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On the optimality of localized distributed controllers

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***Abstract*— Design of optimal distributed controllers with *a priori* assigned localization constraints is, in general, a difficult problem. Optimality of a closed-loop system is desirable because it guarantees, among other properties, favorable gain and phase margins. These margins provide robustness to different types of uncertainty. Alternatively, one can ask a following question: given a localized distributed exponentially stabilizing controller, is it inversely optimal with respect to some physically meaningful cost functional? We study this problem for Linear Spatially Invariant (LSI) systems and establish a frequency domain criterion for inverse optimality (in the LQR sense). We utilize this criterion to separate localized distributed controllers that are never optimal from localized distributed controllers that are optimal. In the latter case, we provide examples to demonstrate optimality with respect to physically appealing cost functionals. These cost functionals are characterized by state penalties that are not fully decentralized. Our results can be used to motivate design of both optimal and inversely optimal controllers for other classes of distributed control problems.**

***Index Terms*— Inverse Optimality; Localized Control; Spatially Invariant Systems.**

I. Introduction

Large arrays of spatially distributed dynamical systems are becoming prevalent in modern technological applications. These systems can range from the macroscopic−such as vehicular platoons [1]–[6], Unmanned Aerial Vehicle (UAV) formations [7]–[9], and satellites constellations [10]–[12]−to the microscopic, for example, arrays of micro-mirrors [13] or micro-cantilevers [14]. Significant potential for research on these systems is due to the field of Micro-Electro-Mechanical Systems (MEMS) where the fabrication of very large arrays of sensors and actuators is now both feasible and economical. The key design issues in the control of these systems are architectural such as the choice of localized versus centralized control.

Design of optimal distributed controllers with pre-specified localization constraints is, in general, a difficult task (we refer the reader to [15]–[19] and references therein for recent efforts in this area). Optimality of a closed-loop system is desirable because it guarantees, among other properties, favorable gain and phase margins. These margins provide robustness to different types of uncertainty [20]. Alternatively, one can ask a following question:

- *Given a localized distributed exponentially stabilizing controller, is it inversely optimal with respect to some physically meaningful performance index?*

We study this problem for Linear Spatially Invariant (LSI) systems [21] and derive a frequency domain condition for inverse optimality. This condition represents an extension of a well-known result for Linear Time Invariant (LTI) systems [22] to a class of systems studied in this paper. We provide examples of localized distributed controllers that are *inversely optimal with respect to meaningful performance criteria*, and examples of localized distributed controllers that are *not optimal in any sense*. Our results can be used to motivate design of both optimal and inversely optimal distributed controllers for other classes of spatio-temporal systems (e.g. spatially varying).

Our presentation is organized as follows: in section II, we setup the problem, introduce necessary background material, and provide two examples of LSI systems to illustrate that LQR design with *fully decentralized performance indices* yields *centralized optimal controllers*. In § III, we establish frequency domain criterion for inverse optimality of spatially invariant controllers. For systems with a single input field, this criterion requires the absolute value of the corresponding return difference to be greater than or equal to one at all spatio-temporal frequencies. In § IV, we provide examples of exponentially stabilizing localized distributed controllers and utilize results of § III to characterize control laws that are optimal (in the LQR sense). We show that optimality of localized distributed controllers can be guaranteed by departing from fully decentralized performance indices. We end our presentation with some concluding remarks in § V.

II. Preliminaries

We consider distributed systems of the form

$$\partial_t \psi(t,\xi) \;=\; [\mathcal{A}\psi(t)](\xi) \;+\; [\mathcal{B}u(t)](\xi). \qquad (1)$$

where operator $\mathcal{A}$ generates a *strongly continuous* (C_o) *semigroup* [23], [24]. We assume that spatial coordinate $\xi := [\,\xi_1 \;\cdots\; \xi_d\,]^*$ belongs to a commutative group $\mathbb{G}$, and that time independent operators $\mathcal{A}$ and $\mathcal{B}$ are invariant with respect to translations in this coordinate. These properties imply spatial invariance of (1). The analysis and design problems for LSI systems are greatly simplified by the application of the appropriate Fourier transform in the spatially invariant directions [21]. By taking a (spatial) Fourier transform of (1) we obtain

$$\dot{\hat{\psi}}_\kappa(t) \;=\; \hat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) \;+\; \hat{\mathcal{B}}_\kappa \hat{u}_\kappa(t), \qquad (2)$$

where $\kappa := [\,\kappa_1 \;\cdots\; \kappa_d\,]^*$ denotes the vector of frequencies corresponding to the spatial coordinates $\xi = [\,\xi_1 \;\cdots\; \xi_d\,]^*$, $\hat{\psi}_\kappa(t) := \hat{\psi}(t,\kappa)$, $\hat{u}_\kappa(t) := \hat{u}(t,\kappa)$, whereas $\hat{\mathcal{A}}_\kappa := \hat{\mathcal{A}}(\kappa)$ and $\hat{\mathcal{B}}_\kappa := \hat{\mathcal{B}}(\kappa)$ denote multiplication operators (i.e. Fourier symbols of operators $\mathcal{A}$ and $\mathcal{B}$, respectively). We note that (2) represents *a finite dimensional family of systems parameterized by* $\kappa \in \hat{\mathbb{G}}$: if $\psi(t,\xi)$ and $u(t,\xi)$ respectively denote fields with n and m components then, for any given $\kappa \in \hat{\mathbb{G}}$, $\hat{\psi}_\kappa \in \mathbb{C}^n$, $\hat{u}_\kappa \in \mathbb{C}^m$, which implies that $\hat{\mathcal{A}}_\kappa$ and $\hat{\mathcal{B}}_\kappa$ respectively denote matrices that belong to $\mathbb{C}^{n\times n}$ and $\mathbb{C}^{n\times m}$. We refer to the systems with $m = 1$ as *single input systems*. It was established in [21] that the dynamical properties of (1) can be inferred by checking the same properties of (2) for all $\kappa \in \hat{\mathbb{G}}$. Similar holds for design problems: for example, the solution to the optimal control problems for system (1) can be obtained by solving the analogous problems for a κ-parameterized family of finite dimensional systems (2).

