

Adaptive Extremum Seeking Control of a Tubular Reactor with Limited Actuation

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Abstract—In this paper, we optimize the productivity of a chemical reaction taking place in a tubular reactor governed by a set of hyperbolic partial differential equations with unknown kinetics. We use only the temperature information along the reactor to estimate the unmeasured concentration at the reactor exit. Using adaptive extremum seeking, we derive a feedback algorithm that steers the system to its optimum using a non-distributed jacket temperature actuation.

I. INTRODUCTION

Some physical applications are better represented by distributed parameters systems (DPS). Examples of such systems in chemical engineering are nonisothermal tubular reactors where states vary in time and space. Those systems are best described by partial differential equations (PDEs). An extended overview of the problem in this context is given in [6]. Several applications of nonlinear control methods to tubular reactors can be found in the literature, for example robust control ([1] and [7]), sliding mode control ([15]) and backstepping ([5]). A very few adaptive control techniques were applied to DPS. Böhm et. al. [4] extended Model Reference Adaptive Control (MRAC) to DPS. Dochain [9] extended adaptive linearizing control to distributed parameter chemical reactors. Persistency of excitation conditions have been developed for DPS in [8]. Those studies assumed known set-points or reference trajectories.

In some applications, however, the control task could be to optimize an unknown objective function. When the objective is available for measurement, classical extremum-seeking control can be used to perform the optimization [3]. Guay et. al. [11], [12] proposed an adaptive extremum seeking scheme based on a Lyapunov-based approach in which the objective function depends on unknown parameters of the system. In [13], we presented the extension of adaptive extremum seeking control to first-order hyperbolic systems using distributed actuation. In practice, this distributed actuation scheme could be hard to implement. A more practical problem is to consider control of DPS subject to actuator limitations.

In this paper, we solve an adaptive extremum seeking control problem for a system described by first order

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hyperbolic PDEs with a non-distributed constrained jacket temperature actuation. We consider the optimization of the reactor exit concentration. The paper is structured as follows. Section II presents the model under consideration and a preliminary analysis of the problem. In section III, the parameter estimation algorithm and the controller are developed. In section IV, we show a numerical application of the resulting feedback control algorithm and compare it to results obtained with distributed actuation.

II. MODELING AND ASSUMPTIONS

We consider the following hyperbolic PDE model of a nonisothermal tubular reactor:

$$\frac{\partial C(z,t)}{\partial t} = -v \frac{\partial C(z,t)}{\partial z} + MR(C(z,t), T(z,t)) \quad (1)$$

$$\frac{\partial T(z,t)}{\partial t} = -v \frac{\partial T(z,t)}{\partial z} + \lambda_1 R(C(z,t), T(z,t)) + \lambda_2 (T_j(t) - T(z,t)) \quad (2)$$

for $z \in [0, L]$. Here, $C(z,t) \in S_C \subset \mathbb{R}^{n \times 1}$ denotes the concentration of chemical components in the reactor. The temperature, denoted $T(z,t)$, takes value in S_T , a compact subset of $\mathbb{R}^+ \forall z, t$. The matrix of stoichiometric coefficients, $M \in \mathbb{R}^{n \times r}$, is assumed to be known. The unknown vector $R \in \mathbb{R}^{r \times 1}$ is the reaction rate vector. The linear fluid velocity, v , and the heat generation/consumption term λ_1 are known. The heat exchange term, λ_2 , is also supposed to be known. The initial conditions $C(z,0)$ and $T(z,0)$ are known. The known boundary conditions for this problem are given by $C(0,t) = C_{in}$ and $T(0,t) = T_{in}$.

In the remainder of the paper, the components concentration profile, $C(z,t)$, will be denoted by C and the temperature profile, $T(z,t)$, by T . Here, we assume that we have access to discrete measurements of the temperature along the reactor. We suppose no real-time knowledge about the concentration profiles or concentration values at the end of the reactor. We use the temperature profile information to reconstruct the estimated concentration profile along the reactor in order to estimate the objective function. Here, the objective is to construct a jacket temperature control, $u(t) \triangleq T_j(t)$, denoted by T_j , that maximizes a given function of the reactor exit components (i.e., at $z = L$):

$$y = HC(L, \cdot) \quad (3)$$

where $H \in \mathbb{R}^{1 \times n}$. Here, T_j is constrained within the range $[T_L, T_U]$. The goal of the extremum seeking design is to steer the system to the reactor temperature profile that achieves the maximum value of y at steady-state.

