2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Fixed Structure Multirate State Estimation

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Abstract— **In most chemical processes only some measurements are available online while other measurements are available infrequently and often with long delays. Multirate state estimation can optimally combine these different classes of measurements to improve the estimation quality compared to the fast measurements alone. The following new aspects of multirate state estimation are discussed in this paper: Firstly, we discuss the behaviour of variable and fixed structure estimation schemes and conclude that a fixed structure is preferable. Secondly, the Moving Horizon State Estimator is extended to the multirate case in a fashion that corresponds to the fixed structure state estimation scheme. Finally, a performance gauge is proposed which measures the maximal improvements that can be obtained by using additional noisy slow measurements. The theoretical findings are supported by a simulation example.**

I. INTRODUCTION

In most chemical processes, only some measurements, e.g. temperatures and flow rates, are available online, whereas other variables such as concentrations are usually measured infrequently and possibly with long delays. Often, measurements of the other state variables of the process are not available at all (inner states). For process operation, optimisation and online control it is often desirable to know all states of the system. State or parameter estimation is therefore part of many advanced control and optimisation strategies for chemical processes. In this paper, the problem of combining measurements which are obtained at different sampling frequencies into one state estimation scheme is tackled. First, we analyse linear multirate systems in the delay-free case. In particular, fixed and variable structure schemes are compared. We show that the two schemes are equivalent with respect to the estimation error at the sampling points where both measurements are available and advocate the use of the fixed structure because it distributes the correction caused by the slow measurements over the slow sampling period, leading to a smoother estimation. Then, we extend the moving horizon estimator (MHE) to the situation with measurements at different sampling rates and with delays and present a moving horizon estimation scheme that corresponds to the fixed structure observer, thus leading to smooth state estimation. Finally, we propose a performance measure that indicates the gain in estimation accuracy that can be expected from the use of an additional slow measurement.

II. LITERATURE REVIEW

Multirate control is well known and has been used for some time. Multirate output control means that several outputs are sampled before a control action is taken, multirate input control means that several control actions are performed to influence one output [1].

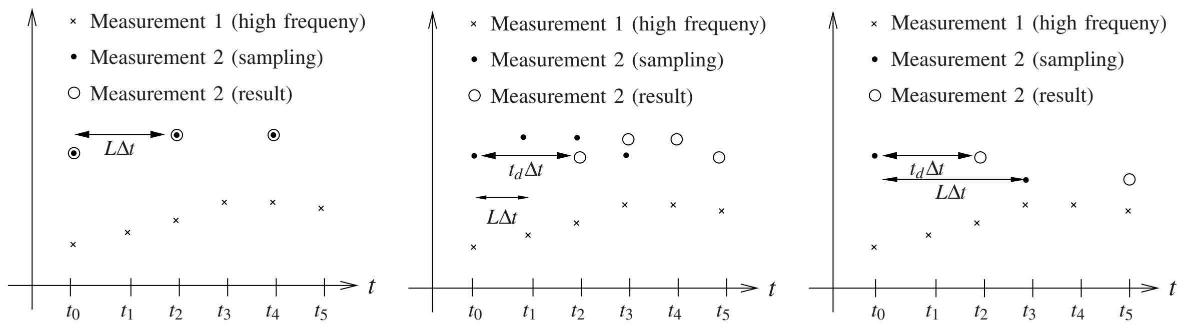
An iterated EKF with variable structure for an augmented linearisation of the nonlinear system was developed and applied to a bioreactor in [2], [3]. The iterated EKF was used as strong nonlinearities are present in the output equation. A similar application, where a variable structure EKF was used and the states were re-estimated from the point in time when the second measurement becomes available, was discussed in [4]. This is a simple approach to the use of infrequent measurements but still results in better estimates as additional information is used. An EKF with fixed lag smoothing where a variable structure EKF is used for the normal and for the augmented system is a more advanced approach [5], [6]. It was shown that better results could be obtained for a process which is not observable from the fast measurements alone.

A different approach based on the concept of the nonlinear reduced observer [7] was reported [8]. In this estimator, the missing samples of the slow measurement are estimated by polynomial extrapolation, which has severe drawbacks, because, in contrast to the model, the extrapolation does not react to input changes. This results in large errors in the case of changes of the inputs. The illustrative example in the paper shows this problem clearly.

III. LINEAR MULTIRATE STATE ESTIMATION

In the literature, multirate systems are defined as systems where the measurements and/or the controls are sampled or applied at different sampling rates. Here we consider multivariate systems where different measurements are available at different sampling rates and possibly with different delays.