A. Distributed LQR

We associate a quadratic performance index

$$J \;=\; \frac{1}{2}\int_0^{\infty}\left(\langle\psi,\mathcal{Q}\psi\rangle \;+\; \langle u,\mathcal{R}u\rangle\right)\,\mathrm{d}t, \tag{3}$$

with (1). If $\mathcal{Q}\geq 0$ and $\mathcal{R}>0$ are translation invariant operators, the application of spatial Fourier transform renders (3) into

$$J \;=\; \frac{1}{2}\int_0^{\infty}\int_{\hat{\mathbb{G}}}\left(\hat{\psi}_\kappa^*(t)\hat{\mathcal{Q}}_\kappa\hat{\psi}_\kappa(t)+\hat{u}_\kappa^*(t)\hat{\mathcal{R}}_\kappa\hat{u}_\kappa(t)\right)\mathrm{d}\kappa\,\mathrm{d}t \tag{4}$$

where $\mathrm{d}\kappa$ denotes the Haar measure. Thus, distributed LQR problem (1,3) amounts to solving the κ-parameterized family of finite dimensional LQR problems (2,4). If pairs $(\mathcal{A},\mathcal{B})$ and $(\mathcal{A}^*,\mathcal{Q}^{1/2})$ are exponentially stabilizable, then the κ-parameterized family of Algebraic Riccati Equations (AREs)

$$\hat{\mathcal{A}}_\kappa^*\hat{\mathcal{P}}_\kappa \;+\; \hat{\mathcal{P}}_\kappa\hat{\mathcal{A}}_\kappa \;+\; \hat{\mathcal{Q}}_\kappa \;-\; \hat{\mathcal{P}}_\kappa\hat{\mathcal{B}}_\kappa\hat{\mathcal{R}}_\kappa^{-1}\hat{\mathcal{B}}_\kappa^*\hat{\mathcal{P}}_\kappa \;=\; 0, \tag{5}$$

has a unique positive definite uniformly bounded solution for every $\kappa\in\hat{\mathbb{G}}$ [21]. This positive definite matrix determines the optimal stabilizing feedback for system (2) for every $\kappa\in\hat{\mathbb{G}}$

$$\hat{u}_\kappa \;:=\; \hat{\mathcal{K}}_\kappa\hat{\psi}_\kappa \;=\; -\hat{\mathcal{R}}_\kappa^{-1}\hat{\mathcal{B}}_\kappa^*\hat{\mathcal{P}}_\kappa\hat{\psi}_\kappa, \quad \kappa\,\in\,\hat{\mathbb{G}}. \tag{6}$$

In this case, there exist an exponentially stabilizing translation invariant feedback for system (1) that minimizes (3) [21]. This optimal stabilizing feedback for (1) is readily obtained by taking an inverse Fourier transform of (6).

B. Return difference equality

System (2) with a state-feedback control law $\hat{u}_\kappa=\hat{\mathcal{K}}_\kappa\hat{\psi}_\kappa$ can be equivalently represented by a feedback arrangement shown in Fig. 1. The so-called *return difference* of system

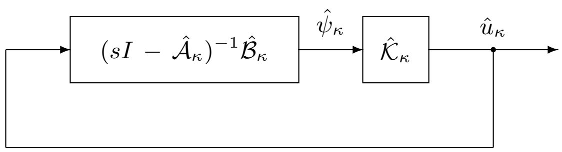

Fig. 1. Block diagram of system (2) with $\hat{u}_\kappa=\hat{\mathcal{K}}_\kappa\hat{\psi}_\kappa$.

whose block diagram is shown in Fig. 1 is defined by [20], [22]

$$\hat{\mathcal{H}}_\kappa(s) \;:=\; I \,-\, \hat{\mathcal{K}}_\kappa(sI\,-\,\hat{\mathcal{A}}_\kappa)^{-1}\hat{\mathcal{B}}_\kappa \;=:\; I\,-\,\hat{\mathcal{K}}_\kappa\hat{\mathcal{G}}_\kappa(s)\hat{\mathcal{B}}_\kappa.$$

This quantity is important because its inverse determines the sensitivity function $\hat{\mathcal{S}}_\kappa(s):=\hat{\mathcal{H}}_\kappa^{-1}(s)$. It is readily established that $\hat{\mathcal{H}}_\kappa(j\omega)$ for every $\omega\in\mathbb{R}$ and $\kappa\in\hat{\mathbb{G}}$ satisfies [20], [22]

$$\hat{\mathcal{R}}_\kappa \,+\, \hat{\mathcal{B}}_\kappa^*\hat{\mathcal{G}}_\kappa^*(j\omega)\hat{\mathcal{Q}}_\kappa\hat{\mathcal{G}}_\kappa(j\omega)\hat{\mathcal{B}}_\kappa \;=\; \hat{\mathcal{H}}_\kappa^*(j\omega)\hat{\mathcal{R}}_\kappa\hat{\mathcal{H}}_\kappa(j\omega), \tag{7}$$

where, for example, $\hat{\mathcal{G}}_\kappa(j\omega) \;:=\; (j\omega I\,-\,\hat{\mathcal{A}}_\kappa)^{-1}$ and $\hat{\mathcal{G}}_\kappa^*(j\omega)\,:=\,-\,(j\omega I+\hat{\mathcal{A}}_\kappa^*)^{-1}$. Equation (7) is usually referred to as the *return difference equality* and it follows directly from the ARE. A straightforward consequence of this equality is

$$\hat{\mathcal{H}}_\kappa^*(j\omega)\hat{\mathcal{R}}_\kappa\hat{\mathcal{H}}_\kappa(j\omega) \;\geq\; \hat{\mathcal{R}}_\kappa. \tag{8}$$

Relationships (7) and (8) are utilized in § III to express a frequency domain condition for inverse optimality of distributed exponentially stabilizing spatially invariant controllers.

