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Robust Stabilization of Uncertain Nonlinear Systems by Nonsmooth Output Feedback

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***Abstract*— We present in this paper some new results on global robust stabilization by *nonsmooth output* feedback, for a class of n-dimensional uncertain systems that are not smoothly stabilizable. To handle the system uncertainty effectively, a rescaling transformation with a suitable dilation is introduced. The rescaling technique, together with the nonsmooth output feedback design method developed recently, leads to a robust output feedback control scheme which achieves, under a homogeneous growth condition, global stabilization for a family of uncertain nonlinear systems with unstabilizable/undetectable linearization. The design of nonsmooth state feedback controllers and homogeneous observers does not require the precise information of the uncertain system and depends only on the knowledge of the bounding system.**

I. Introduction

The purpose of this paper is to investigate the problem of global robust stabilization by nonsmooth output feedback, for a family of uncertain nonlinear systems of the form

$$
\begin{aligned}
\dot{\eta}_1 &= \eta_2^{p_1} + \phi_1(t, \eta, v) \\
&\vdots \\
\dot{\eta}_{n-1} &= \eta_n^{p_{n-1}} + \phi_{n-1}(t, \eta, v) \\
\dot{\eta}_n &= v + \phi_n(t, \eta, v) \\
y &= \eta_1,
\end{aligned}
\tag{1.1}
$$

where $v \in \mathbb{R}$, $\eta = (\eta_1, \cdots, \eta_n)^T$ and $y \in \mathbb{R}$ are the system input, state and output, respectively. For $i = 1, \cdots, n-1$, $p_i \geq 1$ is an *odd* positive integer, and $\phi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, is a C^1 mapping with $\phi_i(0, \cdots, 0) = 0$.

In [2], it was shown that system (1.1) with suitable $\phi_i(\cdot)$ is a generalized *normal form* of affine systems when exact feedback linearization is not possible. Indeed, a necessary and sufficient, coordinate-free geometric condition was given in [2] and then extended in [18], under which an affine system

$$\dot{\xi} = f(\xi) + g(\xi)u \qquad \text{and} \qquad y = h(\xi)$$

is feedback equivalent to the nonlinear system (1.1) with $\phi_i = \sum_{k=0}^{p_i-1} \eta_{i+1}^k a_{i,k}(\eta_1, \cdots, \eta_i)$.

For a nonlinear system (1.1) in the p-normal form [2], [18], global stabilization results by nonsmooth *state* feedback have been obtained under some appropriate conditions; see, for instance, [13] and the references therein. Despite the rapid development of stabilization theory by state feedback, progress in its output feedback counterpart has been slow. There are many reasons for this, such as the lack of the separation principle even assuming that the nonlinear system has a simple structure, the difficulty of designing nonlinear observers, the trouble caused by unstabilizable/undetectable linearization, etc. All of these makes the output feedback control of the nonlinear system (1.1) a difficult problem, and the conventional output feedback design methods based on "Luenberger-type" or "high-gain" observers [7], [10], [11] are no longer applicable.

In the past few years, some preliminary results have been reported on global output feedback stabilization of the nonlinear system (1.1). For example, in [15], [16] it was shown that in the two-dimensional case, global stabilization of (1.1) can be achieved by smooth or nonsmooth output feedback, if the planar system (1.1) has a strict-triangular structure and satisfies suitable growth conditions. In the higher-dimensional case, there are fewer results available in the literature. In the recent paper [20], we have addressed systematically the problem of global stabilization via smooth output feedback, for the n-dimensional system (1.1) with $\phi_i(\cdot) = 0,\ 1 \leq i \leq n$. This was accomplished by employing a new observer design technique for the construction of homogeneous observers, combined with the tool of adding an integrator for the design of smooth state feedback controllers. In addition, it was proved in [20] that global output feedback stabilization of the nonlinear system (1.1) is also possible, as long as (1.1) satisfies a high-order version of global Lipschitz-like condition. The main contribution of the paper [20] was the development of a recursive algorithm for the design of homogeneous observers, enabling one to assign the observer gains step-by-step.

There are two limitations in the work [20]: 1) it requires that $p_1 = \cdots = p_{n-1}$; 2) it cannot be used to deal with a larger class of non-uniformly observable and non-smoothly stabilizable systems. To overcome these shortcomings, a nonsmooth output feedback control scheme was proposed in [17], in which a recursive, non-smooth observer design algorithm was developed for the nonlinear system (1.1) with different $p_i's$. The non-smooth observer thus obtained, together with the nonsmooth state feedback controllers derived in [13], resulted in new results on global stabilization of a subclass of systems (1.1) with different $p_i's$. In the local case, the paper [17] also showed that, by means of the homogeneous systems theory, local stabilization of the nonlinear system (1.1) in the p-normal form is achievable by nonsmooth output feedback, without imposing any growth condition.

Although the observer designs in [20], [17] are substantially different from the traditional ones [7], [9], [10], [11], [12], they still use a copy of the original system and hence require the precise information of the controlled plant. As a result, the output feedback control scheme proposed in [20], [17] is not robust with respect to parametric uncertainty and cannot be applied to nonlinear systems with uncertainty.

Such a robust output feedback control problem was addressed in [14] for the uncertain system (1.1) with $p_1 = \cdots = p_{n-1} = 1$ (i.e., with controllable/observable linearization), and later on in [21] for the uncertain high order system (1.1) when $p_1 = \cdots = p_{n-1} > 1$, under a homogeneous growth condition [21]. In this paper, we shall study the question of when the uncertain system (1.1) with different $p_i's$ is robustly stabilizable by output feedback, and present some preliminary results. Realizing that in the case when $p_i's$ are distinct, it is usually not possible to deal with the nonlinear system (1.1) by smooth feedback, even locally, we shall

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develop a *nonsmooth,* rather than smooth, output feedback design method to tackle the robust output feedback stabilization problem for the uncertain system (1.1).