For clarity, the derivations in this section are presented for sampled systems with two measurement vectors, one of which is available at a fast sampling rate while the other one is available at a slow sampling rate. A slow sampling point always coincides with a fast sampling point. We call the time between two slow sampling points the intersampling period. The slow sampling time and the measurement delay



(a) Measurement 2 is available at longer sampling intervals

(b) Measurement 2 is available with a delay, but at the same sampling rate

(c) Measurement 2 is available with a delay and at longer sampling intervals

Fig. 1. The different measurement cases are represented (L – intersampling rate, t_d – time delay, Δt – sampling time)

are integer multiples of the fast sampling time. Three different situations may occur which are depicted in Figure 1. Case b is not a multirate problem, but it nicely fits into the schemes proposed below. $j \rightarrow j+1$ represents a step from t to $t+\Delta t$ with Δt being the fast sampling interval and L represents the number of steps between the slow sampling points, the class of (linear) systems considered can be written as

$$\mathbf{x}_{j+1} = \mathbf{A}\mathbf{x}_j + \mathbf{B}\mathbf{u}_j \tag{1}$$

$$\mathbf{y}_j = \begin{cases} \mathbf{C}^F\mathbf{x}_j & \text{for} \quad j/L \notin \mathbb{N} \\ \begin{pmatrix} \mathbf{C}^F\mathbf{x}_j \\ \mathbf{C}^S\mathbf{x}_{j-t_d} \end{pmatrix} & \text{for} \quad j/L \in \mathbb{N} \end{cases}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{y} \in \mathbb{R}^m$ is the vector of measurements. ^{F} indicates fast and ^{S} indicates slow measurements. $\mathbf{u} \in \mathbb{R}^p$ is the vector of input variables, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$, $\mathbf{C}^F \in \mathbb{R}^{n\times(m-q)}$ and $\mathbf{C}^S \in \mathbb{R}^{n\times q}$ are the system matrix, the control matrix and the measurement matrices for the fast and slow measurements, respectively. It is assumed that the system is observable.

Traditionally, multirate state estimation is used for the smoothing of control actions, where the controller sampling rate is faster than the measurement sampling rate and the estimator extrapolates by assuming the error as constant for the intersampling points. This method can be extended to multirate measurements. Delays are ignored here for clarity and without loss of generality. The slow measurement error is held constant until the new measurement is available.

Setting $l = L \cdot \lfloor j/L \rfloor$, where $\lfloor . \rfloor$ denotes the nearest smaller integer value, variable and fixed structure state estimators can be written as follows, where $\mathbf{K} \in \mathbb{R}^{n\times m}$ is the estimator gain:

- Variable structure: $\mathbf{K}_{\text{var}} = \begin{pmatrix} \mathbf{K}^F_{\text{var}} & \mathbf{K}^S_{\text{var}} \end{pmatrix}$

$$\hat{\mathbf{x}}_{j+1} = \mathbf{A}\hat{\mathbf{x}}_j + \mathbf{B}\mathbf{u}_j + \begin{cases} \mathbf{K}^F_{\text{var}}\left(\mathbf{y}_j - \mathbf{C}^F\hat{\mathbf{x}}_j\right) & l \neq j \\ \mathbf{K}_{\text{var}}\begin{pmatrix} \mathbf{y}_j - \mathbf{C}^F\hat{\mathbf{x}}_j \\ \mathbf{y}_l - \mathbf{C}^S\hat{\mathbf{x}}_l \end{pmatrix} & l = j \end{cases} \tag{2}$$

- Fixed structure: $\mathbf{K}_{\text{fixed}} = \begin{pmatrix} \mathbf{K}^F_{\text{fixed}} & \mathbf{K}^S_{\text{fixed}} \end{pmatrix}$

$$\hat{\mathbf{x}}_{j+1} = \mathbf{A}\hat{\mathbf{x}}_j + \mathbf{B}\mathbf{u}_j + \mathbf{K}_{\text{fixed}}\begin{pmatrix} \mathbf{y}^F_j - \mathbf{C}^F\hat{\mathbf{x}}_j \\ \mathbf{y}^S_l - \mathbf{C}^S\hat{\mathbf{x}}_l \end{pmatrix}. \tag{3}$$

Let $\hat{\mathbf{e}}_{k,i}$ be the estimation error, k the counter for the major sampling intervals and i the counter for the inner sampling intervals. Then the evolution of the estimation error can be computed for the two structures under the assumption that a steady state is reached within one large sampling interval (∞ indicates the steady state).