C. Distributed controller architectures

Fig. 2 illustrates different control strategies that can be used for control of spatially distributed systems: centralized, localized, and fully decentralized. Centralized controllers require information from all plant units for achieving the desired control objective. On the other hand, in fully decentralized strategies control unit K_n uses only information from the n-th plant unit G_n on which it acts. An example of a localized distributed control architecture with nearest neighbor interactions is shown in Fig. 2.

D. Examples of optimal distributed design

We next provide two examples of spatially invariant systems with fully distributed measurements and controls:

- *diffusion equation over an infinite domain* ($\mathbb{G}:=\mathbb{R}$),
- *mass-spring system on an infinite line* ($\mathbb{G}:=\mathbb{Z}$).

We demonstrate that the LQR design with *fully decentralized performance indices* yields *centralized optimal controllers* for these systems.

1) Diffusion equation: We consider a one-dimensional diffusion equation

$$\psi_t(t,\xi) \;=\; \psi_{\xi\xi}(t,\xi) \;+\; c\psi(t,\xi) \;+\; u(t,\xi), \quad \xi\in\mathbb{R}. \tag{9}$$

The application of the standard spatial Fourier transform yields

$$\begin{aligned}\dot{\hat{\psi}}_\kappa(t) \;&=\; (c-\kappa^2)\hat{\psi}_\kappa(t) \;+\; \hat{u}_\kappa(t),\\ &=:\; \hat{\mathcal{A}}_\kappa\hat{\psi}_\kappa(t) \;+\; \hat{\mathcal{B}}_\kappa\hat{u}_\kappa(t), \quad \kappa\in\mathbb{R},\end{aligned}$$

which implies that (9) is not (open-loop) exponentially stable if $c\geq 0$. Choosing, for example, $\mathcal{Q}:=qI$ and $\mathcal{R}:=rI$ in (3), with ($q=\text{const.}>0$, $r=\text{const.}>0$), yields the following positive definite solution to the κ-parameterized ARE (5):

$$\hat{\mathcal{P}}_\kappa \;=\; r(c-\kappa^2) \;+\; \sqrt{r^2(c-\kappa^2)^2+rq},$$

which gives the optimal control of the form (6) with

$$\hat{\mathcal{K}}_\kappa \;\;=\;\; -\left((c-\kappa^2) \;+\; \sqrt{(c-\kappa^2)^2+q/r}\right).$$

Since $\hat{\mathcal{K}}_\kappa$ is irrational function of κ, it cannot be implemented by a PDE (in t and ξ). Rather, the optimal control in the physical space assumes the form

$$u(t,\xi) \;=\; \int_{\mathbb{R}}\mathcal{K}(\xi-\zeta)\psi(t,\zeta)\,\mathrm{d}\zeta. \tag{10}$$

In [21], it was established that $\mathcal{K}$ decays exponentially fast as a function of its argument which is a desirable property for implementation. Despite this nice feature, (10) represents a centralized controller.

2) Mass-spring system: A system consisting of an infinite number of identical masses and springs on a line is shown in Fig. 3. If restoring forces are considered as linear functions of displacements, the dynamics of the mass indexed by $\xi\in\mathbb{Z}$ are given by

$$\ddot{x}(t,\xi) \;=\; x(t,\xi-1) \;-\; 2x(t,\xi) \;+\; x(t,\xi+1) \;+\; u(t,\xi),$$

where $x(t,\xi)$ represents the displacement from a reference position of the mass ξ, and $u(t,\xi)$ is the control applied on the mass ξ. A state-space representation of this system is given by

$$\begin{aligned}\dot{\psi}(t,\xi) \;&=\; \begin{bmatrix} 0 & 1\\ T_{-1}-2+T_1 & 0\end{bmatrix}\psi(t,\xi) \;+\; \begin{bmatrix}0\\1\end{bmatrix}u(t,\xi),\\ \psi(t,\xi) &:= \begin{bmatrix} x(t,\xi) & \dot{x}(t,\xi)\end{bmatrix}^*, \quad \xi\in\mathbb{Z},\end{aligned} \tag{11}$$

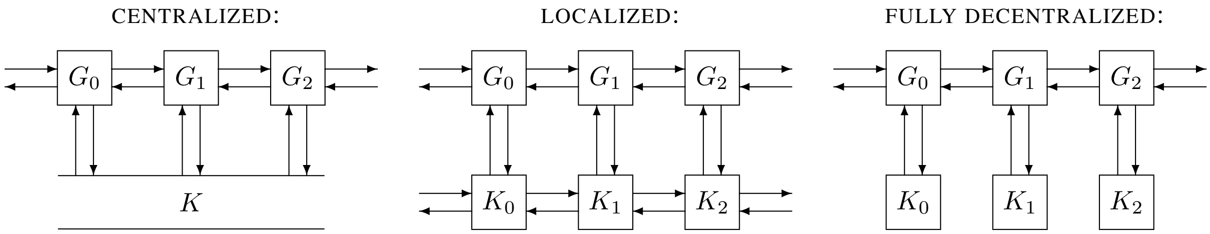

Fig. 2. Distributed architectures for centralized, localized (with nearest neighbor interactions), and fully decentralized control strategies.

