2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

PREDICTIVE CONTROL OF DIFFUSION–REACTION PROCESSES *

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Abstract— **This work focuses on the development of computationally-efficient predictive control algorithms for nonlinear parabolic PDEs with state and control constraints arising in the context of diffusion-reaction processes. Specifically, we consider a diffusion-reaction process described by a nonlinear parabolic PDE and address the problem of stabilization of an unstable steady-state subject to input and state constraints. Galerkin's method is used to derive finite-dimensional systems that capture the dominant dynamics of the parabolic PDE, which are subsequently used for controller design. Various MPC formulations are constructed on the basis of the finite dimensional approximations that differ in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The impact of these differences on the ability of the predictive controller to enforce state constraints satisfaction in the infinite-dimensional system is analyzed. Finally, the MPC formulations are applied, through simulation, to the problem of stabilizing an unstable steady-state of a nonlinear model of a diffusion-reaction process subject to state and control constraints.**

Key words: Diffusion-reaction processes, nonlinear parabolic PDEs, model predictive control.

I. INTRODUCTION

Diffusion-reaction processes are characterized by significant spatial variations and nonlinearities due to the underlying diffusion phenomena and complex reaction mechanisms, respectively. The dynamic models of diffusion-reaction processes typically consist of parabolic partial differential equation (PDE) systems whose spatial differential operators are characterized by a spectrum that can be partitioned into a finite (possibly unstable) slow part and an infinite stable fast complement [7]. The traditional approach to control of linear/quasi-linear parabolic PDEs involves the application of spatial discretization techniques to the PDE system to derive systems of ordinary differential equations (ODEs) that accurately describe the dynamics of the dominant (slow) modes of the PDE system. These finite-dimensional systems are subsequently used as the basis for the synthesis of finite-dimensional controllers (e.g., see [4], [15]). A potential drawback of this approach, especially for quasi-linear parabolic PDEs, is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to the high-order controllers.

Motivated by these considerations, significant recent work has focused on the development of a general framework for the synthesis of low-order controllers for quasi-linear parabolic PDE systems – and other highly dissipative PDE systems that arise in the modeling of spatially-distributed systems including fluid dynamic systems – on the basis of low-order nonlinear ODE models derived through a combination of Galerkin's method (using analytical or empirical basis functions) with the concept of inertial manifolds (e.g., see [6], [3], [2], [1], [10] and the book [5] for results and references in this area).

The control methods proposed in the above works, however, do not address the issue of state constraints in the controller design. Model Predictive Control (MPC), also known as receding horizon control, is a popular control method for handling constraints (both on manipulated inputs and state variables) within an optimal control setting. Numerous research studies have investigated the properties of model predictive controllers and led to a plethora of MPC formulations that focus on a number of control-relevant issues, including issues of closed–loop stability, performance, implementation and constraint satisfaction (e.g., see [11], [14], [13] for results and references in this area).

Few results are available on predictive control of distributed parameter systems. Contributions include analyzing the predictive control problem on the basis of the infinite-dimensional system using control Lyapunov functionals (e.g., [12]), and the use of finite difference method (e.g., [9]) to derive approximate ODE models for MPC design. In [8], we considered linear parabolic PDE systems and derived computationally-efficient predictive control algorithms that systematically handle the objectives of state and input constraints satisfaction and stabilization of the infinite dimensional system.

In this work, we focus on the development of computationally-efficient predictive control algorithms for nonlinear parabolic PDEs with state and control constraints arising in the context of diffusion-reaction processes. The paper is organized as follows: we first present a diffusion-reaction process described by a nonlinear parabolic PDE subject to input and state constraints. Galerkin's method is used to derive finite-dimensional systems that capture the dominant dynamics of the parabolic PDE, which are subsequently used for controller design. Various MPC formulations are constructed on the basis of the finite dimensional approximations that differ in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The various MPC formulations are demonstrated, through simulation, to be successful in

*Financial support by NSF, CTS-0129571, is gratefully acknowledged.

achieving the control objectives.