- Fixed structure:

$$\begin{aligned} \hat{\mathbf{e}}_{k,1} &= \mathbf{A}\hat{\mathbf{e}}_{k,0} - \mathbf{K}^F_{\text{fixed}}\mathbf{C}^F\hat{\mathbf{e}}_{k,0} - \mathbf{K}^S_{\text{fixed}}\mathbf{C}^S\hat{\mathbf{e}}_{k,0} \\ \hat{\mathbf{e}}_{k,2} &= \mathbf{A}\hat{\mathbf{e}}_{k,1} - \mathbf{K}^F_{\text{fixed}}\mathbf{C}^F\hat{\mathbf{e}}_{k,1} - \mathbf{K}^S_{\text{fixed}}\mathbf{C}^S\hat{\mathbf{e}}_{k,0} \\ \Rightarrow \quad \hat{\mathbf{e}}_{k,\infty} &= \left(\mathbf{A} - \mathbf{I} - \mathbf{K}^F_{\text{fixed}}\mathbf{C}^F\right)^{-1}\mathbf{K}^S_{\text{fixed}}\mathbf{C}^S\hat{\mathbf{e}}_{k,0}. \end{aligned}$$

- Variable structure:

$$\begin{aligned} \hat{\mathbf{e}}_{k,1} &= \mathbf{A}\hat{\mathbf{e}}_{k,0} - \mathbf{K}^F_{\text{var}}\mathbf{C}^F\hat{\mathbf{e}}_{k,0} - \mathbf{K}^S_{\text{var}}\mathbf{C}^S\hat{\mathbf{e}}_{k,0} \\ \hat{\mathbf{e}}_{k,2} &= \mathbf{A}\hat{\mathbf{e}}_{k,1} - \mathbf{K}^F_{\text{var}}\mathbf{C}^F\hat{\mathbf{e}}_{k,1} \\ \hat{\mathbf{e}}_{k,i} &= \left(\mathbf{A} - \mathbf{K}^F_{\text{var}}\mathbf{C}^F\right)^i\hat{\mathbf{e}}_{k,0} \\ &\quad - \left(\mathbf{A} - \mathbf{K}^F_{\text{var}}\mathbf{C}^F\right)^{i-1}\mathbf{K}^S_{\text{var}}\mathbf{C}^S\hat{\mathbf{e}}_{k,0} \\ \Rightarrow \quad \hat{\mathbf{e}}_{k,\infty} &= \mathbf{0}, \end{aligned}$$

if the spectral radius of $(\mathbf{A} - \mathbf{K}^F_{\text{var}}\mathbf{C}^F)$ is less than 1

On the one hand, the fixed structure estimator (3) generates a smoother decay of the estimation error than the variable structure estimator (2) and is less susceptible to noisy slow measurements due to smaller values of the gain. On the other hand, the fixed structure estimator will not converge to the correct steady state within the intersampling period as long as the estimation scheme has not converged (i.e. the error at the points when both measurements are available has vanished).

If both slow and fast measurement vectors are considered, it can be shown that both observers show equivalent behaviour at the slow sampling points.

Theorem 1 (Equivalence of fixed and variable structure): For sampled dynamic linear systems (Eq. 1), the fixed structure state estimator as given by (3) and the variable structure estimator as given by (2) yield the same steady state value and the same convergence behaviour in the slow sampling points if

$$\left(\mathbf{A}-\mathbf{K}^F\mathbf{C}^F\right)^{n-1}\mathbf{K}^S_{\text{var}}\mathbf{C}^S=\sum_{j=0}^{n-1}\left(\mathbf{A}-\mathbf{K}^F\mathbf{C}^F\right)^j\mathbf{K}^S_{\text{fixed}}\mathbf{C}^S, \quad (4)$$

where n is the number of intersampling points and $\mathbf{K}$ is separated into $\left(\mathbf{K}^F\ \mathbf{K}^S\right)$.

Proof: Let k be the counter for the major sampling intervals and i for the inner sampling intervals. The correction due to the slow measurement occurs at $i=0$, but becomes effective at $i=1$. Consider the error difference equation of the fixed structure estimator:

$$\begin{aligned}\hat{\mathbf{e}}_{k,i+1}&=\mathbf{A}\hat{\mathbf{e}}_{k,i}-\mathbf{K}_{\text{fixed}}\begin{pmatrix}\mathbf{C}^F\hat{\mathbf{e}}_{k,i}\\ \mathbf{C}^S\hat{\mathbf{e}}_{k,0}\end{pmatrix}\\ \hat{\mathbf{e}}_{k+1,0}&=\hat{\mathbf{e}}_{k,n}.\end{aligned} \quad (5)$$

