \Gamma(t) \Phi(t,0) x_0 < \frac{c_2}{c_1} x_0^T \Gamma(0) x_0 < c_2 .$$

Therefore system (1) is FTS.

$\boxed{i) \Longrightarrow ii)}$ By contradiction. Let us assume that for some $\bar{t}$, $\bar{x}$

$$\bar{x}^T \Phi(\bar{t},0)^T \Gamma(\bar{t}) \Phi(\bar{t},0) \bar{x} \ge \frac{c_2}{c_1} \bar{x}^T \Gamma(0) \bar{x} . \quad (11)$$

Now let

$$x(0) = \lambda \bar{x}$$

where λ is such that

$$x(0)^T \Gamma(0) x(0) = c_1 .$$

Then (11) implies that

$$x(0)^T \Phi(\bar{t},0)^T \Gamma(\bar{t}) \Phi(\bar{t},0) x(0) \ge c_2 .$$

Therefore

$$x(\bar{t})^T \Gamma(\bar{t}) x(\bar{t}) = x(0)^T \Phi(\bar{t},0)^T \Gamma(\bar{t}) \Phi(\bar{t},0) x(0) \ge c_2 ,$$

which contradicts the initial assumption that system (1) be FTS.

$\boxed{iii) \Longrightarrow i)}$ Let $V(t,x) = x^T P(t) x$. Then (9a) implies that $\dot{V}(t,x)$ is negative definite along the trajectories of system (1). Now let $x_0^T \Gamma(0) x_0 \le c_1$; then for a generic t

$$x(t)^T \Gamma(t) x(t) \le x(t)^T P(t) x(t) < x(0)^T P(0) x(0) < \frac{c_2}{c_1} x(0)^T \Gamma(0) x(0) \le c_2$$

$\boxed{i) \Longrightarrow iii)}$ Let us assume that system (1) is FTS. Then by continuity arguments, letting $z = \epsilon x$ for a small ϵ, for all $t \in [0,T]$

$$x(0)^T \Gamma(0) x(0) \le c_1 \Rightarrow x(t)^T \Gamma(t) x(t) + \|z\|_2^2 < c_2 \quad (12)$$

Let $P(\cdot)$ be the solution of

$$\dot{P}(\tau) + A(\tau)^T P(\tau) + P(\tau) A(\tau) = -\epsilon^2 I \,, \quad (13a)$$
$$P(t) = \Gamma(t) \,, \quad (13b)$$

and assume that for some $\bar{x}$

$$\bar{x}^T P(0) \bar{x} \ge \frac{c_2}{c_1} \bar{x}^T \Gamma(0) \bar{x} \,. \quad (14)$$

Now let

$$x(0) = \lambda \bar{x}$$

where λ is such that

$$x(0)^T \Gamma(0) x(0) = c_1 \,.$$

Then (14) implies

$$x(0)^T P(0) x(0) \ge c_2$$

From (13a) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)^T P(\tau) x(\tau) = -\epsilon^2 x(\tau)^T x(\tau) \quad (15)$$

Integrating (15) from 0 to t we have

$$x(t)^T P(t) x(t) - x(0)^T P(0) x(0) = -\epsilon^2 \|x\|_2^2 \,.$$

Therefore

$$x(t)^T \Gamma(t) x(t) \ge x(t)^T P(t) x(t) = x(0)^T P(0) x(0) - \epsilon^2 \|x\|_2^2 \ge c_2 - \|z\|_2^2 \,,$$

which contradicts (12).

$\boxed{iv) \Longrightarrow iii)}$ It is straightforward to check that a matrix function P satisfying conditions (10) also satisfies (9). ■

Remark 3: Note that condition ii) is useful for the analysis; however it cannot be used for design purposes. On the other hand condition iii) requires to check infinitely many linear differential matrix inequalities. Therefore the starting point for the solution of the design problems will be condition iv). △

B. Finite-Time Boundedness

The following Theorem gives a sufficient condition for FTB of system (3), given a specified class of external disturbances.