The goal of this paper is to design a nonsmooth dynamic output compensator of the form

$$\begin{aligned} \dot{\hat{x}} &= \theta(\hat{x}, y), \qquad \hat{x} \in \mathbb{R}^{n-1}, \\ v &= v(\hat{x}, y), \end{aligned} \tag{1.2}$$

achieving global robust stabilization for a family of uncertain systems (1.1).

Throughout this paper, we assume that the uncertain nonlinear system (1.1) satisfies the following condition:

Assumption 1.1: There exists a real constant $C \geq 0$ such that $\forall (t, \eta, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ $(i = 1, \cdots, n)$,

$$|\phi_i(t, \eta, v)| \leq C\Big(|\eta_1|^{1/(p_1 \cdots p_{i-1})} + \cdots + |\eta_i|^{1/p_{i-1}} + |\eta_i|\Big). \tag{1.3}$$

Clearly, Assumption 1.1 reduces to the linear growth condition in [14] when $p_1 = \cdots = p_{n-1} = 1$. In the case of $p_i > 1$, it is weaker than the condition (3.9) in [17], which is nothing but a homogeneous type of the global Lipschitz condition. Moreover, due to the growth condition (1.3), the family of nonlinear systems considered here is larger than those considered in [17], as illustrated in Examples 4.7, 5.4 and 5.5. In this work, we show that for a family of uncertain nonlinear systems (1.1) satisfying Assumption 1.1, global robust stabilization can be achieved by *nonsmooth output* feedback. To deal with the system uncertainty, we introduce a subtle rescaling transformation motivated by the work [21]. The new rescaling technique integrated with the nonsmooth output feedback design method [17] leads to a nonsmooth output feedback control scheme that solves the problem of global robust stabilization. The construction of a nonsmooth state feedback controller and a homogeneous observer relies only on the knowledge of the bounding system rather than on the uncertain system. This is a substantial difference between this paper and the work [17]. Three examples are given to illustrate the applications of the robust output feedback control scheme developed in this paper.

II. Useful Lemmas

This section collects several useful inequalities and technical lemmas that will be used frequently in the sequel.

Lemma 2.1: Given real numbers $x, y, m, n, a, b > 0$, the following inequality holds:

$$ax^m y^n \leq bx^{m+n} + \frac{n}{m+n}\Big(\frac{m+n}{m}\Big)^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} y^{m+n}.$$

Lemma 2.2: Given real numbers $x, y, m, n, a, b > 0$, the following inequality holds:

$$abx^m y^n \leq \frac{m}{m+n} a^{\frac{m+n}{m}} x^{m+n} + \frac{n}{m+n} b^{\frac{m+n}{n}} y^{m+n}.$$

Lemma 2.3: Let $x_1, \cdots, x_n, p > 0$ be real numbers. Then,

$$(x_1 + \cdots + x_n)^p \leq \max(n^{p-1}, 1)(x_1^p + \cdots + x_n^p).$$

Lemma 2.4: Suppose $p \geq 1$ is an odd integer, then the following inequality holds:

$$|a - b|^p \leq 2^{p-1}|a^p - b^p|, \quad \forall a, \ b \in \mathbb{R}.$$

Lemma 2.5: For all $x, y \in \mathbb{R}$ and any odd positive integer p, the following inequality holds:

$$|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1}).$$

The proofs of Lemmas 2.1—2.5 are not difficult and omitted due to the limit of space.

III. Robust Output Feedback Design: A Case Revisit

To tackle the problem of global robust output feedback stabilization for the uncertain nonlinear system (1.1) whose linearization is neither stabilizable nor detectable, we revisit in this section the case where $p_1 = \cdots = p_{n-1} = 1$ in (1.1). In this case, system (1.1) is simplified as

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 + \phi_1(t, \eta, v) \\ &\vdots \\ \dot{\eta}_{n-1} &= \eta_n + \phi_{n-1}(t, \eta, v) \\ \dot{\eta}_n &= v + \phi_n(t, \eta, v) \\ y &= \eta_1. \end{aligned} \tag{3.1}$$

Obviously, the linearized system of (3.1) is controllable and observable. Moreover, the homogeneous growth condition (1.3) reduces to

$$|\phi_i(t, \eta, v)| \leq C\big(|\eta_1| + \cdots + |\eta_i|\big), \quad i = 1, \cdots, n. \tag{3.2}$$

Under the linear growth condition (3.2), it has been proved in [14] that global robust stabilization of the uncertain system (3.1) can be achieved by a linear output feedback controller. The proof was done by means of a coupled controller-observer design [14] which is not based on the separation principle. However, the output feedback control scheme proposed in [14] cannot be easily extended to the nonlinear case, for instance, the case when $p_i \geq 1$, because of the linear nature of the design [14].

In what follows, we seek an alternative output feedback design method that is capable of taking advantage of homogeneity of the uncertain system (3.1). As we shall see later, the robust linear output feedback scheme developed in this section has a valid nonsmooth counterpart that turns out to be effective in dealing with a family of uncertain nonlinear systems (1.1) with $p_i \geq 1$.