$\mathbf{K}_{\text{fixed}}$ is separated into $\left(\mathbf{K}^F_{\text{fixed}}\ \mathbf{K}^S_{\text{fixed}}\right)$ and the step from k to $k+1$ is considered leading to:

$$\hat{\mathbf{e}}_{k+1,0}=\left(\left(\mathbf{A}-\mathbf{K}^F_{\text{fixed}}\mathbf{C}^F\right)^n-\sum_{j=0}^{n-1}\left(\mathbf{A}-\mathbf{K}^F_{\text{fixed}}\mathbf{C}^F\right)^j\mathbf{K}^S_{\text{fixed}}\mathbf{C}^S\right)\hat{\mathbf{e}}_{k,0}. \quad (6)$$

Similarly, the error difference equation of the variable structure state estimator can be computed:

$$\begin{aligned}\hat{\mathbf{e}}_{k,i+1}&=\mathbf{A}\hat{\mathbf{e}}_{k,i}-\begin{cases}\mathbf{K}^F_{\text{var}}\mathbf{C}^F\hat{\mathbf{e}}_{k,i} & i\neq 0\\ \mathbf{K}^F_{\text{var}}\mathbf{C}^F\hat{\mathbf{e}}_{k,i}+\mathbf{K}^S_{\text{var}}\mathbf{C}^S\hat{\mathbf{e}}_{k,i} & i=0\end{cases}\\ \hat{\mathbf{e}}_{k+1,0}&=\hat{\mathbf{e}}_{k,n}.\end{aligned} \quad (7)$$

For the step from k to $k+1$ one obtains

$$\hat{\mathbf{e}}_{k+1,0}=\left(\left(\mathbf{A}-\mathbf{K}^F_{\text{var}}\mathbf{C}^F\right)^n-\left(\mathbf{A}-\mathbf{K}^F_{\text{var}}\mathbf{C}^F\right)^{n-1}\mathbf{K}^S_{\text{var}}\mathbf{C}^S\right)\hat{\mathbf{e}}_{k,0}. \quad (8)$$

Comparing the system matrices of (6) and (8) and requiring

$$\mathbf{K}^F_{\text{var}}=\mathbf{K}^F_{\text{fixed}}=\mathbf{K}^F$$

leads to the desired result. ■

For both the fixed and the variable structure, it is paramount that the last slow measurement is not compared with a newer simulated measurement. This will produce wrong estimation results, especially for the case of an input change and must be avoided.

From this derivation, it is obvious that the gain calculated for the variable structure estimator will be larger than for the fixed structure estimator, i.e. the variable structure will be more susceptible to noisy slow measurements. It is thus recommended to use the fixed structure state estimator unless the intersampling periods are very long and the error vanishes before a new slow measurement is available. As a consequence of Theorem 1 either the fixed structure or the variable structure state estimator can be used for the estimator design. It is then possible to transform one into the other. The following design rules are suggested:

1) Design $\mathbf{K}^F$ independent of the slow measurement vector for the system

$$\hat{\mathbf{x}}_{j+1}=\mathbf{A}\hat{\mathbf{x}}_j+\mathbf{B}\mathbf{u}_j+\mathbf{K}^F\left(\mathbf{y}^F_j-\mathbf{C}^F\hat{\mathbf{x}}_j\right) \quad (9)$$

for example by pole placement or steady state Kalman Filtering.

2) Design $\mathbf{K}^S$ for the system

$$\begin{aligned}\hat{\mathbf{x}}_{k+1}&=\left(\mathbf{A}-\mathbf{K}^F\mathbf{C}^F\right)^n\hat{\mathbf{x}}_k+\mathbf{u}^*_k+\mathbf{K}^S\left(\mathbf{y}^S_{k,0}-\mathbf{C}^S\hat{\mathbf{x}}_k\right)\\ \mathbf{u}^*_k&=\sum_{j=0}^{n-1}\left(\mathbf{A}-\mathbf{K}^F\mathbf{C}^F\right)^j\left(\mathbf{B}\mathbf{u}_{k,j}+\mathbf{K}^F\mathbf{y}^F_{k,j}\right).\end{aligned} \quad (10)$$

System (10) represents the step from one slow sampling point to the next for the designed $\mathbf{K}^F$. This way, the observer for the fast and the slow measurement vector can be designed in two steps. $\mathbf{K}^S$ is now the same as $\mathbf{K}^S_{\text{var}}$.

3) Use the result of Theorem 1 to calculate $\mathbf{K}^S_{\text{fixed}}$.

The above derivation can be directly applied to a Luenberger Observer. If a Kalman Filter (KF) is the preferred state estimator, applying it to a multirate system always yields a variable structure KF. The corresponding fixed structure KF can then be calculated by calculating the fixed structure gain using the variable structure gain and the result of Theorem 1.