We suppose that there exists a vector-valued function of the temperature profile, denoted by $\pi(T)$, solution of the components balance (1) at steady-state:

$$v \frac{d\pi(T)}{dz} = MR(\pi(T)) \quad (4)$$

We assume that the solution $\pi(T)$ exists and is continuous on S_T . We make the following two key assumptions:

Assumption 2.1: The function $H\pi(T)$ is twice continuously differentiable and admits a maximum on S_T .

Assumption 2.2: Consider the mass balance equation dynamics (1). There exists a continuous positive definite functional $V(C)$ such that:

$$a_1 \|C\|^2 \leq V(C) \leq a_2 \|C\|^2$$

$$\dot{V} \leq -a_3 \|C - \pi(T)\| + a_4 \|C\| \|\pi(T)\|$$

$\forall T \in S_T$ and for positive nonzero constants a_1, a_2, a_3, a_4 .

Here, $\|\cdot\|$ is the usual Euclidian norm. Assumption 2.2 provides a minimum-phase property [17] of the reaction kinetics that guarantees the convergence of the composition, C , to a neighborhood of the equilibrium $C = \pi(T)$. From a physical point of view, this assumption states that for a given temperature profile, T , there exists a unique concentration profile $\pi(T)$. Applicability of these two assumptions in the present context is guaranteed by the choice of basis that will be used to parameterize the solution $\pi(T)$.

The goal is to re-formulate the temperature dynamics using the mass balance parameterization (4). Using the chain rule, reaction rates are

$$R(\pi(T), T) = vM^\dagger \left(\frac{\partial \pi(T)}{\partial T} \frac{\partial T}{\partial z} \right) \quad (5)$$

where M^\dagger is the left pseudo-inverse of M (i.e., $M^\dagger M = I_r$ with I_r , the $r \times r$ identity matrix). Adding and subtracting the approximated heat of reaction term on the right-hand side of (2), the temperature dynamics can be re-written as

$$\frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial z} + \lambda_1 R(C, T) + \lambda_2 (T_J - T) + \lambda_1 R(\pi(T), T) - \lambda_1 R(\pi(T), T) \quad (6)$$

or using the expression of $R(\pi(T), T)$ from (5):

$$\frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial z} + v\lambda_1 M^\dagger \left(\frac{\partial \pi}{\partial T} \frac{\partial T}{\partial z} \right) + \lambda_2 (T_J - T) + \lambda_1 (R(C, T) - R(\pi(T), T)) \quad (7)$$

We assume that the following Lipschitz continuity property holds:

Assumption 2.3: $\forall C \in S_C$ and $\forall T \in S_T$, there exists a positive nonzero constant L_1 such that

$$\|R(C, T) - R(\pi(T), T)\| \leq L_1 \|C - \pi(T)\|$$

Assumption 2.3 is trivially met if the reaction kinetics $R(C, T)$ are continuously differentiable with respect to C . In this paper, we approximate the equilibrium concentrations profiles $\pi(T)$ by:

$$\pi(T) = W^T S(T(z)) + \mu_l(z, t) \quad (8)$$

where W are the weight of the approximation, μ_l is the approximation error and

$$S(T(z)) = [s_1(T(z)), s_2(T(z)), \dots, s_N(T(z))]^T \quad (9)$$

$$(10)$$

where $s_i(T)$ are radial basis functions:

$$s_i(T(z)) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[\frac{-(T(z) - \varphi_i)^T (T(z) - \varphi_i)}{\sigma^2} \right] - \frac{(T(z) - \varphi_i)^2}{\sigma^2}, \quad i = 1, 2, \dots, N. \quad (11)$$

The functional approximation of the equilibrium manifold depends on the variance parameter, σ , and the centers of the approximation, φ_i . Here, we choose those centers to be evenly distributed over the temperature domain $T \in [T_L, T_U]$. Universal approximation results stated in [16] and [18] indicate that if N is chosen sufficiently large, then $W^T S(T(z))$ can approximate any continuous function to any desired accuracy on a compact set. We assume that the following holds.