Theorem 2: Consider the following class of signals

$$\mathcal{W} := \left\{ w(\cdot) \,|\, w(\cdot) \in \mathcal{L}^2([0,T]) \,, \int_0^T w(\tau)^T w(\tau) \, \mathrm{d}\tau \le d \right\}$$

where $\mathcal{L}^2([0,T])$ is the set of square integrable vector-valued functions in $[0,T]$ and d is a positive scalar. Then system (3) is FTB wrt to $(c_1, c_2, \mathcal{W}, T, \Gamma(t))$ if there exists a symmetric matrix-valued function $P(\cdot)$ such that

$$\dot{P}(t) + A(t)^T P(t) + P(t) A(t) + \frac{c_1 + d}{c_2} P(t) G(t) G(t)^T P(t) < 0 \quad (16a)$$
$$P(t) \ge \Gamma(t) \,, \quad \forall t \in [0,T] \quad (16b)$$
$$P(0) < \frac{c_2}{c_1 + d} \Gamma(0) \,. \quad (16c)$$

Proof: Let

$$\alpha := \frac{c_1 + d}{c_2} \,. \quad (17)$$

Using (16a) and (3) it is easy to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} x^T P x < -\alpha x^T P G G^T P x + w^T G^T P x + x^T P G w = \frac{1}{\alpha} w^T w - \left(\frac{w}{\sqrt{\alpha}} - \sqrt{\alpha} G^T P x \right)^T \left(\frac{w}{\sqrt{\alpha}} - \sqrt{\alpha} G^T P x \right) . \quad (18)$$

Now integrating (18) we have

$$x(t)^T P(t) x(t) - x(0)^T P(0) x(0) < \frac{1}{\alpha} \int_0^t w^T w \, \mathrm{d}t - \int_0^t \left(\frac{w}{\sqrt{\alpha}} - \sqrt{\alpha} G^T P x \right)^T \left(\frac{w}{\sqrt{\alpha}} - \sqrt{\alpha} G^T P x \right) \mathrm{d}t \le \frac{1}{\alpha} \int_0^T w^T w \, \mathrm{d}t = \frac{1}{\alpha} d \,, \qquad \forall t \in [0,T] \,.$$

Now let $x_0^T \Gamma(0) x_0 \le c_1$; then for a generic t

$$x(t)^T \Gamma(t) x(t) \le x(t)^T P(t) x(t) < x(0)^T P(0) x(0) + \frac{1}{\alpha} d < \frac{1}{\alpha} \left(x(0)^T \Gamma(0) x(0) + d \right) \le c_2 \,, \forall t \in [0,T] \,.$$

Therefore system (3) is FTB. ■

IV. Controller design

First we consider the state feedback case.

Theorem 3: Problem 1 is solvable if there exist a symmetric matrix-valued function $Q(\cdot)$ and a matrix function

$L(\cdot)$ such that

$$-\dot{Q}(t)+A(t)Q(t)+Q(t)A(t)^T + L(t)^TB(t)^T+B(t)L(t)+\frac{c_1+d}{c_2}G(t)G(t)^T<0 \quad (19a)$$

$$Q(t)\le\Gamma^{-1}(t)\,,\quad \forall t\in[0,T] \quad (19b)$$

$$Q(0)>\frac{c_1+d}{c_2}\Gamma^{-1}(0)\,. \quad (19c)$$

In this case a controller gain which solves Problem 1 is $K(t)=L(t)Q^{-1}(t)$.