IV. ILLUSTRATIVE EXAMPLE

Example 1: The linearisation of a three tank process yields the model equations:

$$\dot{\mathbf{x}}=\begin{pmatrix}-1&1&0\\1&-2&1\\0&1&-1.6\end{pmatrix}\mathbf{x}+\begin{pmatrix}2.3&0\\0&0\\0&2.3\end{pmatrix}\mathbf{u}+\begin{pmatrix}-1\\0\\0\end{pmatrix}p$$

$$\mathbf{y}_k=\begin{cases}\begin{pmatrix}0&0&1\end{pmatrix}^T\mathbf{x}_k & k/L\notin\mathbb{N}\\ \begin{pmatrix}0&0&1\\1&0&0\end{pmatrix}^T\mathbf{x}_k & k/L\in\mathbb{N}\end{cases}.$$

Here, $\mathbf{x}$ are the levels in the three tanks, $\mathbf{u}$ are the input flows to tank 1 and 3 and p is a leakage from tank 1. The aim of the estimation is the reconstruction of the missing states and of the unknown parameter p. Therefore the state vector is extended by p and the model is extended by the equation $\dot{p}=0$. The system is of fourth order and has three stable non-zero eigenvalues $\lambda=(-0.1443,-1.3299,-3.1258,0)^T$ and thus three time constants $\mathbf{T}=(6.9218,0.7520,0.3199)^T$. A sampling time of $\Delta t=0.1$ is chosen, $L=10$. A Luenberger and a Kalman Filter multirate state estimator were tuned according to the rules given above and applied to this system.

Figures 2 and 3 clearly illustrate the result of Theorem 1. The estimates at the major sampling points are the same but

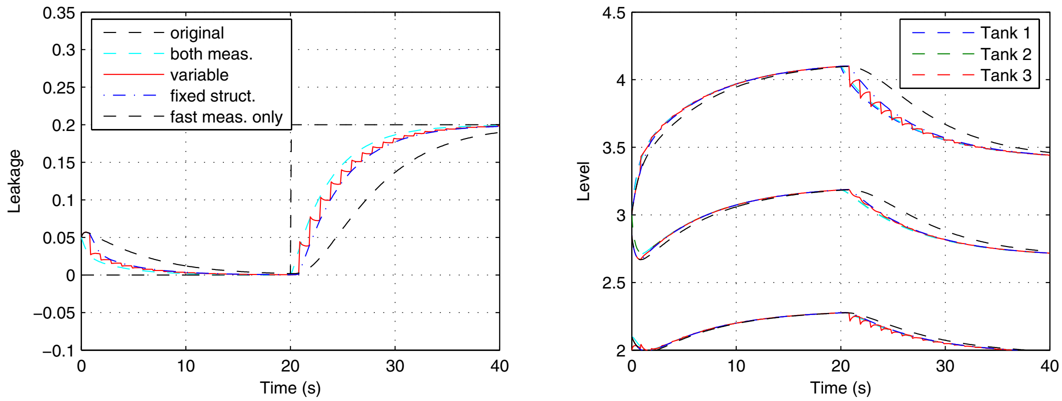


Fig. 2. Luenberger observer: $L = 10$, $t_d = 0$s. Right figure: top line – level tank 1, middle line – level tank 2, bottom line – level tank 3

at the intersampling points different estimates result. The example shows that the fixed structure estimates the parameter and the states more smoothly for both the Luenberger observer and the Kalman Filter. The simulations support the recommendation to use the fixed structure multirate estimator. While the arrival of a slow measurement of tank level 1 corrects the parameter strongly in the right direction for the variable structure, it increases the estimation error of tank levels 2 and 3. When only the fast measurement is available again, the estimator has to correct these levels and thus the estimation error of the parameter increases again.

V. Multirate Moving Horizon Estimation

The above derivation is valid for linear systems where the slow measurement is available without delay. However, slow measurements (e.g. GC analyses) are often also subject to considerable delays, which implies there is a sampling point and a time delay after which the result becomes available. Different options are available to deal with these delays, e.g. the use of an augmented system or the re-calculation of the estimated values after the slow measurement vector has become available. An alternative approach, which incorporates the use of infrequent measurements with delays in a natural fashion is the Moving Horizon Estimator (MHE [9]), which is an extension of the Kalman Filter on a moving past horizon of measurements. Advantages of this approach are the possibility to impose additional constraints on the estimated variables (e.g. that concentrations must be non-negative) and the natural extension to nonlinear systems. A new formulation of a Multirate Moving Horizon Estimator (MMHE) is presented below. From the result for the linear case discussed above, a fixed structure is used in the MMHE as well, rather than simply adding the slow measurement to the available data when it occurs, as one might propose intuitively.