Assumption 2.4: The NN approximation errors satisfies $|\mu_l(t)| \leq \bar{\mu}_l$ with constants $\bar{\mu}_p > 0$ and $\bar{\mu}_l > 0$ over the compact set $\Omega_w \times S_T$.

As a result, the parameterized energy balance becomes:

$$\frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial z} + v\lambda_1 M^\dagger \left(W^T \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} \right) + \lambda_2 (T_J - T) + \lambda_1 \Delta \quad (12)$$

where $\Delta = R(C, T) - R(\pi(T), T)$. Assumption 2.4 is associated with the universal approximation property of the basis function approximation. Given that the underlying non-linearity is continuous over the interval of interest, the property can always be fulfilled.

In most applications, it may be unreasonable to attempt to estimate the steady-state profiles of all chemical species present in the reaction mixture. To overcome this difficulty associated with our formulation, we rewrite the term of the temperature dynamics associated with the equilibrium heat of reaction. Nominally, we parameterize the heat of reaction term as follows:

$$v\lambda_1^T M^\dagger \left(\frac{\partial \pi}{\partial T} \frac{\partial T}{\partial z} \right) = v\lambda_1 M^\dagger W^T \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} + v\lambda_1 M^\dagger \frac{\partial \mu_l}{\partial z} \quad (13)$$

The approximation term in eq. (13) is further decomposed as

$$v\lambda_1 M^\dagger W^T \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} = v\lambda_1 M^\dagger H^T W_b^T \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} + vW_a^T \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} \quad (14)$$

This decomposition highlights the component of the heat of reaction that is associated with the optimization objective.

The second term provides an approximation of the total contribution of the heat of reaction in the energy balance. The temperature dynamics are written as

$$\begin{aligned} \frac{\partial T}{\partial t} = & -v \frac{\partial T}{\partial z} + v \left((\lambda_1 M^\dagger H^T W_b^T + W_a^T) \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} \right) \\ & + v \lambda_1 M^\dagger \frac{\partial \mu_l}{\partial z} + \lambda_2 (T_J - T) + \lambda_1 \Delta. \end{aligned} \quad (15)$$

Equation (15) forms the basis for the development of an adaptive optimizing control algorithm that is presented in the next section.

III. CONTROL DESIGN

In this section, we design a control strategy that tracks the unknown optimum. The objective of the control strategy is to steer the system to the equilibrium temperature profile that maximizes the objective (3) on S_T . Using the parameterization of the last section, the objective function is re-formulated as

$$y = (W_b^T S(T))_L. \quad (16)$$

We first develop the parameter estimation algorithm for the unknown parameter vector W_b . Let \hat{W}_b denotes the estimate of the true optimal weights W_b . The estimate of the objective function y_e is given by:

$$y_e = (\hat{W}_b^T S(T))_L \quad (17)$$

The estimated gradient of y_e with respect to T is

$$\Gamma_1 = \left(\hat{W}_b \frac{\partial S(T)}{\partial T} \right)_L \quad (18)$$

and the Hessian of y_e with respect to T is

$$\Gamma_2 = \left(\hat{W}_b \frac{\partial^2 S(T)}{\partial T^2} \right)_L. \quad (19)$$

We define:

$$z_s = \Gamma_1 - d(t) \quad (20)$$

where $d(t)$ is an excitation signal to be assigned. To address the controller design, we define the following auxiliary signal

$$\eta_1 = z_s - c_1(t)^T \tilde{W} \quad (21)$$

where $W = [W_a^T, W_b^T]^T$, $\tilde{W} = W - \hat{W}$ and $c_1(t)$ are time-varying vector valued functions to be assigned in the design. We propose the Lyapunov function candidate

$$V = \frac{1}{2} \eta_1(t)^2. \quad (22)$$

Taking the time derivative of V , we have

$$\begin{aligned} \dot{V} = & \eta_1 \left[-\dot{d}(t) + \Gamma_2 \left(-v \frac{\partial T}{\partial z} + v \lambda_1 M^\dagger \left(\hat{W} \frac{\partial S}{\partial T} \frac{\partial T}{\partial z} \right) + \lambda_1 \Delta \right. \right. \\ & \left. \left. + \lambda_2 (T_J - T) \right) + c_1(t)^T \dot{\tilde{W}} - \dot{c}_1(t)^T \tilde{W} + \dot{W}_b \frac{\partial S}{\partial T} \right]_L \end{aligned} \quad (23)$$