where T_{-1} and T_1 respectively denote the operators of translation by -1 and 1 (in the mass' index). We utilize the fact that system (11) has spatially invariant dynamics over discrete spatial lattice $\mathbb{Z}$ and apply the appropriate Fourier transform (spatial $\mathcal{Z}$ transform evaluated on a unit circle) to obtain

$$\begin{aligned} \dot{\hat{\psi}}_\kappa(t) &= \begin{bmatrix} 0 & 1 \\ a_\kappa & 0 \end{bmatrix} \hat{\psi}_\kappa(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}_\kappa(t), \\ &=: \hat{\mathcal{A}}_\kappa \hat{\psi}_\kappa(t) + \hat{\mathcal{B}}_\kappa \hat{u}_\kappa(t), \\ & \quad a_\kappa := 2(\cos\kappa - 1), \quad \kappa \in [0, 2\pi), \end{aligned} \tag{12}$$

where, for example,

$$\hat{u}(t,\kappa) := \sum_{\xi \in \mathbb{Z}} u(t,\xi) \mathrm{e}^{-j\kappa\xi}.$$

Selecting, for example, fully decentralized weights

$$\mathcal{Q} := \begin{bmatrix} q_1 I & 0 \\ 0 & q_2 I \end{bmatrix}, \quad \mathcal{R} := rI,$$

with ($q_1 = \text{const.} > 0$, $q_2 = \text{const.} \geq 0$, $r = \text{const.} > 0$), renders (3) into

$$J := \frac{1}{2} \int_0^\infty (\sum_{\xi \in \mathbb{Z}} q_1 x^2(t,\xi) + q_2 \dot{x}^2(t,\xi) + r u^2(t,\xi))\, \mathrm{d}t,$$

and yields the following optimal control

$$\begin{aligned} \hat{u}_\kappa(t) &= \hat{\mathcal{K}}_\kappa \hat{\psi}_\kappa(t) := \begin{bmatrix} \hat{\mathcal{K}}_{1\kappa} & \hat{\mathcal{K}}_{2\kappa} \end{bmatrix} \hat{\psi}_\kappa(t), \\ \hat{\mathcal{K}}_{1\kappa} &= 2(1 - \cos\kappa) - \sqrt{4(\cos\kappa - 1)^2 + q_1/r}, \\ \hat{\mathcal{K}}_{2\kappa} &= -\sqrt{-2\hat{\mathcal{K}}_{1\kappa} + q_2/r}. \end{aligned}$$

Again, since $\hat{\mathcal{K}}_\kappa$ is irrational function of κ, it cannot be implemented by a localized distributed controller. Rather, the optimal control in the physical space is a centralized controller of the form

$$u(t,\xi) = \sum_{\zeta \in \mathbb{Z}} \mathcal{K}(\xi - \zeta) \psi(t,\zeta), \quad \xi \in \mathbb{Z}. \tag{13}$$

In § IV, we illustrate that both spatially localized and fully decentralized exponentially stabilizing controllers for diffusion equation (9) and mass-spring system (11) can be inversely optimal with respect to physically appealing cost functionals. In particular, for a diffusion equation, these cost functionals incorporate penalties on spatial derivatives of ψ (in addition to penalties on ψ), which implies that they are no longer fully decentralized.

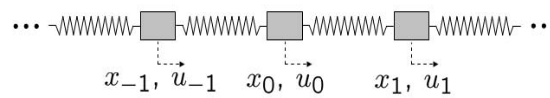

Fig. 3. Mass-spring system.

III. THE INVERSE PROBLEM OF OPTIMAL DISTRIBUTED CONTROL

In this section, we consider the inverse problem of optimal exponential stabilization of LSI system (1). This problem is inverse because we assume that an exponentially stabilizing state-feedback control law for (1) is available and search for performance indices (3) for which this control law is optimal. In other words, operators $\mathcal{Q}$ and $\mathcal{R}$ in (3) are not *a priori* assigned; rather, they are determined *a posteriori* by the exponentially stabilizing state-feedback. We state a frequency domain condition that separates distributed controllers that are never optimal (in the LQR sense) from distributed controllers that are optimal (in the LQR sense). This condition represents an extension of a well-known result for finite dimensional LTI systems [22] to a class of systems considered in this paper. In particular, for single input systems, the inverse optimality of an exponentially stabilizing control law $\mathcal{K}$ is guaranteed if and only if the absolute value of the return difference:

$$\hat{\mathcal{H}}_\kappa(j\omega) := I - \hat{\mathcal{K}}_\kappa (j\omega I - \hat{\mathcal{A}}_\kappa)^{-1} \hat{\mathcal{B}}_\kappa =: I - \hat{\mathcal{K}}_\kappa \hat{\mathcal{G}}_\kappa(j\omega) \hat{\mathcal{B}}_\kappa,$$

is not less than one at any spatio-temporal frequency pair (κ, ω) (see Theorem 1 for precise formulation).

Theorem 1 and Corollary 2 are readily established by recognizing that the application of the appropriate spatial Fourier transform renders LSI systems into a κ-parameterized family of finite dimensional LTI systems. We refer the reader to [20], [22] for finite dimensional LTI results.