As described above, the fixed structure extrapolates the last error of the slow measurement by a zero order hold. This has to be done beyond the sampling point of the next measurement, as it is subject to the delay. At the moment when the new measurement becomes available, the formerly held errors between the corresponding sampling point and the current time in the horizon have to be replaced by the new value of the error. It is advantageous if the horizon is at least as long as the measurement delay ($N \leq t_d$), but due to the fixed structure this is not a requirement. The MMHE

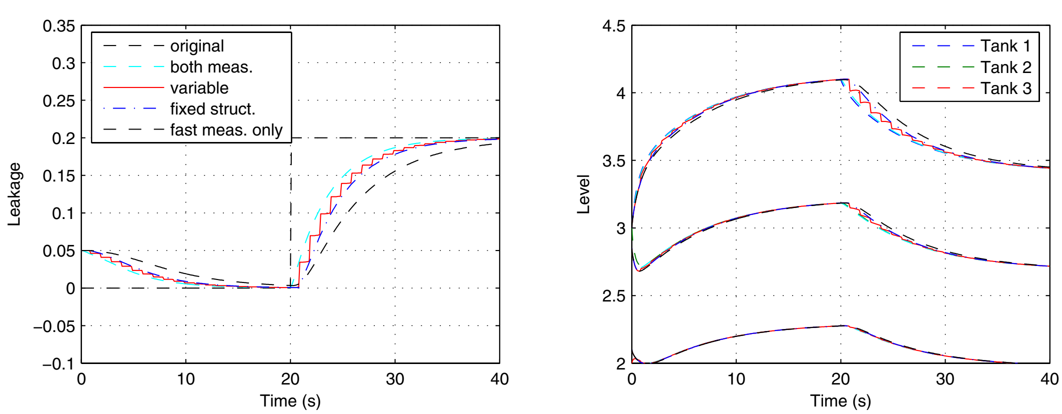


Fig. 3. Kalman Filter: $L = 10$, $t_d = 0$s. Right figure: top line – level tank 1, middle line – level tank 2, bottom line – level tank 3

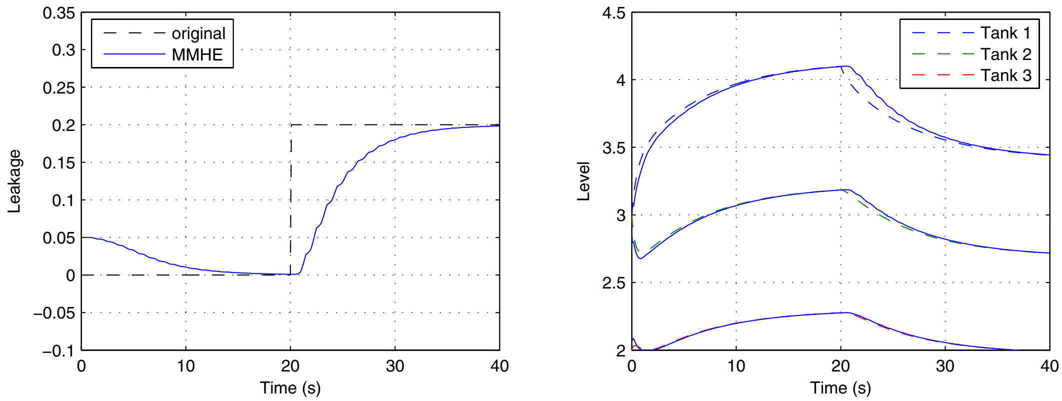


Fig. 4. MMHE: $L = 10$, $t_d = 3$ s. Right figure: top line – level tank 1, middle line – level tank 2, bottom line – level tank 3

formulation is given by

$$\min_{\hat{\xi}} \quad \Psi_k = \hat{\xi}^T_{k-N-1|k} \mathbf{P}^{-1}_{k-N|k-1} \hat{\xi}_{k-N-1|k} + \sum_{j=k-N}^{k-1} \hat{\xi}^T_{j|k} \mathbf{Q}^{-1} \hat{\xi}_{j|k} + \sum_{j=k-N}^{k} \hat{\varphi}^T_{j|k} \mathbf{R}^{-1} \hat{\varphi}_{j|k} \tag{11}$$