We propose to use the following dynamic controller

$$\dot{c}_1(t)^T = -k_z c_1(t)^T + \Gamma_2 F(T)^T \quad (24)$$

$$\dot{d}(t) = -k_d d(t) + a(t) - \frac{\partial S}{\partial T} \dot{W}_b \quad (25)$$

$$\begin{aligned} T_j = & T(L) + \frac{1}{\lambda_2} \left[v \frac{\partial T}{\partial z} \left(1 - (\lambda_1^T M^\dagger H^T \hat{W}_b^T + \hat{W}_a^T) \frac{\partial S}{\partial T} \right) \right. \\ & \left. - \frac{1}{\Gamma_2} \left(-\dot{d} + c_1(t)^T \dot{\tilde{W}} - k_z z_s \right) \right]. \end{aligned} \quad (26)$$

where $k_z > 0$ and $k_d > 0$ are gain function to be assigned and

$$F(T) = v \frac{\partial T}{\partial z} \left[\frac{\partial S^T}{\partial T}, \lambda_1^T M^\dagger H^T \frac{\partial S^T}{\partial z} \right]_L. \quad (27)$$

The signal $a(t)$ acts as a secondary dither signal and is set to

$$a(t) = e^{-0.1t} \sum_{i=1}^{10} (A_1 \sin((10+9i)t) + A_2 \cos((10+9i)t))$$

where A_{1i} and A_{2i} are chosen as unit random numbers. Using eq. (21), we substitute for z_s , eq. (24), eq. (25) and eq. (26) in eq. (23). By Assumption 2.3, we get

$$\begin{aligned} \dot{V} \leq & -k_z \eta_1^2 + \eta_1 \Gamma_2 v \lambda_1 M^\dagger \left(\frac{\partial \mu}{\partial z} \right)_L \\ & + |\eta_1| \|\Gamma_2\| \|\lambda_1\|_L \|C - \pi(T)\| \end{aligned} \quad (28)$$

In addition, it follows from Assumption 2.2 that $\sup_{C \in S_C, T \in S_T} \|C - \pi(T)\| = C_1$ exists and is finite. Completing the squares, we assign the gain function, k_z , as

$$k_z = k_{z0} + \frac{k_1}{2} \|\lambda_1\|^2 \Gamma_2^2 v^2 \|M^\dagger\|^2 + \frac{k_2}{2} \|\lambda_1\|^2 \Gamma_2^2, \quad (29)$$

where $k_{z0} > 0$, $k_1 > 0$ and $k_2 > 0$ are constants. As a result, eq. (28) becomes

$$\dot{V} \leq -k_{z0} \eta_1^2 + \left(\frac{1}{2k_1} \right) \left| \frac{\partial \mu_l}{\partial z} \right|_L^2 + \left(\frac{1}{2k_2} \right) L_1^2 C_1^2. \quad (30)$$

or

$$V(t) \leq V(0) e^{-k_{z0}(t-t_0)} + C_2 \quad (31)$$

where

$$C_2 = \sup_{0 \leq \tau \leq t} \left(\left(\frac{1}{2k_4} \right) \left| \frac{\partial \mu_l(\tau, z)}{\partial z} \right|_L^2 + \left(\frac{1}{2k_5} \right) L_1^2 C_1^2 \right) \quad (32)$$

Note that by assumption, the constant C_2 exists and is finite. It is then straightforward to compute the bound on η_1 given by

$$\|\eta_1\| \leq \alpha_1 e^{-\beta_1(t-t_0)} + \sqrt{2C_2} \quad (33)$$

where $\alpha_1 = \sqrt{2V(t_0)}$ and $\beta_1 = k_{z0}/2$.