Theorem 1: Let a triple $\{\mathcal{A}, \mathcal{B}, \mathcal{K}\}$ for LSI system (1) satisfy: (a) $\mathcal{A}$ is a generator of a C_o semigroup; (b) $(\mathcal{A}, \mathcal{B})$ is exponentially controllable; and (c) $\mathcal{K}$ is an exponentially stabilizing translation invariant state-feedback operator. Then, a necessary and sufficient condition for $\mathcal{K}$ to be an optimal control law with respect to a performance index (3) is that condition

$$\sigma_{\min} \left\{ \hat{\mathcal{R}}_\kappa^{1/2} \hat{\mathcal{H}}_\kappa(j\omega) \hat{\mathcal{R}}_\kappa^{-1/2} \right\} \geq 1, \tag{14}$$

holds for all $\omega \in \mathbb{R}$ and $\kappa \in \hat{\mathbb{G}}$. For a single input LSI system (1), condition (14) simplifies to

$$|\hat{\mathcal{H}}_\kappa(j\omega)| \geq 1, \quad \forall\, \omega \in \mathbb{R}, \quad \forall\, \kappa \in \hat{\mathbb{G}}. \tag{15}$$

Corollary 2: Let a quadruple $\{\mathcal{A}, \mathcal{B}, \mathcal{K}, \mathcal{R}\}$ for LSI system (1) satisfy: (a) $\mathcal{A}$ is a generator of a C_o semigroup; (b)

$(\mathcal{A}, \mathcal{B})$ is exponentially stabilizable; (c) $\mathcal{K}$ is an exponentially stabilizing translation invariant state-feedback operator; (d) $\mathcal{R} > 0$; and (e) return difference inequality (14) holds for all $\omega \in \mathbb{R}$ and $\kappa \in \hat{\mathbb{G}}$. Then, there exist a translation invariant operator $\mathcal{Q} = \mathcal{D}\mathcal{D}^*$ with $(\mathcal{A}^*, \mathcal{D})$ exponentially stabilizable such that the optimal state-feedback operator $\bar{\mathcal{K}}$ associated with the LQR problem (1,3) satisfies $\bar{\mathcal{K}}\mathcal{G}(j\omega)\mathcal{B} = \mathcal{K}\mathcal{G}(j\omega)\mathcal{B}$. If a pair $(\mathcal{A}, \mathcal{B})$ is exponentially controllable, then $\mathcal{K} = \bar{\mathcal{K}}$.

If Corollary 2 is satisfied, then a translation invariant operator $\mathcal{D}$ ($\mathcal{Q} = \mathcal{D}\mathcal{D}^*$) can be determined from return difference equality (7) using polynomial matrix fraction description of $\hat{\mathcal{H}}_\kappa(j\omega)$. This operator can always be selected to guarantee exponential stabilizability of pair $(\mathcal{A}^*, \mathcal{D})$ (we refer the reader to [20], [22] for finite dimensional LTI version). In general, for a given $\mathcal{R}$ there are many different $\mathcal{D}$'s that satisfy (7) and yield $\mathcal{K}$ as a solution to the corresponding LQR problem. For single input systems, $\hat{\mathcal{D}}_\kappa$ can be determined from [20], [22]

$$\|\hat{\mathcal{D}}_\kappa^* \hat{\mathcal{G}}_\kappa(j\omega)\hat{\mathcal{B}}_\kappa\|^2 \;=\; \hat{\mathcal{R}}_\kappa\left(|\hat{\mathcal{H}}_\kappa(j\omega)|^2 \;-\; 1\right). \qquad (16)$$

IV. Examples of inversely optimal distributed design

In this section, we investigate inverse optimality of localized distributed exponentially stabilizing controllers for diffusion equation (9) and mass-spring system (11). We utilize results of § III to distinguish between controllers that are never optimal and controllers that are optimal. In the latter case, we show that inverse optimality is guaranteed with respect to physically appealing performance criteria. For a diffusion equation, we demonstrate that localized distributed and even fully decentralized optimal controllers can be obtained by incorporating spatial derivatives of ψ (in addition to ψ) in the performance index. Similarly, for a mass-spring system, we establish that spatially localized cost functionals can produce controllers with favorable architectures. These observations should be compared to the results of § II-D, where it was shown that LQR design with fully decentralized performance criteria results into centralized optimal controllers for both these systems.

A. Diffusion equation

It is readily established that the following spatially invariant localized distributed controller

$$\begin{aligned} u(t,\xi) \;&=\; -\big(\beta\psi_{\xi\xi}(t,\xi) \;+\; (c\,+\,\alpha)\psi(t,\xi)\big), \\ &\Updownarrow \\ \hat{u}_\kappa(t) \;=\; \hat{\mathcal{K}}_\kappa\hat{\psi}_\kappa(t) \;&=\; -\big(c\,+\,\alpha\,-\,\beta\kappa^2\big)\hat{\psi}_\kappa(t), \end{aligned} \qquad (17)$$

provides exponential stability of (9) so long as κ-independent real design parameters α and β respectively satisfy $\alpha > 0$ and $\beta \in (-\infty, 1]$. Based on Theorem 1, it follows that controller (17) is inversely optimal if and only if

$$\big(\alpha \,-\, c \,+\, (2 \,-\, \beta)\kappa^2\big)\big(\alpha \,+\, c \,-\, \beta\kappa^2\big) \;\geq\; 0,$$

holds for all $\kappa \in \mathbb{R}$. This condition is satisfied for all $\kappa \in \mathbb{R}$ if and only if $\alpha \geq c$ and $\beta \leq 0$. Thus, *if either $\alpha < c$ or $\beta \in (0, 1]$ then controller (17) is never optimal in the LQR sense*. In other words, for this choice of design parameters α and β it is not possible to select a pair $(\mathcal{Q} \geq 0, \mathcal{R} > 0)$ for which (17) is obtained as a solution to the corresponding LQR problem (1,3). This implies that this exponentially stabilizing control law does not have any stability margins: with a slightly perturbed feedback closed-loop system becomes unstable. On the other hand, for $\alpha \geq c$ and $\beta \leq 0$ there always exist $(\mathcal{Q} \geq 0, \mathcal{R} > 0)$ in (3) with respect to which (17) is inversely optimal. Choosing, for example, $\mathcal{R} := rI$ in (3), with $r = \text{const.} > 0$, yields the following state penalty:

$$\begin{aligned} \hat{\mathcal{Q}}_\kappa &= r\left((\alpha^2 - c^2) + 2\big(c + \alpha(1-\beta)\big)\kappa^2 + \beta(\beta-2)\kappa^4\right), \\ &\Updownarrow \\ \mathcal{Q} &= r\left((\alpha^2 - c^2)I - 2\big(c + \alpha(1-\beta)\big)\partial_{\xi\xi} + \beta(\beta-2)\partial_{\xi\xi\xi\xi}\right). \end{aligned}$$

Since for any $\kappa \in \mathbb{R}$, κ-parameterized diffusion equation represents a scalar system, this state penalty is obtained as a unique solution to (16) for any $r > 0$.

