Higher-Order Optimal Control Design via Singular Perturbation

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Abstract— **Control design is generally based on an approximate model of a real-life system. In this paper, a complex dynamical system is depicted using not one but several models of the same system. Based on interest for robust implementation of reduced-order controllers in higher-order models, we study the regulator problem for a dynamical system expressed as a framework of several models where the reduction between them is made by singular perturbation theory. This multiresolutional representation portrays the model and cost at different levels of complexity, allowing to study analytical solutions for the minimization problem at each level to determine cases in which the cost can be minimized without necessarily requiring an exact optimal control. As result, we show that sometimes a small improvement in the reduced-order optimal control may improve the minimum cost value for the high-order model in a significant way. A singularly perturbed quadratic regulator accounting for actuator dynamics expressed as the discontinuous nonlinearity *sign* in its structure is shown as example to illustrate these properties.**

I. INTRODUCTION

In this paper, we propose a new optimal control design method for a model of a dynamical system in terms of its reduced-order optimal control. This can lead to optimal control design for higher-order models of a dynamical system resembling a real-life process. Instead of building the controller design for one particular model of a system, we consider a control design for a multiresolutional framework representation of the dynamical system, described as a hierarchical group of models expressed at different dimensionality. The case we study is one where the model has a small improvement in terms of its complexity.

We express this framework as a reduced-order model representing the lower-resolutional model and a higher-order model obtained by taking into account additional dynamics in the lower dimension. The case we investigate is a singularly perturbed optimal problem corresponding to a dynamical system expressed as a multiresolutional framework where a cost function will be minimized. Considering this representation and introducing a two-channel structure at the control input, we demonstrate the optimal control design for higher-dimensional models can be simplified under modified reduced-order optimal control feedback.

Previously, several articles have appeared in the literature considering the order reduction technique in optimal control. For example, Kokotovic [1] [2], proposed ways to apply order reduction in nonlinear control systems using singular perturbations. O'Malley [3] [4], also found methods to obtain near-optimal solutions for singularly perturbed optimal control problems. In [5], Artstein recently studied the singularly perturbed optimal control problem, finding a way to extract near-optimal solutions for the original system from optimal solutions of the reduced-order one, investigating the case considering first order fast dynamics.

However, for the order reduction to apply, the optimal solutions must also satisfy restrictive conditions. In particular, the fast scale dynamics has to converge to a stationary point. Thus, many systems fail to have these variational limit properties needed for the system to have a continuous plant trajectory. In the following sections we investigate the singularly perturbed optimal regulator problem for a case that considers two-channel dynamics.

The problem we study is described next: Assuming that a dynamical system can be expressed as different models having certain level of approximation, we introduce the notation of a multiresolutional framework [6] [7] in which the models are connected through a small parameter ε, and where their limit is obtained as $\varepsilon \to 0$.

Let

$$g : \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \ , \quad n_2 > n_1 \quad ; \quad x = g(z) \tag{1}$$

be a map between two spaces with different dimensions.
A low resolutional model is

$$\dot{x} = f_x(x) + B_x(x)u \quad ; \quad x \in \mathbb{R}^{n_1} \tag{2}$$

$$J_{i-1}(x,u) = \int_0^\infty L_{i-1}(x,u)dt. \tag{3}$$

Let $u_{\text{opt}}^{i-1} = u_{\text{opt}}^{i-1}(x)$ be an optimal controller for (2).
A higher-resolutional model is

$$\dot{z} = f_z(z) + B_z(z)u \quad ; \quad z \in \mathbb{R}^{n_2} \tag{4}$$

$$J_i(z,u) = \int_0^\infty L_i(x,u)dt \tag{5}$$

with optimal control $u_{\text{opt}}^{i} = u_{\text{opt}}^{i-1}(z)$. Using map (1) and optimal solution u_{opt}^{i-1} we express u_{opt}^{i} as

$$u_{\text{opt}}^{i} = u_{\text{opt}}^{i-1}(g(z)) + \delta(z) \tag{6}$$

where $\delta(z)$ is a function of the additional variables in the higher level.