Eq. (33) establishes that the state, η_1 , converges to a small neighborhood of the origin. It remains to show that the original state variables, z_s , and the parameter estimation errors \tilde{W} also converge to a small neighborhood of the origin. Note that it is not sufficient to check that z_s can be

made small since the value of z_s depends on the parameter estimates, \hat{W}_b . To this end, we derive a persistency of excitation condition that guarantees the convergence of the parameter estimates to the ideal weights, W_b . Consider the following vector,

$$\Upsilon(t) = c_1(t)^T$$

By construction, this vector solves the system of differential equations

$$\dot{\Upsilon}(t) = -K(t)\Upsilon(t) + B(t) \quad (34)$$

where $K(t) = k_z$ and $B(t) = \Gamma_2 F(T)^T$. The elements of $B(t)$ are such that,

$$\|B(t)\|^2 \leq \Gamma_2^2 v^2 \left(\frac{\partial T}{\partial z} \right)_L^2 (1 + \lambda_1 M^\dagger H^T)^2 \frac{\partial S^T}{\partial T} \frac{\partial S}{\partial T}. \quad (35)$$

For the particular choice of basis functions proposed in this study (Eq. 11), we have $\|S\| \leq \sqrt{N}$ where N is the number of weights used in the approximation. The term Γ_2 is bounded by

$$\|\Gamma_2\| \leq \|\hat{W}\| \|d^2 S\|$$

For the specific choice of basis functions, the derivatives dS and $d^2 S$ are given by

$$\frac{\partial S_i}{\partial T} = -2 \frac{(T - \phi_i)}{\sigma_i^2} \exp \left[-\frac{(T - \phi_i)^2}{\sigma_i^2} \right] \quad (36)$$

and

$$\frac{\partial^2 S_i}{\partial T^2} = \left(-2 \frac{1}{\sigma^2} + 4 \frac{(T - \phi_i)^2}{\sigma^4} \right) \exp \left[-\frac{(T - \phi_i)^2}{\sigma_i^2} \right] \quad (37)$$

for $i = 1, \dots, l$. From eqs. (36)-(37), we get that $\|dS\| \leq 2\sqrt{\frac{NT_U}{\sigma_m^2}}$ and $\|d^2 S\| \leq 2\sqrt{\frac{N}{\sigma_m^2}} + 4\sqrt{N} \frac{T_U}{\sigma_m^2}$ where $\sigma_m = \min_{1 \leq i \leq N} \{\sigma_i\}$ and T_U is the upper bounded in T on the compact set S_T . It is assumed that $\phi_i \in S_T$, $1 \leq i \leq N$. A bound on the parameter estimates \hat{W} can be ensured by choosing the following parameter update law,

$$\dot{\hat{W}} = \begin{cases} \gamma_w \Gamma & \text{if } \|\hat{W}\| < w_m \text{ or} \\ & \text{if } \|\hat{W}\| = w_m \text{ and } \hat{W}^T \Gamma \leq 0 \\ \gamma_w \left(I - \frac{\hat{W} \hat{W}^T}{\hat{W}^T \hat{W}} \right) \Gamma & \text{otherwise} \end{cases} \quad (38)$$

(39)

where $\Gamma = \Upsilon(t)^T z_s(t)$. Eq. (38) is a projection algorithm which ensures that $\|\hat{W}\| \leq w_m$ (see [14] for details). From the above analysis, it follows that the norm of $B(t)$ can be bounded as follows

$$\|B(t)\| \leq \left(w_m \left[2\sqrt{\frac{N}{\sigma_m^2}} + 4\sqrt{N} \frac{T_U}{\sigma_m^2} \right] \right) |v| \left((1 + |\lambda^T M^\dagger H^T|) 2\sqrt{\frac{NT_U}{\sigma_m^2}} \right) \left| \frac{\partial T}{\partial z} \right|_L = B_M \left| \frac{\partial T}{\partial z} \right|_L. \quad (40)$$