Thus, we have established optimality of spatially invariant localized distributed controller (17) with respect to the following performance index

$$\begin{aligned} J \;=\; & \frac{r}{2}(\alpha^2 - c^2)\int_0^\infty \langle\psi,\psi\rangle \; \mathrm{d}t \;+ \\ & \frac{r}{2}2\big(c + \alpha(1-\beta)\big)\int_0^\infty \langle\psi_\xi,\psi_\xi\rangle \; \mathrm{d}t \;+ \\ & \frac{r}{2}\beta(\beta-2)\int_0^\infty \langle\psi_{\xi\xi},\psi_{\xi\xi}\rangle \; \mathrm{d}t \;+ \\ & \frac{r}{2}\int_0^\infty \langle u,u\rangle \; \mathrm{d}t, \quad r > 0, \quad \alpha \geq c, \quad \beta \leq 0. \end{aligned} \qquad (18)$$

In particular, for $\beta = 0$ controller (17) is fully decentralized and (18) simplifies to

$$\begin{aligned} J \;=\; & \frac{r}{2}(\alpha^2 - c^2)\int_0^\infty \langle\psi,\psi\rangle \; \mathrm{d}t \;+ \\ & \frac{r}{2}2(\alpha + c)\int_0^\infty \langle\psi_\xi,\psi_\xi\rangle \; \mathrm{d}t \;+ \\ & \frac{r}{2}\int_0^\infty \langle u,u\rangle \; \mathrm{d}t, \quad r > 0, \quad \alpha \geq c. \end{aligned} \qquad (19)$$

To recap:

- *fully decentralized controller*

  $$u(t,\xi) \;=\; -\,(c\,+\,\alpha)\psi(t,\xi),$$

  with $\alpha \geq c$ represents exponentially stabilizing solution to the LQR problem (9,19);
- *localized distributed controller*

  $$u(t,\xi) \;=\; -\big(\beta\psi_{\xi\xi}(t,\xi) \;+\; (c\,+\,\alpha)\psi(t,\xi)\big),$$

  with $\alpha \geq c$, $\beta < 0$ represents exponentially stabilizing solution to the LQR problem (9,18).

Remark 1: Our analysis indicates that a choice of the state-space on which optimal control problems are formulated can significantly influence localization properties of resulting distributed optimal controllers. Example of § II-D.1 illustrates that the spatially invariant LQR problem (for a diffusion equation) formulated on the space of square integrable functions $L_2(-\infty, \infty)$ yields centralized controllers. On the other hand, the LQR design performed on the Sobolev spaces $H_1(-\infty, \infty)$ or $H_2(-\infty, \infty)$ can result into localized distributed and fully decentralized controllers provided that the penalties on $\langle\psi,\psi\rangle$, $\langle\psi_\xi,\psi_\xi\rangle$, and $\langle\psi_{\xi\xi},\psi_{\xi\xi}\rangle$ are appropriately selected.

B. Mass-spring system

It is easily shown that the exponential stability of (11) is guaranteed with the spatially invariant localized distributed controller of the form

$$
\begin{aligned}
u(t,\xi) &= -\left[\begin{array}{cc} \alpha + \beta(T_{-1} - 2 + T_1) & \gamma \end{array}\right] \psi(t,\xi) \\
&= -(\alpha - 2\beta)x(t,\xi) - \gamma\dot{x}(t,\xi) - \\
&\qquad \beta(x(t,\xi-1) + x(t,\xi+1)), \quad \xi \in \mathbb{Z}, \\
&\qquad\qquad \Updownarrow \\
\hat{u}_\kappa(t) &= \hat{\mathcal{K}}_\kappa \hat{\psi}_\kappa(t) \\
&= -\left[\begin{array}{cc} \alpha + \beta a_\kappa & \gamma \end{array}\right] \hat{\psi}_\kappa(t), \quad \kappa \in [0,\, 2\pi),
\end{aligned}
\tag{20}
$$

so long as κ-independent real design parameters α, β, and γ satisfy

$$
\gamma > 0 \;\text{ and }\; \left\{ \begin{array}{rl} \alpha > 0, & \beta \in (-\infty,\, 1], \\ \alpha > 4(\beta - 1), & \beta > 1. \end{array} \right.
$$

Based on Theorem 1, it follows that controller (20) is inversely optimal if and only if

$$
\big(\gamma^2 - 2(\alpha + \beta a_\kappa)\big)\omega^2 + \big(\alpha + \beta a_\kappa\big)\big(\alpha + (\beta - 2)a_\kappa\big) \geq 0,
$$

holds for all $\omega \in \mathbb{R}$, $\kappa \in [0,\, 2\pi)$. This criterion for inverse optimality is satisfied for all $\omega \in \mathbb{R}$, $\kappa \in [0,\, 2\pi)$ if and only if either

$$
\{\alpha > 0,\ \beta \leq 0,\ \gamma \geq \sqrt{2(\alpha - 4\beta)}\},
$$

or

$$
\{\alpha \geq 4\beta,\ \beta > 0,\ \gamma \geq \sqrt{2\alpha}\}.
$$

Hence, *controller (20) is never optimal in the LQR sense if parameters α, β, and γ fail to satisfy either of these two conditions*. On the other hand, if α, β, and γ satisfy either of these two conditions than there always exist $(\mathcal{Q} \geq 0, \mathcal{R} > 0)$ in (3) with respect to which (20) is optimal. Selecting, for example,