Some of the results are known. For example, O'Malley [8] considered the following

$$u_{\text{opt}}^{i}(t) = u_{\text{opt}}^{i-1}(t) + \sum_{j=1}^{\infty} c_j(t)\cdot\varepsilon^j \ ; \ \delta(z) = \sum_{j=1}^{\infty} c_j(t)\cdot\varepsilon^j \tag{7}$$

where the additional function $\delta(z)$ in (6) represents the difference between u_{opt}^{i} and u_{opt}^{i-1} expanded with respect to the small parameter ε as is shown above.

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In general, the minimized cost for each model as considered in the multiresolutional framework, is found to be

$$J_i(u^i_{\text{opt}}(t)) \leq J_i(u^{i-1}_{\text{opt}}(t)). \tag{8}$$

Assuming the controls u_{i-1} and u_i that minimize the cost at each resolution are bounded in magnitude, and contained in the admissible control set $\mathcal{U}$, in the control aspect of the problem we analyze, we can think about the following

- If (2), (3) is the limit for a high-resolution model (4), (5) as $\varepsilon \to 0$, a) is it true that $J_i^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} J_{i-1}$ for an optimal solution? b) does the optimal control converge $u_i^\varepsilon \to u_{i-1}$?
- If a way to improve the lower resolution optimal control optimality without considering the exact design of a higher-resolutional problem exists, is it possible to find a case

$$J_i^\varepsilon(u^*_{i-1}) < J_i^\varepsilon(u^*_i)\ , \quad \text{as } \varepsilon \to 0, \tag{9}$$

where the minimized cost value J_i is significantly improved by using the improved lower optimal control?

For the quadratic regulator minimization problem, we are also interested to find if the minimum cost J^* and optimal control u^* in each model follow the next consideration

- If $\varepsilon \to 0$, is it true that $J_i^* \to J_{i-1}^*$?
- If $\varepsilon \to 0$, $u_i^* \to u_{i-1}^*$?

We study this problem by accounting for additional second-order actuator dynamics at the lower-dimensional model input to construct a higher-dimensional model. Thus, we prove that considering this framework under reduced-order optimal control feedback will satisfy the conditions mentioned previously.

II. Optimal control formulation in multiresolutional framework

A. Multiresolutional framework from singular perturbation

As shown in the introduction, the limit between the state variables from two different models can be expressed as a mapping g_i. However, this may not always satisfy the overall limit between models; for instance, when a time varying system under feedback control is considered. In this case, the mapping function has to satisfy other restrictions in order to provide sufficient conditions for a limit value to exist between the state trajectory in both models.

A dynamical system expressed as the model and cost written in terms of the system states at two different levels of resolution defines the multiresolutional formulation for an optimal control problem as

$$\dot{x}_i^\varepsilon = f_i(x_i^\varepsilon, u_i) \tag{10}$$

$$J_i^\varepsilon = \int_0^\infty J(x_i, u_i)dt \tag{11}$$

$$\dot{x}_{i-1} = f_{i-1}(x_{i-1}, u_{i-1}) \tag{12}$$

$$J_{i-1}^\varepsilon = \int_0^\infty J(x_{i-1}, u_{i-1})dt. \tag{13}$$

Assuming that the ith and $(i-1)$th model depend on x, and u, and a parameter ε in the ith-order system connects both models, cases exist where the ith model is reduced to $(i-1)$th-order in the multiresolutional framework by singular perturbation as $\varepsilon \to 0$. Mathematically, this is described as two models of different dimension from the same dynamical system expressed as a singular perturbation problem where the higher-order system is reduced as $x_i^\varepsilon \to x_{i-1},\ \varepsilon \to 0$.

Consider a manifold $\mathcal{M}_i$ corresponding to the function $\sigma_i(x_i, u_i) = 0$ is a subset of the extended space $\mathcal{M}_i(x_i, u_i) \subset \mathrm{X}_i \times \mathrm{U}_i$. Assuming the initial state $x(t_0)$ lies within an ε-neighborhood of $\mathcal{M}_i$, the mapping from the ith to the $(i-1)$th resolution is described by an instantaneous motion onto $\sigma_i(x_{i-1}, u_{i-1}) = 0$ as $\lim_{\varepsilon\to 0} \sigma_i(x_{i-1}, u_{i-1}) = x_{i-1}$. For this to occur, we also require a map in the control providing the limit value for model reduction under feedback as $\varepsilon \to 0$. Thus, the state trajectory map from the ith model onto the manifold $\mathcal{M}_i$ is a dual map, where g in (1) accounts for the state variables, and h for the trajectory and control

$$(g_i \times h_i) \quad : \quad \lim_{\varepsilon\to 0}(x_i) = x_{i-1}. \tag{14}$$

The lower-order model (12) is isomorphic to the manifold $\mathcal{M}_i$ where (14) represents an instantaneous switching in the trajectory between the set describing the extended system, and $\mathcal{M}_i$.