By eq. (34) and the bound on $B(t)$, an explicit bound for the solution of eq. (34) can be obtained

$$\|\Upsilon(z, t)\| \leq \alpha_2 e^{-\beta_2(t-t_0)} + \frac{B_M}{\beta_2} \int_{t_0}^t \left| \frac{\partial T(z, \sigma)}{\partial z} \right|_L d\sigma \quad (41)$$

where $\alpha_2 = \|\Upsilon(0)\|$ and $\beta_2 > 0$ is a positive constant. The term $\left| \frac{\partial T}{\partial z} \right|_L$ is such that

$$\left| \frac{\partial T(z, \sigma)}{\partial z} \right|_L \leq L |T_U - T_L| \quad (42)$$

and, hence,

$$\|\Upsilon(t)\| \leq \alpha_2 e^{-\beta_2(t-t_0)} + \frac{B_M}{\beta_2} L |T_U - T_L|. \quad (43)$$

Next, we show that the parameter estimation error \tilde{W} converges to a neighborhood of the origin. Upon substitution of $z_s = \eta + \Upsilon(t)\tilde{W}$, the dynamics of the parameter estimation error can be written as

$$\dot{\tilde{W}} = -\gamma_w \left(\Upsilon(t)^T \Upsilon(t) \tilde{W}(t) + \Upsilon(t)^T \eta(t) \right) + \begin{cases} 0 & \text{if } \|\hat{W}\| < w_m \text{ or} \\ & \text{if } \|\hat{W}\| = w_m \\ & \text{and } \hat{W}^T \Gamma \leq 0 \\ \gamma_w \frac{\hat{W} \hat{W}^T}{\hat{W}^T \hat{W}} \left(\Upsilon(t)^T \Upsilon(t) \tilde{W} + \Upsilon(t)^T \eta(t) \right) & \text{otherwise} \end{cases}$$

In the following, we show that, under suitable assumptions, the non-perturbed (*i.e.*, $\eta(t) \equiv 0$) dynamics of the parameter estimation errors are exponentially stable. To do this, we require the following assumption.

Assumption 3.1: The solution of eq. (34) is such that there exists, a positive constant $\tau > 0$ and $k_l > 0$ such that

$$\int_t^{t+\tau} \Upsilon(\sigma)^T \Upsilon(\sigma) d\sigma \geq k_l I_l \quad (44)$$

where I_l is the l -dimensional identity matrix.

The following Lemma ([2, Theorem 2.2]) will be used to establish the convergence of \tilde{W} to a neighborhood of the origin.

Lemma 3.1: Consider the differential equation

$$\dot{x}(t) = -\phi(t)^T \phi(t) x(t) \quad (45)$$

and assume that there exists a $T > 0$ and a $k > 0$ such that

$$\int_t^{t+T} \phi(\tau)^T \phi(\tau) d\tau \geq kI \quad (46)$$

then the origin is a globally exponentially stable equilibrium of the system.

Lemma 3.1 establishes that the origin of the differential equation

$$\dot{\tilde{W}} = -\gamma_w \left(\Upsilon(t)^T \Upsilon(t) \right) \tilde{W} \quad (47)$$

is an exponentially stable equilibrium. In fact, it follows from the proof of Lemma 3.1 that the Lyapunov function

$$V_w = \frac{1}{2\gamma_w} \tilde{W}^T \tilde{W} \quad (48)$$

is such that

$$\dot{V}_w = -\tilde{W}^T \left(\Upsilon(t)^T \Upsilon(t) \right) \tilde{W} \leq -c_3 \|\tilde{W}\|^2 \quad (49)$$

for $c_3 > 0$ a positive constant. It then follows that the rate of change of V_w along the trajectories of the system eq. (47) is given by

$$\dot{V}_w = -\tilde{W}^T (\Upsilon(t)\Upsilon(t)) \tilde{W} - \tilde{W}^T \Upsilon(t)^T \eta_1(t) + \begin{cases} 0 & \text{if } \|\tilde{W}\| < w_m \text{ or} \\ \gamma_w \tilde{W}^T \frac{\dot{\tilde{W}} \tilde{W}^T}{\tilde{W}^T \tilde{W}} (\Upsilon(t)^T \Upsilon(t)) \tilde{W} + \Upsilon(t)^T \eta_1(t) & \text{otherwise} \end{cases} \quad (50)$$