$$\begin{aligned}
\text{s.t.} \quad & \hat{\mathbf{x}}_{k-N|k} = \hat{\mathbf{x}}_{k-N|k-1} + \hat{\xi}_{k-N-1|k} \\
& \hat{\mathbf{x}}_{j+1|k} = \mathbf{A}\hat{\mathbf{x}}_{j|k} + \mathbf{B}u_j + \hat{\xi}_{j|k}, \quad j = k-N, \ldots, k-1 \\
& \hat{\varphi}_{j|k} = \begin{pmatrix} \mathbf{y}^F_j - \mathbf{C}^F \hat{\mathbf{x}}_{j|k} \\ \mathbf{y}^S_l - \mathbf{C}^S \hat{\mathbf{x}}_{l|\bar{k}} \end{pmatrix}, \quad j = k-N, \ldots, k \\
& l = \begin{cases} L(\lfloor j/L \rfloor - 1) \\ \quad \forall (j \geq L\lfloor k/L \rfloor) \wedge (k < L\lfloor k/L \rfloor + t_d) \\ L\lfloor j/L \rfloor \quad \text{otherwise} \end{cases} \\
& \bar{k} = \begin{cases} k & \forall l \leq k-N \\ l+N & \text{otherwise} \end{cases} \\
& \xi_{\min} \leq \hat{\xi}_{j-1|k} \leq \xi_{\max} \\
& \varphi_{\min} \leq \hat{\varphi}_{j|k} \leq \varphi_{\max} \\
& \mathbf{x}_{\min} \leq \hat{\mathbf{x}}_{|k} \leq \mathbf{x}_{\max}, \quad j = k-N, \ldots, k.
\end{aligned}$$

$\hat{x}_{j|k}$ denotes the estimate of x at time t_j based on the measurements up to time t_k. The value $\bar{k}$ (as in $\hat{\mathbf{x}}_{l|\bar{k}}$) is necessary if l lies outside the horizon

$\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{P}$ are weighting matrices and are chosen according to [9] similar to the covariance matrices of the Kalman Filter, i.e. $\mathbf{Q}$ is the covariance matrix of the model error, $\mathbf{R}$ denotes the covariance matrix of the measurement error and $\mathbf{P}$ defines the covariance matrix of the estimation error. The prediction of $\mathbf{P}$ assumes that all measurements are always available and it is calculated from the algebraic matric Riccati equation:

$$\hat{\mathbf{x}}_{i+1|k} = \mathbf{A}\hat{\mathbf{x}}_{i|k} + \mathbf{B}\mathbf{u}_i \tag{12}$$

$$\mathbf{P}_{i+1|k} = \mathbf{A}\left(\mathbf{P}_{i|k-1} - \mathbf{P}_{i|k-1}\mathbf{C}^T\left(\mathbf{C}\mathbf{P}_{i|k-1}\mathbf{C}^T + \mathbf{R}\right)^{-1} \mathbf{C}\mathbf{P}_{i|k-1}\right)\mathbf{A}^T + \mathbf{Q}, \quad i = k-N. \tag{13}$$

The above equations are written for an equidistantly sampled slow measurement. This is not required if the estimator is implemented. However, if the second measurement is not available regularly, but only at randomly distributed large intervals, a fixed weighting matrix $\mathbf{Q}$ might not be suitable for all cases.

Figure 4 shows the results of the estimation using the MMHE. The conditions were the same as for the KF and the Luenberger, however, a time delay of $t_d = 3$ was added. The horizon length chosen was $N = 7$. The estimation results are as good as those of the variable structure KF and therefore the MMHE can be used well for multirate systems with measurement delays.

VI. Multirate Estimator Performance Gauge

If both the system with the fast measurements alone and the multirate system are observable, a performance gauge is needed to justify a possibly expensive extra measurement device. The distance to unobservability can be used as such a measure. However, this measure only reflects the process dynamics and does not include the quality (errors) of the measurements.

The Kalman Filter of which the MMHE is a variant provides such a quality indicator. In a Kalman filter the covariance of the estimation error ($\mathbf{P}$) is estimated in parallel to the state estimation and its calculation is independent of the state estimation process. In time invariant single rate systems $\mathbf{P}$ converges to a steady state value. This steady state value can be used to gauge the quality of the estimation. For the multirate KF, a variable structure results and $\mathbf{P}$ reaches a cyclic steady state as illustrated in Figure 5. The Riccati equation for the variable structure scheme is given by

$$\mathbf{P}_{k+1} = \mathbf{A}(\mathbf{P}_k - \mathbf{P}_k\mathbf{C}^T(\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \mathbf{R})^{-1}\mathbf{C}\mathbf{P}_k)\mathbf{A}^T + \mathbf{Q} \tag{14}$$

$$\mathbf{C} = \begin{cases} \mathbf{C}^F & k/L \notin \mathbb{N} \\ \mathbf{C}^{FS} & k/L \in \mathbb{N} \end{cases}$$

$$\mathbf{R} = \begin{cases} \mathbf{R}^F & k/L \notin \mathbb{N} \\ \mathbf{R}^{FS} & k/L \in \mathbb{N} \end{cases},$$

where ^{FS} indicates the use of both measurement vectors.