In the regulator problem, the extended cost J_i is also reduced to the lower resolution cost J_{i-1} as $(x_i^\varepsilon, J_i^\varepsilon) \to (x_{i-1}, J_{i-1}),\ \varepsilon \to 0$. The extended cost is minimized for an admissible control $u \in \mathcal{U}, u = u(t)$, as $\varepsilon \to 0$. In this case, the state trajectories will be mapped between the full-order model and the reduced model by restricting the cost value J_i onto the state and control values in manifold $\mathcal{M}_i$. In that way, any variables in the extended model depending on the control will be eliminated from the extended cost, restricting it to only take values for x and u, thus it becomes equivalent to the reduced cost. By excluding the dependent variables in the extended model, its cost J_i is reduced to J_{i-1}, as $\varepsilon \to 0$, restricting the extended cost to be

$$J_{i-1} = J_i \Bigg|^{\text{restricted}}_{(x,u)\in\mathcal{M}_i} \quad \text{for } (x,u) \in \mathcal{M}_i. \tag{15}$$

B. Two-channel control structure

The *two-channel control structure* shown in Figure 1,

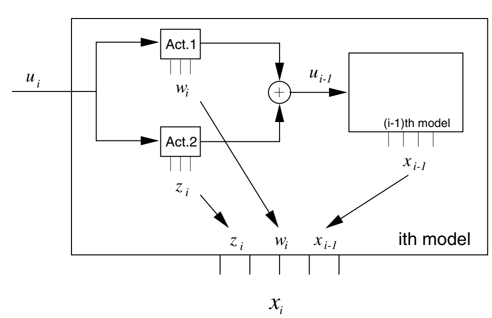

Fig. 1. Two-channel structure

considers two paths at the control input providing a way to introduce additional dynamics into the reduced model to obtain a full-order model. Thus, the multiresolutional framework is expressed as a singular perturbation problem that becomes reduced as $\varepsilon \to 0$.

In general, from the model representation at each resolution level, the state vector in each model may be expressed as

$$x_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} \quad ; \quad x_{i-1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{bmatrix}. \qquad (16)$$

The state vector for the higher resolution model can also be expressed as a function of the additional states introduced by the actuator dynamics as

$$x_i = \begin{bmatrix} x_{i-1} \\ x_N \end{bmatrix} \quad ; \quad x_N = \begin{bmatrix} w_i \\ z_i \end{bmatrix}. \qquad (17)$$

Considering additional dynamics are introduced, two new additional states w_i, z_i appear, thus

$$\tilde{x}_i = \begin{bmatrix} x_{i-1} \\ w_i \\ z_i \\ u_i \end{bmatrix} \quad ; \quad \tilde{x}_{i-1} = \begin{bmatrix} x_{i-1} \\ u_{i-1} \end{bmatrix}. \qquad (18)$$

By using this configuration, the state vector mapping between models $g_i \times h_i : \tilde{x}_i \to \tilde{x}_{i-1}$ provides limit properties between the additional state variables from the extended model x_N, u_i and the reduced-order control variable u_{i-1}

$$g_i \times h_i \left(\begin{bmatrix} w_i \\ z_i \\ u_i \end{bmatrix} \right) = u_{i-1} \qquad (19)$$

such that the additional states w_i and z_i and the extended control u_i become u_{i-1} as $\varepsilon \to 0$.