$$
\left. \begin{array}{rcl} \mathcal{R} & := & rI \\ r & = & \text{const.} > 0 \end{array} \right\}, \quad \mathcal{Q} := \left[\begin{array}{cc} \mathcal{Q}_{11} & 0 \\ 0 & \mathcal{Q}_{22} \end{array} \right], \tag{21}
$$

in (3) yields the following state penalty:

$$
\begin{aligned}
\hat{\mathcal{Q}}_{11\kappa} &= r\left(\alpha^2 + 2\alpha(\beta - 1)a_\kappa + \beta(\beta - 2)a_\kappa^2\right), \\
\hat{\mathcal{Q}}_{22\kappa} &= r\left(\gamma^2 - 2(\alpha + \beta a_\kappa)\right), \\
&\qquad \Updownarrow \\
\mathcal{Q}_{11} &= r\big(\alpha^2 + 2\alpha(\beta - 1)(T_{-1} - 2 + T_1) + \\
&\qquad \beta(\beta - 2)(T_{-2} - 4T_{-1} + 6 - 4T_1 + T_2)\big), \\
\mathcal{Q}_{22} &= r\left(\gamma^2 - 2(\alpha + \beta(T_{-1} - 2 + T_1))\right).
\end{aligned}
$$

Thus, we have established optimality of spatially invariant localized distributed controller (20) with respect to the following performance index

$$
\begin{aligned}
J &= \frac{1}{2}\int_0^\infty \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \psi^*(t,n) Q_{n-m} \psi_m(t,m)\, \mathrm{d}t + \\
&\quad \frac{1}{2}\int_0^\infty \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} u^*(t,n) R_{n-m} u(t,m)\, \mathrm{d}t,
\end{aligned}
\tag{22}
$$

where

$$
\begin{gathered}
\{R_0 = r = \text{const.} > 0;\ R_n = 0,\ \forall\, n \in \mathbb{Z} \setminus \{0\}\}, \\
\{Q_n = 0,\ \forall\, n \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}\}, \\
Q_0 = r \left[\begin{array}{cc} \alpha^2 - 4\alpha(\beta - 1) + 6\beta(\beta - 2) & 0 \\ 0 & \gamma^2 - 2(\alpha - 2\beta) \end{array} \right], \\
Q_{\pm 1} = r \left[\begin{array}{cc} 2\alpha(\beta - 1) - 4\beta(\beta - 2) & 0 \\ 0 & -2\beta \end{array} \right], \\
Q_{\pm 2} = r \left[\begin{array}{cc} \beta(\beta - 2) & 0 \\ 0 & 0 \end{array} \right].
\end{gathered}
\tag{23}
$$

In particular, for $\beta = 0$ controller (20) is fully decentralized and (23) simplifies to

$$
\begin{gathered}
\{R_0 = r = \text{const.} > 0;\ R_n = 0,\ \forall\, n \in \mathbb{Z} \setminus \{0\}\}, \\
\{Q_n = 0,\ \forall\, n \in \mathbb{Z} \setminus \{0, \pm 1\}\}, \\
Q_0 = r \left[\begin{array}{cc} \alpha^2 + 4\alpha & 0 \\ 0 & \gamma^2 - 2\alpha \end{array} \right], \\
Q_{\pm 1} = r \left[\begin{array}{cc} -2\alpha & 0 \\ 0 & 0 \end{array} \right].
\end{gathered}
\tag{24}
$$

To recap:

- *fully decentralized controller*

  $$
  u(t,\xi) = -(\alpha x(t,\xi) + \gamma \dot{x}(t,\xi)),
  $$

  with $\{\alpha > 0,\ \gamma > \sqrt{2\alpha}\}$ represents exponentially stabilizing solution to the LQR problem (11,22,24);
- *nearest neighbor interaction controller*

  $$
  \begin{aligned}
  u(t,\xi) &= -(\alpha - 2\beta)x(t,\xi) - \gamma\dot{x}(t,\xi) - \\
  &\qquad \beta(x(t,\xi-1) + x(t,\xi+1)),
  \end{aligned}
  $$

  with either

  $$
  \{\alpha > 0,\ \beta < 0,\ \gamma \geq \sqrt{2(\alpha - 4\beta)}\},
  $$

  or

  $$
  \{\alpha \geq 4\beta,\ \beta > 0,\ \gamma \geq \sqrt{2\alpha}\},
  $$

  represents exponentially stabilizing solution to the LQR problem (11,22,23).

Remark 2: The above penalties on $\{x(t,\xi)\}_{\xi\in\mathbb{Z}}$ and $\{\dot{x}(t,\xi)\}_{\xi\in\mathbb{Z}}$ represent unique solutions to (16) provided that (21) is satisfied (that is, $\mathcal{Q}_{12} \equiv 0$). However, for given $\mathcal{R} := rI > 0$, there are many other operators $\mathcal{Q} = \mathcal{Q}^* \geq 0$ with non-zero off-diagonal elements (that is, $\mathcal{Q}_{12} \neq 0$) that satisfy (16) and give controller (20) as a solution to the corresponding LQR problem.

Remark 3: Example of § II-D.2 illustrates that the spatially invariant LQR design (for a mass-spring system) with fully decentralized performance indices yields centralized controllers. On the other hand, spatially localized performance indices (with penalties on positions and velocities of several neighboring masses) can yield localized distributed and fully decentralized controllers, provided that these penalties are appropriately assigned. However, it is very difficult to choose these cost functionals *a priori*. Rather, they have been determined *a posteriori* by the exponentially stabilizing control law using the return difference equality.