Proof: From Theorem 2 it follows that Problem 1 admits a solution if there exist a symmetric matrix function $P(\cdot)$ and a matrix function $K(\cdot)$ such that

$$\dot{P}(t)+(A(t)+B(t)K(t))^TP(t)+P(t)(A+BK(t)) + \frac{c_1+d}{c_2}P(t)G(t)G(t)^TP(t)<0 \quad (20a)$$

$$P(t)\ge\Gamma(t)\,,\quad \forall t\in[0,T] \quad (20b)$$

$$P(0)<\frac{c_2}{c_1+d}\Gamma(0)\,. \quad (20c)$$

Now pre- and post-multiply (20a) by $P^{-1}(t)$ and let $Q(t)=P^{-1}(t)$. Condition (19a) is obtained noticing that

$$\dot{Q}(t)=-Q(t)\dot{P}(t)Q(t)$$

and letting $L(t)=K(t)Q(t)$ according to [5]. Conditions (19b) and (19c) are easily derived from (20b) and (20c) respectively. ■

The corresponding result for finite-time stabilization can be easily derived from Theorem 3.

Theorem 4: System (4) (with $w(t)=0$) is finite-time stabilizable via state feedback with respect to $(c_1,c_2,T,\text{blockdiag}\big(\Gamma(t),\Gamma_K(t)\big))$ if there exist a symmetric matrix-valued function $Q(\cdot)$ and a matrix function $L(\cdot)$ such that

$$-\dot{Q}(t)+A(t)Q(t)+Q(t)A(t)^T + L(t)^TB(t)^T+B(t)L(t)<0 \quad (21a)$$

$$Q(t)\le\Gamma^{-1}(t)\,,\quad \forall t\in[0,T] \quad (21b)$$

$$Q(0)>\frac{c_1}{c_2}\Gamma^{-1}(0)\,. \quad (21c)$$

In this case a controller gain which solves Problem 1 is $K(t)=L(t)Q^{-1}(t)$.

Next, we consider output feedback control. First we need the following auxiliary lemma.

Lemma 1 ([4]): Given symmetric matrices $S\in\mathbb{R}^{n\times n}$ and $Q\in\mathbb{R}^{n\times n}$, the following statements are equivalent.

i) There exist a symmetric matrix $T\in\mathbb{R}^{n\times n}$ and nonsingular matrices $M\in\mathbb{R}^{n\times n}$, $N\in\mathbb{R}^{n\times n}$ such that

$$P:=\begin{pmatrix}S & M\\ M^T & T\end{pmatrix}>0\,,\qquad P^{-1}=\begin{pmatrix}Q & N\\ N^T & \star\end{pmatrix}\,, \quad (22)$$

where $\star$ denotes a 'don't care' block.

ii)

$$\begin{pmatrix}Q & I\\ I & S\end{pmatrix}>0\,. \quad (23)$$

The following is the main result of the section.

Theorem 5: Problem 2 is solvable if there exist continuously differentiable symmetric positive definite matrix-valued functions $Q(\cdot)$ and $S(\cdot)$, a nonsingular matrix-valued function $N(\cdot)$ and matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that (the time argument is omitted for brevity)

$$\begin{pmatrix}\Theta_{11} & \Theta_{12} & \Theta_{13}\\ \Theta_{12}^T & \Theta_{22} & \Theta_{23}\\ \Theta_{13}^T & \Theta_{23}^T & -I\end{pmatrix}<0 \quad (24a)$$

$$\begin{pmatrix}Q & \Psi_{12} & \Psi_{13} & \Psi_{14}\\ \Psi_{12}^T & \Psi_{22} & 0 & 0\\ \Psi_{13}^T & 0 & I & 0\\ \Psi_{14}^T & 0 & 0 & I\end{pmatrix}\ge 0\,,\quad t\in[0,T] \quad (24b)$$

$$\begin{pmatrix}Q(0) & I\\ I & S(0)\end{pmatrix}\le\frac{c_2}{c_1+d}\begin{pmatrix}\Delta_{11} & Q(0)\Gamma(0)\\ \Gamma(0)Q(0) & \Gamma(0)\end{pmatrix}\,, \quad (24c)$$