The optimal control formulation in the multiresolutional framework is made by minimizing the cost for each particular model of the dynamical system using some permissible control u in each case. In particular, consider the cost functions J_i, and J_{i-1}. Let (11) be the extended cost and (13) be the reduced cost. By replacing the additional states from (17) into (11) we have that

$$J_i^\varepsilon = \int_0^\infty J(x_{i-1}, w_i, z_i, u_i)dt. \qquad (20)$$

Thus, we can write the cost as a limit between the two models of the multiresolutional framework for $\varepsilon \to 0$ as

$$\lim_{\varepsilon\to 0} J_i^\varepsilon(x_i, u_i) = \lim_{\varepsilon\to 0} J_i^\varepsilon(x_{i-1}, w_i, z_i, u_i). \qquad (21)$$

The ith model trajectory maps onto the surface $\sigma_i = 0$ under the following condition

$$\lim_{\varepsilon\to 0} \quad \mathrm{dist}[(x_i(\cdot), u_i(\cdot)), \sigma_i] = 0 \qquad (22)$$

where $\sigma_i = x_N - u_i = 0$, and x_N in (17) represents the additional states introduced into x_i by the actuator dynamics mapped by (19) to the manifold $\mathcal{M}_i : \sigma = 0$. These variables represent the internal dynamics contributed by each channel of the two-channel structure. The extended cost in (20) is also expressed as the solution of the variables that define $\sigma = 0$, where σ_i is a function of the additional states x_N and extended optimal control u_i.

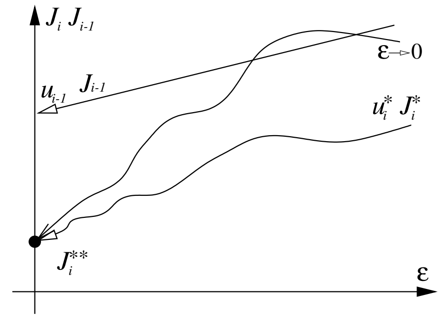

Fig. 2. Relationship between reduced and extended problems

In general, the numerical value of the minimized cost for the reduced system (u_{i-1}^*, J_{i-1}^*) is $J_{i-1} > J_i$. We look to minimize the cost using reduced-order optimal control schemes. Although this could happen only approximately, considering the analytical solution is found for $\varepsilon \to 0$, the minimized value for the extended cost J^{**} will be equal to the minimized cost for the reduced model, as $\varepsilon \to 0$.

Figure 2 shows the limit of the cost function J_i^ε for the extended model, will be exactly equal to the reduced cost as $\varepsilon \to 0$. This is formally stated as

$$\lim_{\varepsilon\to 0} J_i^{**}(\varepsilon) = \lim_{\varepsilon\to 0} J_i^*. \qquad (23)$$

Sometimes this solution may not guarantee a minimization on the whole interval $\varepsilon \neq 0$, taking only a near-minimum value in certain sections.

Assuming the trajectory of the extended system reaches an ε-vicinity of the manifold $\mathcal{M}_i$ at time moment t_1, the model of the dynamical system at this time is expressed by

$$x_i(t_1) = x_{i-1}(t_1) \qquad (24)$$
$$z_i^\varepsilon(t_1) = u(t_1) \ , \quad \text{as } \varepsilon \to 0. \qquad (25)$$

This guarantees the ith-order system will reduce to the $(i-1)$th-order system as $\varepsilon \to 0$. The ith-order states are also reduced to the $(i-1)$th-order states as we find the limit[1] when $\varepsilon \to 0$. In mathematical terms, this is expressed by the following limit functions

$$\lim_{\varepsilon\to 0} x_i^\varepsilon = x_{i-1} \qquad (26)$$
$$\lim_{\varepsilon\to 0} z_i^\varepsilon = u. \qquad (27)$$

[1] *note:* the limit is understood in L_2-sense, i.e.

$$\lim_{\varepsilon\to 0} x_i^\varepsilon = x_{i-1} \iff \|x_i^\varepsilon - x_{i-1}^\varepsilon\|_{L_2} \to 0 \quad (\varepsilon \to 0)$$
$$\|x\|_{L_2}^2 = \int_0^T \|x\|^2 dt$$

Therefore, during the reduction stage, the extended model x_i tends to the reduced model x_{i-1}, because $z \to u$ as $\varepsilon \to 0$. The mapping from the manifold $\mathcal{M}_i$ onto the reduced space is expressed by the following identity

$$g_i(x,u,u) \equiv g_{i-1}(x,u) \quad (28)$$

$$g_i \times h_i(x,u,u) \equiv g_{i-1} \times h_{i-1}(x,u). \quad (29)$$

III. EXAMPLE

To demonstrate we can improve the minimum cost value in the extended system, we must compare its performance in two different situations. We will first use the reduced-order control to show that the optimal cost and state converge to the cost of the reduced system as $\varepsilon \to 0$. After that, we will show that the optimal solution for the full-order system tends to a completely different value (improved) as $\varepsilon \to 0$. Simulations are included to illustrate our theoretical results.