To begin with, we first introduce the rescaling transformation for the uncertain system (3.1):

$$x_i = \frac{\eta_i}{M^{i-1}}, \quad i = 1, \cdots, n, \quad \text{and} \quad u = \frac{v}{M^n}, \tag{3.3}$$

where $M \geq 1$ is a rescaling constant to be assigned later.

Under the new coordinates x_i's, the uncertain system (3.1) is represented as

$$\begin{aligned} \dot{x}_1 &= Mx_2 + f_1(t, x, u) \\ &\vdots \\ \dot{x}_{n-1} &= Mx_n + f_{n-1}(t, x, u) \\ \dot{x}_n &= Mu + f_n(t, x, u) \\ y &= x_1, \end{aligned} \tag{3.4}$$

where the uncertain functions $f_i(t, x, u), \ i = 1, \cdots, n$, satisfy the linear growth condition (by (3.2))

$$\begin{aligned} |f_i(t, x, u)| = \frac{|\phi_i(t, \eta, v)|}{M^{i-1}} &\leq \frac{C(|\eta_1| + \cdots + |\eta_i|)}{M^{i-1}} \\ &\leq C\left(|x_1| + \cdots + |x_i|\right). \end{aligned} \tag{3.5}$$

For the rescaled uncertain system (3.4) with the constraints (3.5), it is easy to design recursively, in a fashion similar to [14], a linear state feedback controller

$$u^* = -\beta_n \xi_n = -(b_n x_n + \cdots + b_1 x_1), \tag{3.6}$$

such that

$$\dot{U} \leq M\Big[-3(\xi_1^2 + \cdots + \xi_n^2) + \xi_n(u - u^*)\Big], \tag{3.7}$$

where $U = \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2)$, $\xi_i = x_i - x_i^*,\ i = 1, \cdots, n$, and

$$x_1^* = 0, \quad x_2^* = -\beta_1 \xi_1, \quad \cdots, \quad x_n^* = -\beta_{n-1}\xi_{n-1}$$

with $\beta_i > 0$ and $b_i > 0$ being known constants independent of M.

Next, we shall design a linear observer for the rescaled system (3.4). Because $y = x_1$ is measurable and only unmeasurable states of (3.4) are $(x_2, \cdots, x_n)$, it is natural to design an $(n-1)$-dimensional observer rather than a full-order observer. Motivated by the reduced-order observer design for linear systems, we build an $(n-1)-th$ order linear observer to estimate, instead of the states $(x_2, \cdots, x_n)$, the unmeasurable variables $(z_2, \cdots, z_n)$ defined by

$$z_2 = x_2 - L_2 x_1, \quad \cdots \quad, z_n = x_n - L_n x_{n-1}, \tag{3.8}$$

where L_i's> 0 are gain constants to be determined later.

From (3.8) it is clear that

$$\begin{aligned} \dot{z}_2 &= Mx_3 + f_2(\cdot) - ML_2x_2 - L_2 f_1(\cdot) \\ &\vdots \\ \dot{z}_{n-1} &= Mx_n + f_{n-1}(\cdot) - ML_{n-1}x_{n-1} - L_{n-1}f_{n-2}(\cdot) \\ \dot{z}_n &= Mu + f_n(\cdot) - ML_n x_n - L_n f_{n-1}(\cdot). \end{aligned} \tag{3.9}$$

In view of (3.9), we construct the $(n-1)$-th order linear observer (regardless of the uncertain terms $f_i(\cdot), 1 \le i \le n$):

$$\begin{aligned} \dot{\hat{z}}_2 &= M\hat{x}_3 - ML_2\hat{x}_2, & \hat{x}_2 &= z_2 + L_2 x_1 \\ &\vdots & \hat{x}_3 &= z_3 + L_3\hat{x}_2 \\ \dot{\hat{z}}_{n-1} &= M\hat{x}_n - ML_{n-1}\hat{x}_{n-1}, & &\vdots \\ \dot{\hat{z}}_n &= Mu - ML_n\hat{x}_n, & \hat{x}_n &= z_n + L_n\hat{x}_{n-1}, \end{aligned} \tag{3.10}$$

where $(\hat{z}_2, \cdots, \hat{z}_n)$ and $(\hat{x}_2, \cdots, \hat{x}_n)$ are the estimates of the unmeasurable state $(z_2, \cdots, z_n)$ and $(x_2, \cdots, x_n)$.

Let $e_i = z_i - \hat{z}_i,\ 2 \le i \le n$, be the estimate errors. It follows that $x_i - \hat{x}_i = e_i + L_{i-1}e_{i-1} + \cdots + L_{i-1}\cdots L_2 e_2$. Thus, the error dynamics are

$$\begin{aligned} \dot{e}_2 &= M(e_3 + L_2e_2) + f_2(\cdot) - ML_2e_2 - L_2 f_1(\cdot) \\ &\vdots \\ \dot{e}_n &= f_n(\cdot) - ML_n(e_n + L_{n-1}e_{n-1} \\ &\qquad + \cdots + L_{n-1}\cdots L_2e_2) - L_n f_{n-1}(\cdot). \end{aligned} \tag{3.11}$$