V. Concluding remarks

This paper deals with the inverse problem of optimal distributed stabilization of LSI systems. We establish a frequency domain criterion that separates controllers that are never optimal (in the LQR sense) from controllers that are optimal (in the LQR sense). This criterion is expressed in terms of return difference and, for systems with a single input field, the return difference is required to be at least equal to one at all spatial and temporal frequencies. We provide examples of localized distributed controllers that are inversely optimal with respect to physically appealing performance indices. A distinctive feature of these indices is the absence of fully decentralized state penalties that seem to yield centralized optimal controllers.

Our ongoing research effort is directed towards finding out whether distributed backstepping controllers for nonlinear systems on lattices [25]–[28] are inversely optimal, and if not whether they could be appropriately modified such that inverse optimality is guaranteed.

References

[1] P. Varaiya, "Smart cars on smart roads: problems of control," *IEEE Transactions on Automatic Control*, vol. 38, no. 2, pp. 195–207, February 1993.

[2] H. Raza and P. Ioannou, "Vehicle following control design for automated highway systems," *IEEE Control Systems Magazine*, vol. 16, no. 6, pp. 43–60, December 1996.

[3] D. Swaroop and J. K. Hedrick, "Constant spacing strategies for platooning in automated highway systems," *Transactions of the ASME. Journal of Dynamic Systems, Measurement and Control*, vol. 121, no. 3, pp. 462–470, September 1999.

[4] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1835–1842, October 2004.

[5] M. R. Jovanović and B. Bamieh, "On the ill-posedness of certain vehicular platoon control problems," *submitted to IEEE Transactions on Automatic Control*, 2003, available at http://www.ece.umn.edu/users/mihailo/.

[6] M. R. Jovanović, J. M. Fowler, B. Bamieh, and R. D'Andrea, "On avoiding peaking in the control of vehicular platoons," *submitted to Automatica*, 2005, available at http://www.ece.umn.edu/users/mihailo/.

[7] D. Chichka and J. Speyer, "Solar-powered, formation-enhanced aerial vehicle systems for sustained endurance," in *Proceedings of the 1998 American Control Conference*, 1998, pp. 684–688.

[8] J. M. Fowler and R. D'Andrea, "Distributed control of close formation flight," in *Proceedings of the 41st IEEE Conference on Decision and Control*, 2002, pp. 2972–2977.

[9] ——, "A formation flight experiment," *IEEE Control Systems Magazine*, vol. 23, no. 5, pp. 35–43, October 2003.

[10] V. Kapila, A. G. Sparks, J. M. Buffington, and Q. Yan, "Spacecraft formation flying: dynamics and control," *Journal of Guidance, Control, and Dynamics*, vol. 23, no. 3, pp. 561–564, May-June 2000.

[11] R. W. Beard, J. Lawton, and F. Y. Hadaegh, "A coordination architecture for spacecraft formation control," *IEEE Transactions on Control Systems Technology*, vol. 9, no. 6, pp. 777–790, November 2001.

[12] H. Wong, V. Kapila, and A. G. Sparks, "Adaptive output feedback tracking control of spacecraft formation," *International Journal of Robust and Nonlinear Control*, vol. 12, no. 2-3, pp. 117–139, February-March 2002.

[13] D. T. Neilson, "MEMS subsystems for optical networking," in *Proceedings of 8th Microoptics Conference (MOC 01)*, Osaka, Japan, 2001.

[14] M. Napoli, B. Bamieh, and M. Dahleh, "Optimal control of arrays of microcantilevers," *Transactions of the ASME. Journal of Dynamic Systems, Measurement and Control*, vol. 121, no. 4, pp. 686–690, December 1999.

[15] G. Ayres and F. Paganini, "Convex synthesis of localized controllers for spatially invariant systems," *Automatica*, vol. 38, p. 445456, 2002.

[16] M. Rotkowitz and S. Lall, "Decentralized control information structures preserved under feedback," in *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, 2002, pp. 569–575.

[17] P. G. Voulgaris, G. Bianchini, and B. Bamieh, "Optimal $\mathcal{H}_2$ controllers for spatially invariant systems with delayed communication requirements," *Systems and Control Letters*, vol. 50, no. 5, pp. 347–361, December 2003.

[18] G. Ayres and F. Paganini, "A method for decentralized control design in spatially invariant systems," in *Proceedings of the 2003 American Control Conference*, Denver, CO, 2003, pp. 125–130.

[19] C. Langbort, R. S. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1502–1519, September 2004.

[20] B. D. O. Anderson and J. B. Moore, *Optimal Control; Linear Quadratic Methods*. New York, NY: Prentice Hall, 1990.

[21] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1091–1107, July 2002.

[22] R. E. Kalman, "When is a linear control system optimal?" *Trans. ASME Ser. D: J. Basic Eng.*, vol. 86, pp. 1–10, March 1964.

[23] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

[24] S. P. Banks, *State-Space and frequency-domain methods in the control of distributed parameter systems*. London, UK: Peter Peregrinus Ltd., 1983.

[25] M. R. Jovanović and B. Bamieh, "Lyapunov-based distributed control of systems on lattices," *IEEE Transactions on Automatic Control*, vol. 50, no. 4, April 2005.

[26] ——, "Lyapunov-based state-feedback distributed control of systems on lattices," in *Proceedings of the 2003 American Control Conference*, Denver, CO, 2003, pp. 101–106.

[27] ——, "Lyapunov-based output-feedback distributed control of systems on lattices," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, HI, 2003, pp. 1333–1338.

[28] M. R. Jovanović, "Modeling, analysis, and control of spatially distributed systems," Ph.D. dissertation, University of California, Santa Barbara, 2004, available at http://www.ece.umn.edu/users/mihailo/.