Consider a multiresolutional framework described as two models of the same dynamical system expressed at different level of resolution satisfying the limit properties. The actuator dynamics are introduced in the reduced-order model by considering a two-channel structure at the control input. Based on this multiresolutional representation, we will study the cost-minimization problem in systems that have the form (10)-(13) using reduced-order optimal control schemes.
The extended model is

$$\dot{x}(t) = \frac{1}{2}z(t) + \frac{1}{2}u(t) \quad (30)$$

$$\varepsilon\dot{z}(t) = -\text{sign } (z(t) - u(t)) \quad (31)$$

$$J_1 = \int_0^\infty (x^2(t) + z^2(t))dt. \quad (32)$$

The reduced-order model is

$$\dot{x}(t) = u(t) \quad (33)$$

$$J_0 = \int_0^\infty (x^2(t) + u^2(t))dt. \quad (34)$$

In fact, because the sign function is not smooth, an exact optimal control design for this particular model becomes even more difficult. In our approach, we assume certain initial conditions and perform an analysis based on the behavior of the state trajectories as $t \to \infty$.

Assuming the initial state is close to an ε-vicinity of manifold $\mathcal{M}_i$, the system trajectories will reach the ε-vicinity at t_1. At this time, when $\varepsilon \to 0$, the state trajectory is described as the mapping in (14) from $x(t_1)$ to $\sigma = 0$.

From time t_1 until reaching the origin, the nonlinear actuator dynamics drives the argument of the sign function (31), which is the sliding surface σ into sliding mode.

The two cases shown next illustrate a way to improve the optimality of the lower-resolution optimal control without considering the exact design for a higher-resolutional problem, where it is possible to find a case such that

$$\min J_i^\varepsilon(u_{i-1}^*) < \min J_i^\varepsilon(u_i^*) \ , \quad \text{as } \varepsilon \to 0 \quad (35)$$

in which the minimized cost value J_i is significantly improved by using a modified lower optimal control.

A. Minimization using a reduced-order optimal control

The cost-minimization analysis for the multiresolutional framework is divided into two time intervals

$$\int_0^\infty = \int_0^{t_1} + \int_{t_1}^\infty \quad (36)$$

where the first stage is a transition between the initial time at the starting point in the trajectory in the extended space and the time when the trajectory reaches the sliding manifold, as $\varepsilon \to 0$; and the final stage in the sliding manifold, that describes the system in the reduced-order space. Consider the multiresolutional framework in (30)-(34). The optimal control for the reduced-order model is

$$u(t) = -x(t). \quad (37)$$

Substituting (37) into (30)-(32) we obtain

$$\dot{x}(t) = \frac{1}{2}z(t) - \frac{1}{2}x(t) \quad (38)$$

$$\varepsilon\dot{z}(t) = -\text{sign } (z(t) + x(t)). \quad (39)$$

The lower equation (39) can also be expressed as

$$\dot{z}(t) = -\varepsilon^{-1}\text{sign } \sigma(t), \quad \text{where } \sigma(t) = z(t) + x(t) \quad (40)$$

and $\mathcal{M}_i = \{\sigma = 0\}$ is a sliding surface. At $t_0 = 0$, $\sigma(0) = z(0) - u(0)$. Assuming the trajectory reaches the manifold $\mathcal{M}_i$ at t_1, $\sigma(t_1) = 0$, thus $z(t_1) = u(t_1)$.

To find the reaching time t_1 we obtain the derivative

$$\dot{\sigma}_i(t) = \dot{z}_i(t) - \dot{u}_i(t). \quad (41)$$

Replacing $\dot{z}$ from (40) into (41), the integral on $[0, t_1)$ is

$$\int_0^{t_1} \dot{\sigma}\ dt = -\frac{1}{\varepsilon}\int_0^{t_1} \text{sign } \sigma\ dt - \int_0^{t_1} \dot{u}\ dt. \quad (42)$$