To obtain an implementable controller, we simply replace the unmeasurable states $(x_2, \cdots, x_n)$ in (3.6) by its estimates $(\hat{x}_2, \cdots, \hat{x}_n)$, which are generated by the linear observer (3.10). In this way, we arrive at

$$u = -(b_n\hat{x}_n + \cdots + b_2\hat{x}_2 + b_1x_1). \tag{3.12}$$

Now, consider the Lyapunov function

$$V(x, e) = U(x) + \frac{1}{2}\Big(e_2^2 + \cdots + e_n^2\Big).$$

Substituting (3.12) into (3.7) and using the completion of square, it can be shown from the linear growth condition (3.5) that

$$\begin{aligned} \dot{V} \quad \le \quad & -M\bigg\{\Big[2 - \frac{L_2 + \cdots + L_n}{M}\Big](\xi_1^2 + \cdots + \xi_n^2) \\ & + \Big[L_2 - c_2(L_3, \cdots, L_n) - \frac{K_2L_2}{M}\Big]e_2^2 \\ & + \cdots + \Big[L_{n-1} - c_{n-1}(L_n) - \frac{K_{n-1}L_{n-1}}{M}\Big]e_{n-1}^2 \\ & + \Big[L_n - c_n - \frac{K_nL_n}{M}\Big]e_n^2\bigg\}, \end{aligned} \tag{3.13}$$

where $c_2(L_3, \cdots, L_n), \cdots, c_{n-1}(L_{n-1}, L_n), c_{n-1}(L_n)$ are positive constants independent of M, $c_n > 0$ and $K_i > 0, 2 \le i \le n$ are known constants independent of M and all L_i's.

Choosing the gain parameters L_i and M one-by-one, in the following manner:

$$\begin{aligned} L_n &\ge 2 + c_n \\ L_{n-1} &\ge 2 + c_{n-1}(L_n) \\ &\vdots \\ L_2 &\ge 2 + c_2(L_3, \cdots, L_n) \\ M &\ge \max\{L_2 + \cdots + L_n, K_2L_2, \cdots, K_nL_n\}, \end{aligned} \tag{3.14}$$

we have

$$\dot{V}(\cdot) \le -M\left[(\xi_1^2 + \cdots + \xi_n^2) + (e_2^2 + \cdots + e_n^2)\right].$$

Therefore, the uncertain system (3.1) is globally robustly stabilized by the output feedback controller (3.12)-(3.10).

IV. Nonsmooth Output Feedback Stabilization of Uncertain Nonlinear Systems

The robust output feedback design method presented in the last section for the uncertain system (3.1) with controllable/observable linearization can be generalized, with a suitable twist, to system (1.1). In this section, we show that under Assumption 1.1, a robust output feedback control strategy can be developed for a family of uncertain nonlinear systems (1.1) whose linearization is neither stabilizable nor detectable. In particular, with the aid of Lemmas 2.1—2.5, we can prove the following important result.

Theorem 4.1: Under Assumption 1.1, the uncertain nonlinear systems (1.1) can be globally robustly stabilized by a nonsmooth dynamic output compensator of the form (1.2).

Proof: This result will be proved by explicitly designing a nonsmooth state feedback controller and a robust homogeneous observer, both of them do not depend on the uncertain functions $\phi_i(t, \eta, v)$. Note that the construction of the nonsmooth observer is differently from the one in [17]. In particular, no copy of $\phi_i(t, \eta, v)$ is needed here for the design of a robust observer, while the observer in [17] involved a copy of $\phi_i(t, \eta, v)$. Thus, $\phi_i(t, \eta, v)$ in [17] must be time independent and precisely known. That is, no system uncertainty is allowed.

We begin with the proof by observing that according to the homogeneous system theory [3], [6], [8], system (1.1) with $\phi_i(\cdot) = 0$ is a homogeneous system with the dilation $(1, \frac{1}{p_1}, \cdots, \frac{1}{p_1\cdots p_{n-1}}; \frac{1}{p_1\cdots p_{n-1}})$ and degree 0.

With this in mind, we introduce the rescaling transformation

$$\begin{aligned} x_i &= \frac{\eta_i}{M^{q_i}}, \quad i = 1, \cdots, n, \qquad u = \frac{v}{M^{q_{n+1}}} \\ q_1 &= 0,\ q_2 = \frac{q_1 + 1}{p_1},\ \cdots,\ q_n = \frac{q_{n-1} + 1}{p_{n-1}},\ q_{n+1} = q_n + 1. \end{aligned} \tag{4.1}$$

which is a natural extension of (3.3) proposed in the last section. The rescaling transformation (4.1) has a dilation $(q_1, \cdots, q_n; q_{n+1})$, and $M \ge 1$ in (4.1) is a rescaling factor to be determined later.

In the new coordinate $(x_1, \cdots, x_n)$, the uncertain nonlinear system (1.1) is rewritten as

$$\begin{aligned} \dot{x}_1 &= Mx_2^{p_1} + f_1(t, x, u) \\ &\vdots \\ \dot{x}_{n-1} &= Mx_n^{p_{n-1}} + f_{n-1}(t, x, u) \\ \dot{x}_n &= Mu + f_n(t, x, u) \end{aligned} \tag{4.2}$$

where $f_i(\cdot) = \phi_i(\cdot)/M^{q_i},\ i = 1, \cdots, n$.