Using the property of the projection algorithm, it follows that

$$\dot{V}_w \leq -\tilde{W}^T (\Upsilon(t)^T \Upsilon(t)) \tilde{W} - \tilde{W}^T \Upsilon(t)^T \eta_1(t)$$

Completing the squares and by exponential stability of the nominal system eq. (47), we get

$$\dot{V}_w \leq -\frac{a_3}{2} \|\tilde{W}\|^2 + \frac{1}{2} \eta_1^2 = -a_3 \gamma_w V_w + \frac{1}{2} \eta_1^2. \quad (51)$$

Substitution of eq. (33) and integration of (51) yield

$$V_w(t) \leq \theta_1 e^{-\theta_2(t-t_0)} + \frac{C_2}{a_3 \gamma_w}$$

where $\theta_1 = \max\left[\frac{V(t_0)}{a_3 \gamma_w - k_{z0}}, \frac{C_2}{a_3 \gamma_w}, V_w(t_0)\right]$ and $\theta_2 = \min[k_{z0}, a_3 \lambda_w]$. Consequently, the parameter estimation error is guaranteed to decay exponentially as

$$\|\tilde{W}\|^2 \leq 2\gamma_w \theta_1 e^{-\theta_2(t-t_0)} + \frac{2C_2}{a_3} \quad (52)$$

or

$$\|\tilde{W}\| \leq \alpha_3 e^{-\beta_3(t-t_0)} + \sqrt{\frac{2C_2}{a_3}} \quad (53)$$

where $\alpha_3 = \sqrt{\theta_1}$, $\beta_3 = \theta_2/2$. Taking the limit as $t \rightarrow \infty$ confirms that the estimation error converges to a small adjustable neighborhood of the origin given by

$$\lim_{t \rightarrow \infty} \|\tilde{W}\| \leq \sqrt{\frac{2C_2}{a_3}}. \quad (54)$$

The main result of this study is the following. The proof follows the lines of the one presented in [11], [13].

Theorem 3.1: Consider the nonisothermal plug-flow reactor model eqs. (1)-(2) in closed-loop with the controller eq. (24)-(26) and the adaptive learning law eq. (38). Assume that the signal $a(t)$ in eq. (25) is such that,

$$\int_t^{t+\tau} \Upsilon(\sigma)^T \Upsilon(\sigma) d\sigma \geq k_l I_l \quad (55)$$

for positive constants $\tau > 0$ and $k_l > 0$ where $\Upsilon(t)$ is the solution of eq. (34). Then

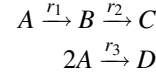
- the parameter estimation errors \tilde{W} converge exponentially to a small neighbourhood of the origin
- the tracking error from the unknown steady-state profile, z_s , converges exponentially to a small neighbourhood of the origin.

TABLE I
SIMULATION PARAMETERS - VAN DE VUSSE REACTION

$k_{10} = 1.287 \cdot 10^{12} \text{ h}^{-1}$	$k_{20} = 1.287 \cdot 10^{12} \text{ h}^{-1}$
$k_{30} = 9.034 \cdot 10^9 \text{ L molA}^{-1} \text{ h}^{-1}$	$E_1 = 9758.3 \text{ K}$
$E_2 = 9758.3 \text{ K}$	$E_3 = 8560.0 \text{ K}$
$\Delta H_{R1} = -4.20 \text{ kJ molA}^{-1}$	$\Delta H_{R2} = 11.00 \text{ kJ molB}^{-1}$
$\Delta H_{R3} = 41.85 \text{ kJ molA}^{-1}$	$\rho = 0.9342 \text{ kg / L}$
$C_p = 3.01 \text{ kJ kg}^{-1} \text{ K}^{-1}$	$h = 4032 \text{ KJ / (h m}^2 \text{ K)}$
$d = 0.1 \text{ m}$	$L = 1 \text{ m}$
$v = 5.0 \text{ m h}^{-1}$	$C_{A,in} = 5.1 \text{ molA / L}$
$C_{B,in} = 0 \text{ molB / L}$	$T_{in} = 298 \text{ K}$
$C_A(0, z) = 5.1 \text{ molA / L}$	$C_B(0, z) = 0 \text{ molB / L}$
$T(0, z) = 298 \text{ K}$	$\lambda_1 = \frac{\Delta H_R}{\rho C_p}$
$\lambda_2 = \frac{4h}{\rho C_p}$	