For $\sigma_i(t_1) = 0$ and $x(t_1) = z(t_1)$, the solution to (42) is

$$-\sigma(0) = \pm\frac{1}{\varepsilon}t_1 - u(t_1) + u(0). \quad (43)$$

Finally, solving for t_1 the reaching time is found to be

$$t_1 = \varepsilon\Big|u(t_1) - z(0)\Big| = C\varepsilon. \quad (44)$$

Now, we consider the multiresolutional framework (30)-(34) to demonstrate that $\min J_i^\varepsilon \to \min J_{i-1}$, as $\varepsilon \to 0$. Assuming the trajectory is above the switching surface at initial time, $\sigma(0) > 0$, we obtain $z(t)$ in the reaching phase using (37)

$$z(t) = -\varepsilon^{-1}t + z(0) \quad (45)$$

$$z^2(t) = \frac{1}{\varepsilon^2}t^2 - \frac{2z(0)}{\varepsilon}t + z^2(0). \quad (46)$$

We substitute (45) into (38) and find the solution for $x(t)$

$$x(t) = e^{-\frac{t}{2}}x(0) + \frac{t}{\varepsilon}\psi_\varepsilon(t) \quad (47)$$

$$x^2(t) = e^{-t}x^2(0) + 2e^{-\frac{t}{2}}x(0)\frac{t}{\varepsilon}\psi_\varepsilon(t) + \frac{t^2}{\varepsilon^2}\psi_\varepsilon^2(t) \quad (48)$$

where $|\psi_\varepsilon(t)| \leq C$ is bounded for every $\varepsilon > 0$, $0 \leq t \leq t_1$.

Replacing (46) and (48) into the cost (32), and evaluating (47) at t_1 from (44), results an integral of order $\mathcal{O}(\varepsilon)$, as $\varepsilon \to 0$. Thus, the cost in that interval will be insignificant.

$$\int_0^{t_1} = \int_0^{C\varepsilon} \left[e^{-t} x^2(0) + 2e^{-\frac{t}{2}} x(0) \frac{t}{\varepsilon} \psi_\varepsilon(t) + \frac{t^2}{\varepsilon^2} \psi_\varepsilon^2(t) \right] dt = \mathcal{O}(\varepsilon) \tag{49}$$

Since at t_1 the ith-order system is reduced to $(i-1)$th-order, its total cost expressed as J_1 in the simulations will be

$$J_1 = \mathcal{O}(\varepsilon) + \int_{t_1(\varepsilon)}^{\infty} J_{\text{low resolution cost.}} \tag{50}$$

In this way we prove that J tends to the low resolutional optimal cost, shown as J_0 in the simulations, as $\varepsilon \to 0$.

B.Minimization using modified reduced-order optimal control

This section shows a case where the limit of the full-order optimal cost is actually much smaller that the lower-resolution optimal cost. We will demonstrate a control $u_\varepsilon(t)$, although not optimal for $\varepsilon > 0$ but for which the cost J_ε has the same limit as the full-order optimal cost as $\varepsilon \to 0$, will significantly improve the minimum value of the cost, as $\varepsilon \to 0$.

Analysis for $0 \le t \le t_1$

The first time interval occurs while the extended model states approach the sliding manifold. Our analysis of the state trajectories is made assuming that the states should be $x(t_1) = z(t_1) = 0$ at the end of this interval. Singular perturbation during the first stage causes $z \to u$, as $\varepsilon \to 0$. The higher-order cost is expressed as a function of x and z as is shown in (32). The control during the first stage, is an open-loop high magnitude control with constant value

$$u(t) = -\frac{2}{t_1} x(0). \tag{51}$$

Analytically, choosing (51) allows us to cancel out the initial condition $x(0)$ in the reduced model after substituting it into state equation (30) and solving for $x(t)$. Thus, the cost-minimization problem will be dependent only on the extended state initial conditions $z(0)$.