Using Assumption 1.1 and the fact that $M \geq 1$, it is easy to see from (4.1) that for $i = 1, \cdots, n$,

$$|f_i(\cdot)| \leq C(|x_1|^{1/(p_1\cdots p_{i-1})} + \cdots + |x_{i-1}|^{1/p_{i-1}} + |x_i|). \quad (4.3)$$

Next, we use the nonsmooth feedback design method [13] to derive a robust state feedback controller for the rescaled system (4.2) under the growth condition (4.3). In fact, using an inductive argument as done in [13], there are a C^1 Lyapunov function $U(x_1, \cdots, x_n)$, which is positive definite and proper, and a set of C^0 virtual controllers $x_1^*, \cdots, x_n^*, u^*$, defined by

$$\begin{array}{ll} x_1^* = 0, & \xi_1 = x_1 - x_1^* = x_1 \\ x_2^{*p_1} = -\beta_1\xi_1, & \xi_2 = x_2^{p_1} - x_2^{*p_1} = x_2^{p_1} + \beta_1 x_1 \\ \vdots & \vdots \\ x_n^{*p_1\cdots p_{n-1}} = -\beta_{n-1}\xi_{n-1}, & \xi_n = x_n^{p_1\cdots p_{n-1}} - x_n^{*\,p_1\cdots p_{n-1}} \end{array}$$

$$u^* = -\left(b_n x_n^{p_1\cdots p_{n-1}} + \cdots + b_2 x_2^{p_1} + b_1 x_1\right)^{\frac{1}{p_1\cdots p_{n-1}}} \quad (4.4)$$

with constants $\beta_i, b_i > 0$ are constants independent of M such that

$$\dot{U}(\xi) \leq M\left[-4(\xi_1^2 + \cdots + \xi_n^2) + \xi_n^{2-1/(p_1\cdots p_{n-1})}(u - u^*)\right]. \quad (4.5)$$

Because the states $(x_2, \cdots, x_n)$ are unmeasurable, the virtual controller u^* in (4.4) cannot be directly implemented. To obtain an implementable controller, a natural thing to do is to design an $(n-1)$-dimensional observer for recovering $(x_2, \cdots, x_n)$ of the rescaled system (4.2). Notably, the observer design method proposed in [17] is not applicable to the uncertain system (4.2) since it requires a copy of $f_i(\cdot), i = 1, \cdots, n$, which are unknown in the current case. Inspired by the robust observer design approach reviewed in the last section, we build an $(n-1)$-dimensional robust homogeneous observer to estimate, instead of the states $(x_2, \cdots, x_n)$, the unmeasurable variables [17]:

$$z_2 = x_2^{p_1} - L_2 x_1, \quad \cdots, \quad , z_n = x_n^{p_{n-1}} - L_n x_{n-1}, \quad (4.6)$$

where L_i's> 0 are gain parameters to be determined later.

Using (4.6), it is easy to see that

$$\begin{aligned} \dot{z}_2 &= p_1 x_2^{p_1-1}(Mx_3^{p_2} + f_2(\cdot)) - L_2(Mx_2^{p_1} + f_1(\cdot)) \\ &\vdots \\ \dot{z}_n &= p_{n-1} x_n^{p_{n-1}-1}(Mu + f_n(\cdot)) - L_n(Mx_n^{p_{n-1}} + f_{n-1}(\cdot)). \end{aligned} \quad (4.7)$$

As done in the last section, we ignore the uncertain terms $f_i(t, x, u)$ in system (4.7) and simply design the following $(n-1)$–dimensional homogeneous observer similar to [17]:

$$\begin{array}{ll} \dot{\hat{z}}_2 = -ML_2\hat{x}_2^{p_1}, & \hat{x}_2^{p_1} = z_2 + L_2 x_1 \\ \dot{\hat{z}}_3 = -ML_2\hat{x}_3^{p_2}, & \hat{x}_3^{p_2} = z_3 + L_3\hat{x}_2 \\ \vdots & \vdots \\ \dot{\hat{z}}_n = -ML_n\hat{x}_n^{p_{n-1}}, & \hat{x}_n^{p_{n-1}} = z_n + L_n\hat{x}_{n-1} \end{array} \quad (4.8)$$

where $(\hat{z}_2, \cdots, \hat{z}_n)$ and $(\hat{x}_2, \cdots, \hat{x}_n)$ are the estimates of the unmeasurable states $(z_2, \cdots, z_n)$ and $(x_2, \cdots, x_n)$.

Let $e_i = z_i - \hat{z}_i,\ i = 2, \cdots, n$, be the estimate errors. Note that

$$e_{i+1} = x_{i+1}^{p_i} - L_{i+1}x_i - \hat{x}_{i+1}^{p_i} + L_{i+1}\hat{x}_i. \quad (4.9)$$

Thus,

$$x_{i+1}^{p_i} - \hat{x}_{i+1}^{p_i} = e_{i+1} + L_{i+1}(x_i - \hat{x}_i). \quad (4.10)$$

Therefore, the error dynamics are given by

$$\begin{aligned} \dot{e}_2 =& p_1 x_2^{p_1-1}(Mx_3^{p_2} + f_2(\cdot)) - ML_2 e_2 - L_2 f_1(\cdot) \\ \dot{e}_3 =& p_2 x_3^{p_2-1}(Mx_4^{p_3} + f_3(\cdot)) \\ & - ML_3(e_3 + L_3(x_2 - \hat{x}_2)) - L_3 f_2(\cdot) \\ \vdots& \\ \dot{e}_n =& p_{n-1} x_n^{p_{n-1}-1}(Mu + f_n(\cdot)) \\ & - ML_n(e_n + L_n(x_{n-1} - \hat{x}_{n-1})) - L_n f_{n-1}(\cdot). \end{aligned} \quad (4.11)$$