where

$$\begin{aligned}
\Theta_{11}&=-\dot{Q}+AQ+QA^T+B\hat{C}_K+\hat{C}_K^TB^T\\
\Theta_{12}&=A+\hat{A}_K^T+BD_KC\\
\Theta_{13}&=\alpha^{1/2}(G+BD_KH)\\
\Theta_{22}&=\dot{S}+SA+A^TS+\hat{B}_KC+C^T\hat{B}_K^T\\
\Theta_{23}&=\alpha^{1/2}(SG+\hat{B}_KH)\\
\Psi_{12}&=I-Q(t)\Gamma(t)\\
\Psi_{13}&=Q(t)\Gamma^{1/2}\\
\Psi_{14}&=N(t)\Gamma_K^{1/2}(t)\\
\Psi_{22}&=S(t)-\Gamma(t)\\
\Delta_{11}&=Q(0)\Gamma(0)Q(0)+N(0)\Gamma_K(0)N^T(0)
\end{aligned}$$

Proof: The connection between systems (7) and (8) reads

$$\begin{pmatrix}\dot{x}\\ \dot{x}_c\end{pmatrix}=\begin{pmatrix}A(t)+B(t)D_K(t)C(t) & B(t)C_K(t)\\ B_K(t)C(t) & A_K(t)\end{pmatrix}\begin{pmatrix}x\\ x_c\end{pmatrix} + \begin{pmatrix}G(t)+B(t)D_K(t)H(t)\\ B_K(t)H(t)\end{pmatrix}w =: A_{CL}(t)\begin{pmatrix}x\\ x_c\end{pmatrix}+G_{CL}(t)w\,, \quad (25)$$

with obvious meaning of the new symbols.

According to Theorem 2 we have that system (25) is FTB wrt $(c_1,c_2,\mathcal{W},T,\text{blockdiag}(\Gamma(t),\Gamma_K(t))$ if there exists a positive definite matrix-valued function $P(\cdot)$ such that

$$\dot{P}(t)+A_{CL}^T(t)P(t)+P(t)A_{CL}(t) + \alpha P(t)G_{CL}(t)G_{CL}^T(t)P(t) \quad <0 \quad (26a)$$

$$P(t)\ge\Gamma_{CL}(t)\,,\quad t\in[0,T] \quad (26b)$$

$$P(0)<\alpha\Gamma_{CL}(0)\,, \quad (26c)$$

where α has been defined in (17), and $\Gamma_{CL}(t)=\mathit{blockdiag}\,(\Gamma(t)\,,\Gamma_K(t))$.

Now let us define

$$P(t) = \begin{pmatrix} S(t) & M(t) \\ M^T(t) & T(t) \end{pmatrix}, \; P^{-1}(t) = \begin{pmatrix} Q(t) & N(t) \\ N^T(t) & \star \end{pmatrix},$$

$$\Pi_1(t) = \begin{pmatrix} Q(t) & I \\ N^T(t) & 0 \end{pmatrix} \quad \Pi_2(t) = \begin{pmatrix} I & S(t) \\ 0 & M^T(t) \end{pmatrix}.$$

Note that by definition

$$S(t)Q(t) + M(t)N^T(t) = I \tag{27a}$$

$$Q(t)\dot{S}(t)Q(t) + N(t)\dot{M}^T(t)Q(t) + Q(t)\dot{M}(t)N^T(t) + N(t)\dot{T}(t)N^T(t) = -\dot{Q}(t) \tag{27b}$$

$$P(t)\Pi_1(t) = \Pi_2(t). \tag{27c}$$

Now note that using Schur complements (26a) can be rewritten as

$$\begin{pmatrix} \dot{P}(t) + A_{CL}^T(t)P(t) + P(t)A_{CL}(t) & \alpha^{1/2}P(t)G_{CL}(t) \\ \alpha^{1/2}G_{CL}^T(t)P(t) & -I \end{pmatrix} < 0.$$