IV. NUMERICAL APPLICATION

This section presents the application of the proposed adaptive extremum seeking control to the van de Vusse reaction using numerical values derived from [10] and summarized in Table I. We consider the van de Vusse reaction:



The objective is to maximize the exit concentration of the intermediate component, B , hence $y = C_B$. The unknown reaction rates are :

$$R(C, T) = [k_{10} e^{-E_1/T} C_A, k_{20} e^{-E_2/T} C_B, k_{30} e^{-E_3/T} C_A^2]^T$$

Resulting profiles for temperature and concentration for limited actuation are given in Fig. 1. Those results compared well with the distributed actuation results from [13] presented in Fig. 2. Comparisons of both objective functions (concentration of components B at the reactor output) with respect to time are shown in Fig. 3. Here, we used the same controller and estimator gains. In both cases, discretization grid was set to 20 points along the reactor and the approximation used 10 basis evenly spaced between $T = [298 \ 400]$ K. Since the estimated weights are no longer distributed, computation time is greatly reduced in the case of limited actuation. The main drawback in the limited actuation case resides in the design of the dither signal which has a greater effect in the limited actuation case, since it affects the actuation directly.

V. CONCLUSIONS

We solved a class of extremum seeking control problems for plug flow reactors with unknown reaction kinetics. Assuming temperature measurements along the reactor and a single jacket temperature control, it has been shown that if a persistent of excitation condition is satisfied locally, then the proposed adaptive extremum seeking controller guarantees the convergence to a small neighborhood of the optimum. The optimal profiles were comparable to the results previously obtained with distributed actuation.

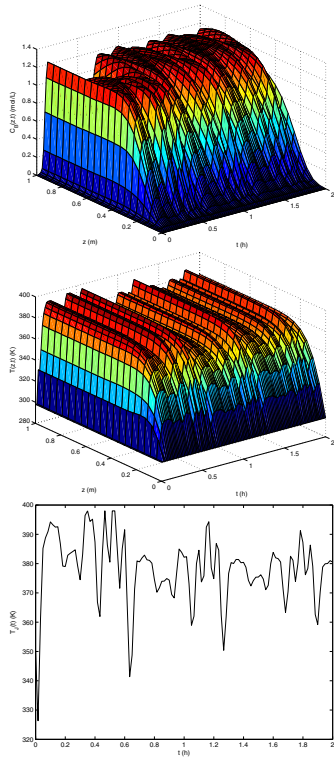


Fig. 1. Desired Product Concentration, Reactor Temperature Profile and Jacket Temperature Profile - Limited Actuation

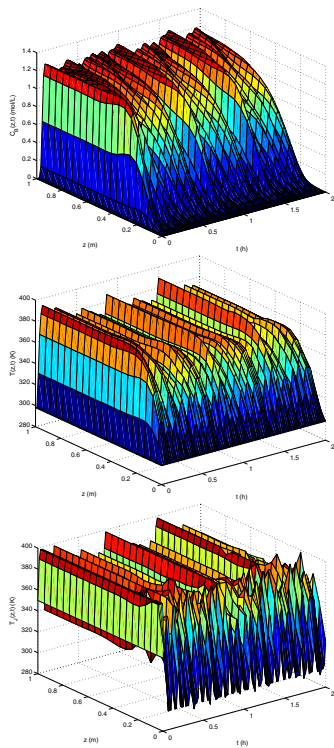


Fig. 2. Desired Product Concentration, Reactor Temperature Profile and Jacket Temperature Profile - Distributed Actuation

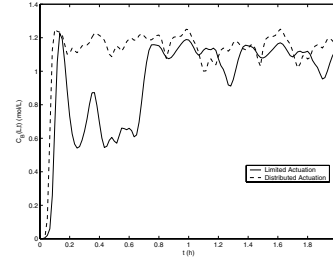


Fig. 3. Objective Function

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