Now, consider the Lyapunov function in [17]

$$V(e_2, \cdots, e_n) = \frac{e_2^2}{2} + \frac{e_3^{2p_1}}{2p_1} + \cdots + \frac{e_n^{2p_1\cdots p_{n-2}}}{2p_1\cdots p_{n-2}}, \quad (4.12)$$

which is positive definite and proper. Clearly,

$$\begin{aligned} \dot{V} =& p_{n-1} M e_n^{2p_1\cdots p_{n-2}-1} x_n^{p_{n-1}-1}(u - u^*) \\ & + p_{n-1} e_n^{2p_1\cdots p_{n-2}-1} x_n^{p_{n-1}-1}\Big(Mu^* + f_n(\cdot)\Big) \\ & + \sum_{i=2}^{n-1} p_{i-1} e_i^{2p_1\cdots p_{i-2}-1} x_i^{p_{i-1}-1}\Big(Mx_{i+1}^{p_i} + f_i(\cdot)\Big) \\ & - M\sum_{i=3}^{n} L_i^2 e_i^{2p_1\cdots p_{i-2}-1}(x_{i-1} - \hat{x}_{i-1}) \\ & - M\sum_{i=2}^{n} L_i e_i^{2p_1\cdots p_{i-2}} - \sum_{i=2}^{n} L_i e_i^{2p_1\cdots p_{i-2}-1} f_{i-1}(\cdot). \end{aligned} \quad (4.13)$$

In order to estimate the terms on the right-hand side of (4.13), we introduce the following propositions whose proofs are similar to [17] and involve tedious calculations but nevertheless can be carried out using Lemmas 2.1–2.5 and are omitted due to the limited space.

Proposition 4.2: There is a real constant $K > 0$ independent of M such that

$$\begin{aligned} &\Bigg| p_{n-1} e_n^{2p_1\cdots p_{n-2}-1} x_n^{p_{n-1}-1}\Big(Mu^* + f_n(\cdot)\Big) \\ &\quad + \sum_{i=2}^{n-1} p_{i-1} e_i^{2p_1\cdots p_{i-2}-1} x_i^{p_{i-1}-1}\Big(Mx_{i+1}^{p_i} + f_i(\cdot)\Big)\Bigg| \\ \leq\ & M\Big[K(\sum_{i=2}^{n} e_i^{2p_1\cdots p_{i-2}}) + (\xi_1^2 + \cdots + \xi_n^2)\Big]. \end{aligned} \quad (4.14)$$

Proposition 4.3: There exist constants $c_i(L_{i+1}, \cdots, L_n) > 0,\ i = 2, \cdots, n-1$, independent of M and $c_n > 0$ independent of all the L_i's and M such that

$$\begin{aligned} \left|\sum_{i=3}^{n} L_i^2 e_i^{2p_1\cdots p_{i-2}-1}(x_{i-1} - \hat{x}_{i-1})\right| &\leq c_2(L_3, \cdots, L_n)e_2^2 \\ & + \cdots + c_{n-1}(L_n)e_{n-1}^{2p_1\cdots p_{n-3}} + c_n e_n^{2p_1\cdots p_{n-2}}. \end{aligned} \quad (4.15)$$

Proposition 4.4: There exist constants $K_i > 0,\ i = 2, \cdots, n$, independent of M such that

$$\begin{aligned} \left|\sum_{i=2}^{n} L_i e_i^{2p_1\cdots p_{i-2}-1} f_{i-1}(\cdot)\right| &\leq (\sum_{i=2}^{n} K_i L_i e_i^{2p_1\cdots p_{i-2}}) \\ & + (L_2 + \cdots + L_n)(\xi_2^2 + \cdots + \xi_n^2). \end{aligned} \quad (4.16)$$

With the help of Propositions 4.2–4.4, the following inequality can be obtained

$$
\begin{aligned}
\dot{V} \quad \le \quad & M\Big\{\Big[1+\frac{L_2+\cdots+L_n}{M}\Big](\xi_1^2+\cdots+\xi_n^2) \\
& -\Big[L_2-K-c_2(L_3,\cdots,L_n)-\frac{K_2L_2}{M}\Big]e_2^2-\cdots \\
& -\Big[L_{n-1}-K-c_{n-1}(L_n)-\frac{K_{n-1}L_{n-1}}{M}\Big]e_{n-1}^{2p_1\cdots p_{n-3}} \\
& -\Big[L_n-K-c_n-\frac{K_nL_n}{M}\Big]e_n^{2p_1\cdots p_{n-2}} \\
& +p_{n-1}e_n^{2p_1\cdots p_{n-2}-1}x_n^{p_{n-1}-1}(u-u^*)\Big\}. \qquad (4.17)
\end{aligned}
$$

Now, we apply the certainty equivalence principle to obtain an implementable output feedback controller. Observe that the reduced-order observer (4.8) has provided an estimation for the unmeasurable states $(x_2,\cdots,x_n)$. Keeping this in mind, we simply replace $(x_2,\cdots,x_n)$ in the controller (4.4) by its estimate $(\hat{x}_2,\cdots,\hat{x}_n)$ generated from the observer (4.8). Thus,

$$
u=-\left(b_n\hat{x}_n^{p_1\cdots p_{n-1}}+\cdots+b_2\hat{x}_2^{p_1}+b_1x_1\right)^{\frac{1}{p_1\cdots p_{n-1}}}. \quad (4.18)
$$

The next proposition gives an estimation for the terms involving with $u-u^*$ in (4.5) and (4.17). Its proof is similar to [17] and omitted due to the limited space.