By pre- and post-multiplying the last inequality by $\mathrm{blockdiag}(\Pi_1^T(t), I)$ and $\mathrm{blockdiag}(\Pi_1(t), I)$ respectively, pre- and post-multiplying (26b) and (26c) by $\Pi_1^T(t)$ and $\Pi_1(t)$ respectively, taking into account (27) and Lemma 1 the proof follows once we let (time is omitted for brevity)

$$\hat{B}_K = MB_K + SBD_K \tag{28a}$$

$$\hat{C}_K = C_K N^T + D_K CQ \tag{28b}$$

$$\hat{A}_K = \dot{S}Q + \dot{M}N^T + MA_K N^T + SBC_K N^T + MB_K CQ + S\big(A + BD_K C\big)Q. \tag{28c}$$

Note that (23) need not to be explicitly imposed since it is implied by (24b). ■

Remark 4: The statement of Theorem 5 requires to find a nonsingular N; this can be obtained by adding a further LMI constraint requiring positive definiteness of N for all t.

Remark 5 (Controller design): Assume now that the hypothesis of Theorem 5 are satisfied; in order to design the controller the following steps have to be followed:

i) Find $Q(\cdot)$, $S(\cdot)$, $N(\cdot)$, $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that (24) is satisfied.
ii) Find matrix function $M(\cdot)$ such that $M(t) = (I - S(t)Q(t))N^{-T}(t)$.
iii) Obtain $A_K(\cdot)$, $B_K(\cdot)$, $C_K(\cdot)$ and $D_K(\cdot)$ by inverting (28).

△

Note that (24a) and (24b) are linear differential matrix inequalities; the initial condition (24c) has to be *a posteriori* checked.

The following result, dealing with finite-time stabilization, can be readily obtained from Theorem 5.

Theorem 6: System (7) is finite-time stabilizable via dynamic output feedback wrt $(c_1, c_2, T, \mathrm{blockdiag}\big(\Gamma(t), \Gamma_K(t)\big))$, if there exist continuously differenziable symmetric positive definite matrix-valued functions $Q(\cdot)$, $S(\cdot)$, a nonsingular matrix $N(\cdot)$ and matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that (the time argument is omitted for brevity)

$$\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{pmatrix} < 0 \tag{29a}$$

$$\begin{pmatrix} Q & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{12}^T & \Psi_{22} & 0 & 0 \\ \Psi_{13}^T & 0 & I & 0 \\ \Psi_{14}^T & 0 & 0 & I \end{pmatrix} \geq 0, \quad t \in [0, T] \tag{29b}$$

$$\begin{pmatrix} Q(0) & I \\ I & S(0) \end{pmatrix} \leq \frac{c_2}{c_1} \begin{pmatrix} \Delta_{11} & Q(0)\Gamma(0) \\ \Gamma(0)Q(0) & \Gamma(0) \end{pmatrix}, \tag{29c}$$

(29d)

where the symbols have the same meaning of Theorem 5.

Note that the feasibility of (29a) is equivalent to the existence of $Q(\cdot)$, $S(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ such that

$$\Theta_{11}(t) < 0 \tag{30a}$$

$$\Theta_{22}(t) < 0. \tag{30b}$$

Indeed it is obvious that an admissible solution of (29a) satisfies also (30). Conversely, let us assume that (30) are feasible for some $Q(\cdot)$, $S(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$. Then it is clear that the same quadruple together with $D_K(t) = 0$ and $\hat{A}_K(t) = -A^T(t)$ also satisfies (29a), since the anti-diagonal terms vanish and it reduces to (30).

V. Conclusions

In this paper we have dealt with the finite-time control of linear time-varying systems. Some necessary and sufficient conditions for finite-time stability and a sufficient condition for finite-time boundedness have been provided; then we have moved to the state and output feedback problems. The proposed design conditions are expressed in terms of linear differential matrix inequalities.

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