Without the slow measurement, $\mathbf{P}$ would converge to a positive definite steady state value ($\mathbf{P}^F$). This steady state is also reached within the intersampling period if it is large compared to the fast sampling period. At a major sampling point, a step change of $\mathbf{P}$ occurs. For the calculation of the "steady state" $\mathbf{P}$, the change from one major sampling to the next ($k \rightarrow k+L$) is considered. Once that difference vanishes, the maximum improvement by the second measurement has been reached. $\mathbf{R}$ is simply extended by the variance of the slow measurement vector. A scalar factor defining the improvement can be calculated similar to [10].

A suitable measure should be between 0 and 1, where 1 represents the best possible case (the slow measurement is always available) and 0 represents the worst possible case (the slow measurement is never available). From Figure 5 the distance measure can be defined as

$$a = \frac{\left\|\mathbf{W}\mathbf{S}\mathbf{P}^F\mathbf{S}\right\|_2 - \left\|\mathbf{W}\mathbf{S}\mathbf{P}^{FS}_{mr,\min}\mathbf{S}\right\|_2}{\left\|\mathbf{W}\mathbf{S}\mathbf{P}^F\mathbf{S}\right\|_2} \qquad (15)$$

where $\mathbf{W}$ is a diagonal weighting matrix defining the importance of each state, $\mathbf{S}$ a diagonal weighting matrix for the normalization of the covariances $\mathbf{P}$, $\mathbf{P}^F$ is the steady state covariance matrix for the fast measurement vector and $\mathbf{P}^{FS}_{mr,\min}$ is the result of the convergence of $\mathbf{P}_{k,1}$ to $\mathbf{P}_{k+1,1}$. $\mathbf{P}^{FS}$ would be the steady state covariance matrix if all measurements ware always available. The value of a in (15) indicates which improvement of the estimates can be obtained by using the slow measurement in addition to the fast ones.

Figure 6 shows the improvement factor a due to multirate sampling plotted against the different entries in the covariance matrix $\mathbf{R}$ for example 1.

The plot should be interpreted as: For the given system with a fast sampled measurement y_1 with a variance of r_{fast}, adding a second slowly sampled measurement with a variance r_{slow} results in a relative improvement a as indicated by the mesh.

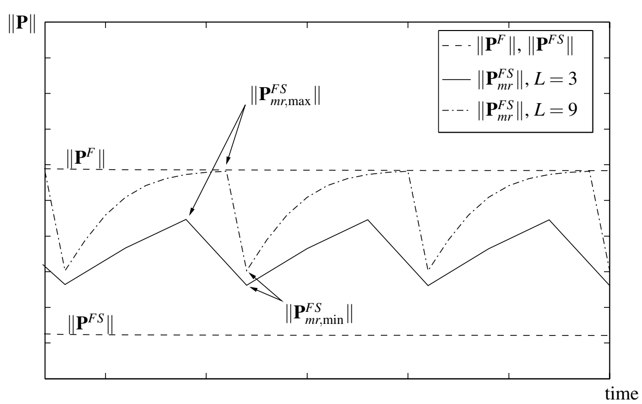


Fig. 5. Examplary result of variable structure multirate KF for a linear system ($\|\mathbf{P}\|_2$)

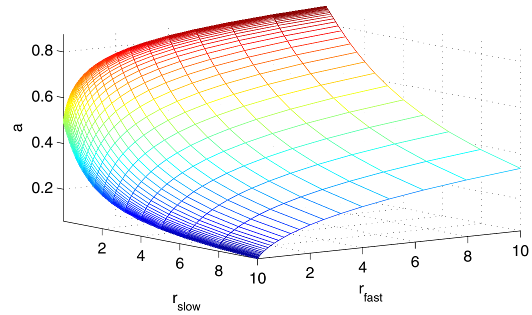


Fig. 6. Improvement of estimation quality due to multirate sampling for an exemplary linear system. $_{mr}$ denotes multirate.

VII. Summary

In this paper, we investigated the use of multirate delayed measurements in state estimation schemes. We discussed variable and fixed structure linear schemes and concluded that a fixed structure scheme should be preferred. The fixed structure distributes the amplification of the error between the predicted and the measured slow measurements over several fast measurements between two slow samples and this results in a smoother decay of the error and a reduces the vulnerability to measurement noise. As a consequence, we presented the corresponding Multirate Moving Horizon Estimator which can handle measurements at different sampling intervals and time delays. Finally, a performance measure was developed which assesses the improvement that can be obtained from the use of a second slow measurement.

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