Proposition 4.5: There exist constants $\bar{c}_i(L_{i+1},\cdots,L_n)>0$, $i=2,\cdots,n-1$, independent of M and a real constant $\bar{c}_n>0$ independent of all the L_i's and M such that

$$
\begin{aligned}
& \Big|(\xi_n^{2-1/(p_1\cdots p_{n-1})}+p_{n-1}e_n^{2p_1\cdots p_{n-2}-1}x_n^{p_{n-1}-1})(u-u^*)\Big| \\
& \le (\xi_2^2+\cdots+\xi_n^2)+\bar{c}_2(L_3,\cdots,L_n)e_2^2+\cdots \\
& \quad +\bar{c}_{n-1}(L_n)e_{n-1}^{2p_1\cdots p_{n-3}}+\bar{c}_ne_{n-1}^{2p_1\cdots p_{n-2}}. \qquad (4.19)
\end{aligned}
$$

Immediately, the inequality (4.19), together with (4.5) and (4.17), yields

$$
\begin{aligned}
& \dot{U}+\dot{V}\le -M\Big\{\Big[2-\frac{L_2+\cdots+L_n}{M}\Big](\xi_2^2+\cdots+\xi_n^2) \\
& +\Big[L_2-K-c_2(L_3,\cdots,L_n)-\bar{c}_2(L_3,\cdots,L_n)-\frac{K_2L_2}{M}\Big]e_2^2 \\
& +\cdots+\Big[L_n-K-c_n-\bar{c}_n-\frac{K_nL_n}{M}\Big]e_n^{2p_1\cdots p_{n-2}}\Big\}. \qquad (4.20)
\end{aligned}
$$

From (4.20), it is clear that choosing the gain parameters L_i one-by-one, in the following manner:

$$
\begin{aligned}
L_n \quad & \ge \quad K+c_n+\bar{c}_n+2 \\
L_{n-1} \quad & \ge \quad K+c_{n-1}(L_n)+\bar{c}_{n-1}(L_n)+2 \\
& \vdots \qquad\qquad\qquad\qquad\qquad\qquad (4.21) \\
L_2 \quad & = \quad K+c_2(L_3,\cdots,L_n)+\bar{c}_2(L_3,\cdots,L_n)+2, \\
M \quad & \ge \quad \max\{L_2+\cdots+L_n,K_2L_2,\cdots,K_nL_n\},
\end{aligned}
$$

we immediately have

$$
\dot{U}+\dot{V}\le -M\Big[(\xi_1^2+\cdots+\xi_n^2)+\Sigma_{i=2}^n e_i^{2p_1\cdots p_{i-2}}\Big],
$$

which is negative definite. Therefore, the closed-loop system (1.1)-(4.8)-(4.18) is globally stable in the sense of Kurzweil [13]. ∎

As a consequence, we have the following important corollary.

Corollary 4.6: If there exists a constant $C>0$ such that

$$
|\phi_i(\cdot)|\le C(|\eta_i|+\cdots+|\eta_{i-k_i}|), \qquad i=1,\cdots,n, \quad (4.22)
$$

where $k_i=\max\{j\ge 0 \mid p_{i-1}=\cdots=p_{i-j}=1\}$, system (1.1) is globally stabilizable by nonsmooth output feedback.

Example 4.7: Consider the uncertain nonlinear system

$$
\begin{aligned}
\dot{\eta}_1 &= \eta_2 \\
\dot{\eta}_2 &= \eta_3^3+\eta_2+d(t)\ln(1+\eta_1^2) \\
\dot{\eta}_3 &= \eta_4 \\
\dot{\eta}_4 &= u+\eta_4+\eta_3 \\
y &= \eta_1, \qquad\qquad\qquad (4.23)
\end{aligned}
$$

where $d(t)$ is a C^0 time-varying parameter satisfying $|d(t)|\le 1$. It is easy to verify that (4.23) is of the form (4.22). In fact, it is not difficult to see that $k_2=k_4=C=1$. By Corollary 4.6, there is a nonsmooth output feedback controller globally stabilizing (4.23). Notably, system (4.23) is non-homogeneous and involves the time-varying parameter $d(t)$. Thus, it cannot be dealt with by the output feedback scheme suggested in [17].

V. Extension and Discussion

In this section, we discuss briefly how the output feedback stabilization result for the uncertain system (1.1) can be extended to a family of C^1 uncertain cascade systems of the form

$$
\begin{aligned}
\dot{\zeta} &= F(t,\zeta,\eta,v), \qquad \zeta\in\mathbb{R}^{n-r}, \\
\dot{\eta}_1 &= \eta_2^{p_1}+\phi_1(t,\zeta,\eta,v) \\
& \vdots \\
\dot{\eta}_{r-1} &= \eta_r^{p_{r-1}}+\phi_{r-1}(t,\zeta,\eta,v) \\
\dot{\eta}_r &= v+\phi_r(t,\zeta,\eta,v) \\
y &= \eta_1, \qquad\qquad\qquad (5.1)
\end{aligned}
$$

where $v\in\mathbb{R}$, $(\zeta,\eta)\in\mathbb{R}^n$ and $y\in\mathbb{R}$ are the system input, state and output, respectively, and $p_i\ge 1$ are *odd* integers.

To achieve global robust stabilization by nonsmooth output feedback, we make the following assumptions.

Assumption 5.1: For $i=1,\cdots,r$,

$$
\begin{aligned}
|\phi_i(t,\zeta,\eta,v)| \quad \le \quad & C(||\zeta||^{\frac{1}{p_1\cdots p_{i-1}}}+|\eta_1|^{\frac{1}{p_1\cdots p_{i-1}}} \\
& +\cdots+|\eta_{i-1}|^{\frac{1}{p_{i-1}}}+|\eta_i|). \quad (5.2)
\end{aligned}
$$

Assumption 5.2: There is a C^2 Lyapunov function $U_0(\zeta)$, which is positive definite and proper, such that

$$
\frac{\partial U_0}{\partial\zeta}F(t,\zeta,\eta,v)\le -||\zeta||^2+K_0\eta_1^2, \quad (5.3)
$$

where $K_0>0$ is a fixed known constant.