We obtain t_1 for a constant control (51) by considering the switching surface $\sigma(t)$ is described by a linear interval. Initially, we know that $z(t_0) \neq u(t_0)$, however, as $t_0 \to t_1$, $z \to u$ as $\varepsilon \to 0$, thus

$$t_1 = \varepsilon\sigma(0). \tag{52}$$

Replacing (52) in (45) we obtain $z(t_1)$, which into (30) is

$$\dot{x}(t) = -\frac{t_1}{2\varepsilon} + \frac{z(0)}{2} + \frac{1}{2} u(t). \tag{53}$$

Then substituting (51) into (53), and solving for $x(t)$ gives

$$x(t) - x(0) = -\frac{t_1\ t}{2\varepsilon} + \frac{t z(0)}{2} + \frac{1}{2}\left(\frac{-2x(0)}{t_1} \right) t. \tag{54}$$

Assuming $t = t_1$ will cancel the terms expressed as a function of the initial condition $x(0)$, thus $x(t_1)$ becomes

$$x(t_1) = -\frac{t_1^2}{2\varepsilon} + \frac{t_1 z(0)}{2}. \tag{55}$$

Analysis for $t > t_1$

At time t_1, the full-order model and cost become reduced in order. Therefore, on the interval $[t_1, \infty)$, for $t > t_1$, the system model that must be considered in the cost-minimization analysis is the reduced system and its optimal control.

Again, in order to analyze the state trajectories to find the total minimized cost value and to verify we obtain an improvement in the cost, as $\varepsilon \to 0$, consider dividing the cost integral (32) in two intervals as is shown in (36).

Analysis of the Cost-Minimization Problem for $0 \le t \le t_1$

As it was previously mentioned, the full-order model in (30)-(32) becomes reduced at t_1, following the singular perturbation of (31), as $\varepsilon \to 0$. During this reduction stage, the penalization J_1 from (34) in the state z will become one in the control u.

The state trajectory reaches the sliding manifold at $\mathcal{M}_i = \{\sigma_i(z_i, u_i) = (0)\}$ when the full-order system has been reduced. Its motion is generated by the function σ in (40). Replacing (45) and u from (51) into (30), we obtain

$$\dot{x}(t) \;=\; -\frac{1}{2\varepsilon} t + \frac{1}{2} z(0) - \frac{1}{t_1} x(0). \tag{56}$$

Integrating 56, the solution for $x(t)$ is

$$x(t) = -\frac{1}{4\varepsilon} t^2 + \left(\frac{z(0)}{2} - \frac{1}{t_1} x(0) \right) t + x(0). \tag{57}$$

We simplify (57) for reaching time $t = t_1 = \varepsilon\sigma(0)$

$$x(t_1) \;=\; -\frac{1}{4\varepsilon} \varepsilon^2 \sigma^2(0) + \frac{1}{2} z(0) \varepsilon \sigma(0). \tag{58}$$

The state $x^2(t)$ is shown in (59) and $z^2(t)$ is shown in (46).

$$\begin{aligned} x^2(t) \;&=\; \frac{1}{16\varepsilon^2} t^4 - \frac{1}{2\varepsilon} \left(\frac{z(0)}{2} - \frac{1}{t_1} x(0) \right) t^3 \\ &+\; \left(\frac{-1}{2\varepsilon} x(0) + \left(\frac{z(0)}{2} - \frac{1}{t_1} x(0) \right)^2 \right) t^2 \\ &+\; 2x(0) \left(\frac{z(0)}{2} - \frac{1}{t_1} x(0) \right) t + x^2(0) \end{aligned} \tag{59}$$

The time t_1 represents the end of the reduction interval, thus the upper limit value of the integral will be $\lim J(\varepsilon) \to 0$, as $\varepsilon \to 0$. Replacing (59) and (46) into the cost (32) in $[0, t_1)$ and simplifying we find every term is a function of ε, as $\varepsilon \to 0$

$$J^0 = \int_0^{t_1} \Big(x^2(t) + z^2(t) \Big) dt = \mathcal{O}(\varepsilon) \to 0,\ J^0 \Big|_0^{t_1} \to 0. \tag{60}$$

Analysis of the Cost-Minimization Problem for $t > t_1$

Following from our explanation of the state trajectory behavior our assumption was that at time t_1, the extended model x_i would become the reduced model x_{i-1}, as $\varepsilon \to 0$.

$$\dot{x}(t) \;=\; u(t) \tag{61}$$

$$J^1 \;=\; \int_{t_1}^{\infty} (x^2(t) + u^2(t)) dt \tag{62}$$

The reduced-order cost is expressed in terms of the reduced state x and the control u as (62). By replacing the reduced-order optimal control (37) into the cost (62) we obtain

$$J^1 = \int_{t_1}^{\infty} 2x^2(t)dt. \qquad (63)$$

Solving (63) and evaluating the state trajectory on $[t_1, \infty)$, $x(t) \to 0$, as $t \to \infty$. Thus, the cost will tend to zero

$$J^1 = -2x^2(t_1) = \mathcal{O}(\varepsilon^2) \to 0 \ , \quad \text{as } \varepsilon \to 0. \qquad (64)$$