Then, the following output feedback stabilization result can be proved by means of an argument similar to that of Theorem 4.1. The details are omitted for the sake of space.

Theorem 5.3: Under Assumptions 5.1 and 5.2, the uncertain cascade system (5.1) is globally robustly stabilized by a nonsmooth output feedback controller of the form (1.2).

For the purpose of illustration, we present two examples to demonstrate how the nonsmooth output feedback control scheme developed so far can be employed to achieve global robust stabilization for uncertain nonlinear systems with *unstabilizable/undetectable* linearization.

Example 5.4: Consider the uncertain planar system

$$
\begin{aligned}
\dot{\eta}_1 &= \eta_2^3+\frac{1}{3}\eta_1e^{d(t)\sin\eta_2} \\
\dot{\eta}_2 &= v \\
y &= \eta_1, \qquad\qquad\qquad (5.4)
\end{aligned}
$$

where $d(t)$ is a C^0 time-varying parameter with $|d(t)|\le 1$.

Clearly, global output feedback stabilization of the uncertain system (5.4) is a nontrivial problem for two reasons: 1) it requires the design of a single output feedback controller to stabilize a family of nonlinear systems, due to the presence of the unknown parameter $d(t)$; 2) most of the existing output feedback control schemes including the one proposed recently [17] is hard to be applied to the uncertain system (5.4), because of the lack of effective design methods for the construction of *robust* observers and/or output compensators for uncertain nonlinear systems with unstabilizable/undetectable linearization.

However, a simple calculation shows that the uncertain system (5.4) satisfies the homogeneous growth condition (1.3). Indeed,

$$|\frac{1}{3}\eta_1 e^{d(t)\sin\eta_2}| \le |\eta_1|, \qquad \forall d(t) \in [-1, 1].$$

By Theorem 4.1, a reduced-order dynamic output compensator can be designed as

$$\begin{aligned} \dot{\hat{z}}_2 &= -106.3(\hat{z}_2 + 3y) \\ v &= -100(5.4\hat{z}_2 + 82y)^{1/3}, \end{aligned} \tag{5.5}$$

which globally robustly stabilizes the uncertain system (5.4).

It should be pointed out that unlike in [17], the design of the output feedback controller (5.5) does not use the knowledge of the uncertain term $\eta_1 e^{d(t)\sin\eta_2}$. This is substantially different from the work [17], where a copy of the term $\eta_1 e^{d(t)\sin\eta_2}$ is needed for the construction of a nonlinear observer. As a result, the output feedback control scheme proposed in [17] is not robust and cannot be applied to *uncertain* systems such as (5.4).

Example 5.5: Consider a cascade system of the form

$$\begin{aligned} \dot{\zeta} &= -\zeta + \ln(1 + \eta_1^2) \\ \dot{\eta}_1 &= \eta_2^3 + \theta\eta_1\cos(\zeta\eta_2) \\ \dot{\eta}_2 &= v \\ y &= \eta_1. \end{aligned} \tag{5.6}$$

where θ is an unknown constant satisfying $\theta \in [\frac{1}{2}, 1]$.

Note that this nonlinear system has a significant features that make global output feedback stabilization of (5.6) difficult. Indeed, the linearized system of (5.6) is given by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0 \ 0],$$

which is neither stabilizable nor detectable $\forall \theta \in [\frac{1}{2}, 1]$. Moreover, the uncontrollable mode is associated with a positive eigenvalue θ. For this reason, system (5.6) must be stabilized by nonsmooth feedback. However, due to the presence of the unknown parameter θ, the nonsmooth output feedback control scheme proposed in [17] is inapplicable to the uncertain system (5.6).

On the other hand, it is easy to see that the cascade system (5.6) is of the form (5.1) with $r = 2$ and $p_1 = 3$, $p_2 = 1$, and satisfies Assumption 5.1. Moreover, the ISS-like inequality (5.3) holds for the z-subsystem of (5.6) by choosing $U_0(z) = \frac{z^2}{2}$. By Theorem 5.3, one can design the dynamic output compensator

$$\begin{aligned} \dot{\hat{z}}_2 &= -45.5(\hat{z}_2 + 52y) \\ v &= -29[12(\hat{z}_2 + 25y)]^{1/3}, \end{aligned} \tag{5.7}$$

such that the closed-loop system (5.6)-(5.7) is globally robustly stable $\forall \theta \in [\frac{1}{2}, 1]$.

VI. Conclusions

In this paper, we have presented a robust nonsmooth output feedback design method, which integrates the nonsmooth state feedback design [13] and the recursive algorithm for the construction of nonsmooth nonlinear observers [17]. The new ingredient is the introduction of a rescaling technique combined with the idea of the non-separation principle design, making it possible to recursively construct both robust state feedback controllers and homogeneous observers that do not depend on the uncertainty of the system. Using this new output feedback control strategy that is nonsmooth in nature, we have identified appropriate conditions under which the problem of global robust stabilization is solvable by nonsmooth output feedback for a class of uncertain nonlinear systems, although they cannot be stabilized, even locally, by any smooth feedback. The significance of the nonsmooth output feedback control schemes proposed in this paper have been illustrated by three examples, which cannot be dealt with by the non-smooth output feedback design method in [17].

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