On the second interval, after integrating, the cost is also found to be a function of t_1. The cost is actually given in the order $\mathcal{O}(\varepsilon^2)$, therefore as result and knowing (52), $x(t_1) \to 0$, as $\varepsilon \to 0$. Thus, the total cost for the whole interval expressed as J_2 in the simulations is zero, as $\varepsilon \to 0$

$$J_2 = J^0 + J^1 = \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^2) \to 0, \quad \lim_{\varepsilon \to 0} J_2(\varepsilon) \to 0. \quad (65)$$

Therefore, we show that using control (51) on the initial interval $[0, t_1)$ we drastically reduce the cost. In fact, as $\varepsilon \to 0$ it tends to zero, which is the best possible value.

IV. Conclusions

The main contribution in this paper was that we found a control design method based on optimal schemes for reduced-order models of a dynamical system, that exhibits good optimal performance and provides an improvement over the actual minimum cost value known for an exact optimal design in a similar higher-order model. As optimal control design usually becomes more complicated for higher-order models, our motivation for this approach was that a reduced-order control may contain part of the information found in a higher-order control of the same dynamical system. The questions made in the introduction are clearly proved theoretically as is shown in the simulations.

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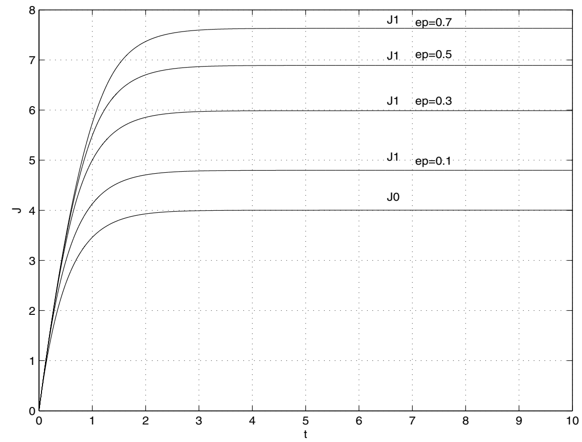

Fig. 3. $J_1 \to J_0$ as $\varepsilon \to 0$, x(0)=2, z(0)=1

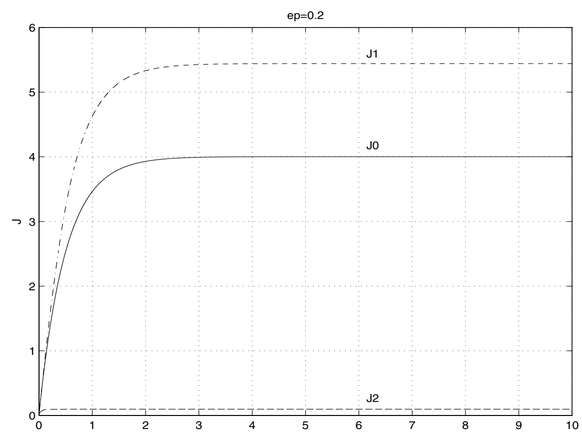

Fig. 4. Improvement in cost for x(0)=2, z(0)=1

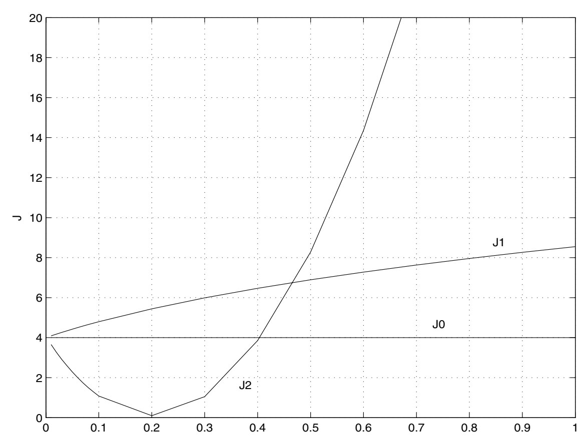

Fig. 5. Variation in J vs variation of ε for x(0)=2, z(0)=1