II. Predictive control of diffusion-reaction processes

Motivating example. In this section, we consider a representative example of a diffusion-reaction system described by a parabolic PDE of the following form:

$$\begin{aligned}\frac{\partial \bar{x}}{\partial t} &= \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T(e^{-\dfrac{\gamma}{1+\bar{x}}} - e^{-\gamma}) - \beta_U \bar{x} \\ &+ \beta_U \sum_{i=1}^{m} b_i(z) u_i(t) \\ \bar{x}(0,t) &= 0,\ \bar{x}(\pi,t) = 0,\ \bar{x}(z,0) = x_0(z)\end{aligned} \tag{1}$$

where $\bar{x}$ denotes the dimensionless state of the system, β_T denotes a dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, $u_i(t)$ denotes the manipulated input and $b_i(z)$ is the actuator distribution function of the i-th actuator, chosen to be $b_i(z) = 1/\mu$ for $z \in [z_{ai}-\mu, z_{ai}+\mu]$ and $b_i(z) = 0$ elsewhere in $[0,\pi]$, where μ is a small positive real number and z_{ai} is the center of the interval where actuation is applied. The following typical values are given to the process parameters: $\beta_T = 50$, $\beta_U = 2$ and $\gamma = 4$. For these values, the operating steady-state, $\bar{x}(z,t) = 0$, is an unstable one, as can be seen from Fig.1. The control objective is to stabilize the state profile at the unstable zero steady-state subject to the following input and state constraints

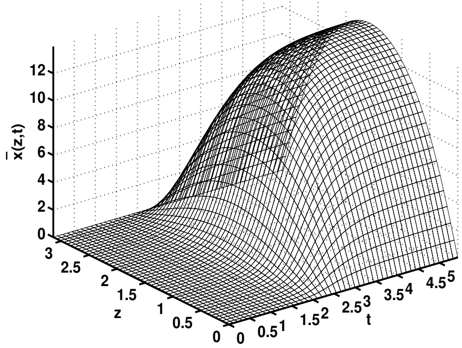

Fig. 1. Open-loop profile showing the instability of the $\bar{x}(z,t) = 0$ steady-state.

$$u_i^{min} \leq u_i \leq u_i^{max} \tag{2}$$

$$\chi^{min} \leq \int_0^{\pi} r(z)\bar{x}(z,t)dz \leq \chi^{max} \tag{3}$$

where $u_i^{min} = -10$, $u_i^{max} = 10$, for $i = 1,2$, $\chi^{min} = -0.035$ and $\chi^{max} = 2$. The state constraints distribution function, $r(\cdot)$, is chosen to be $r(z) = \delta(z-z_c)$ for $z \in [0,\pi]$ and $z_c = 1.156$, which implies that the state constraints are to be enforced only at a single point in the spatial domain, i.e., $-0.035 \leq \bar{x}(z_c,t) \leq 2$. For this system, we consider the first two eigenvalues as the dominant ones and use two point control actuators ($m = 2$), with finite support, centered at $z_{a1} = \pi/3$ and $z_{a2} = 2\pi/3$, to achieve the control objective subject to the constraints of Eqs.2-3.

Galerkin's method. To present our results, we first formulate the PDE of Eq.1 as an infinite dimensional system in the Hilbert space $\mathcal{H}([0,\pi];\mathbb{R})$, with $\mathcal{H}$ being the space of measurable functions defined on $[0,\pi]$, with inner product and norm:

$$(\omega_1,\omega_2) = \int_0^{\pi} (\omega_1(z),\omega_2(z))_{\mathbb{R}^n} dz, ||\omega_1||_2 = (\omega_1,\omega_1)^{\frac{1}{2}} \tag{4}$$

where $\omega_1,\ \omega_2$ are two elements of $\mathcal{H}([0,\pi];\mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function $x(t)$ on the state-space $\mathcal{H} = L_2(0,\pi)$ as

$$x(t) = \bar{x}(z,t), \quad t > 0, \quad 0 \leq z \leq \pi, \tag{5}$$

the operator $\mathcal{A}$ as

$$\mathcal{A}\phi = \frac{d^2\phi}{dz^2}, \quad 0 \leq z \leq \pi, \tag{6}$$

where $\phi(z)$ is a smooth function on $(0,\pi)$ with $\phi(0) = 0$ and $\phi(\pi) = 0$, with the following dense domain:

$$\begin{aligned}&\mathcal{D}(\mathcal{A}) = \{\phi(z) \in L_2(0,\pi) : \\ &\phi(z), \frac{d\phi(z)}{dz} \text{ are absolutely continuous,} \\ &\mathcal{A}\phi \in L_2(0,\pi), \phi(0) = 0 \text{ and } \phi(\pi) = 0\},\end{aligned} \tag{7}$$

and the input operator as:

$$\mathcal{B}u = \sum_{i=1}^{m} b_i u_i, \tag{8}$$

the system of Eq.1 takes the form:

$$\dot{x} = \mathcal{A}x + \mathcal{F}(x) + \mathcal{B}u, \quad x(0) = x_0 \tag{9}$$

where $x_0 = x_0(z)$. For the operator $\mathcal{A}$, the eigenvalue problem takes the form:

$$\frac{d^2\phi_j}{dz^2} = \lambda_j \phi_j \tag{10}$$

subject to

$$\phi_j(0) = \phi_j(\pi) = 0 \tag{11}$$

The above eigenvalue problem can be solved analytically and its solution yields

$$\lambda_j = -j^2,\ \phi_j(z) = \sqrt{\frac{2}{\pi}} sin(j\ z),\ j = 1,\ldots,\infty \tag{12}$$

Throughout the rest of the paper, the notation $|\cdot|$ will be used to denote the standard Euclidian norm in $\mathbb{R}^n$, while the notation $|\cdot|_Q$ will be used to denote the weighted norm defined by $|x|_Q^2 = x'Qx$, where Q is a positive-definite matrix and x' denotes the transpose of x. Finally the notation $\|\cdot\|_2$ will be used to denote the L_2 norm (as defined in Eq.4 above) associated with a finite or infinite dimensional Hilbert space.

Next, we apply standard Galerkin's method to the infinite-dimensional system of Eq.9 to derive a finite-dimensional system. Let $\mathcal{H}_s$, $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = span\{\phi_1,\phi_2,\ldots,\phi_m\}$ and $\mathcal{H}_f =$

$span\{\phi_{m+1}, \phi_{m+2}, \ldots\}$ (the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from the properties of $\mathcal{A}$). Defining the orthogonal projection operators, P_s and P_f, such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq.9 can be decomposed as

$$x = x_s + x_f = P_s x + P_f x \tag{13}$$

Applying P_s and P_f to the system of Eq.9 and using the above decomposition for x, the system of Eq.9 can be re-written in the following equivalent form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{F}_s(x_s, x_f) + \mathcal{B}_s u, \\ \frac{dx_f}{dt} &= \mathcal{A}_f x_f + \mathcal{F}_f(x_s, x_f) + \mathcal{B}_f u \\ x_s(0) &= P_s x(0) = P_s x_0 \\ x_f(0) &= P_f x(0) = P_f x_0 \end{aligned} \tag{14}$$

where $\mathcal{A}_s = P_s\mathcal{A}$, $\mathcal{B}_s = P_s\mathcal{B}$, $\mathcal{A}_f = P_f\mathcal{A}$ and $\mathcal{B}_f = P_f\mathcal{B}$. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = diag\{\lambda_j\}$ (λ_j include all the possibly unstable eigenvalues of $\mathcal{A}_s$) and $\mathcal{A}_f$ is an unbounded exponentially stable differential operator. In the remainder of the paper, we will refer to the x_s- and x_f-subsystems in Eq.14 as the slow and fast subsystems, respectively.

Control problem formulation. We consider the problem of asymptotic stabilization of the origin of the system of Eq.9, subject to the following control and state constraints:

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{F}(x(t)) + \mathcal{B}u(t), \quad x(0) = x_0 \tag{15}$$

$$u_i^{min} \leq u_i(t) \leq u_i^{max} \tag{16}$$

$$\chi^{min} \leq (r, x(t)) \leq \chi^{max} \tag{17}$$

This problem will be addressed within an MPC framework where the control, at state x and time t, is conventionally obtained by solving, on-line, a finite-horizon constrained optimal control problem of the form

$$P(x,t) \;:\; \min\{J(x,t,u(\cdot)) \mid u(\cdot) \in S\} \tag{18}$$

$$\begin{aligned} s.t.\ \dot{x}(\tau) &= \mathcal{A}x(\tau) + \mathcal{F}(x(\tau)) + \mathcal{B}u(\tau) \\ \chi^{min} &\leq (r, x(\tau)) \leq \chi^{max},\ \tau \in [t, t+T] \end{aligned} \tag{19}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathcal{U} := \{u \in \mathbb{R}^m : u_i^{min} \leq u_i \leq u_i^{max},\ i = 1, \cdots, m\}$, and T is the specified horizon. A control $u(\cdot)$ in S is characterized by the sequence $u[k]$, where $u[k] := u(k\Delta)$, and satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. The performance index is given by

$$\int_t^{t+T} \left[q\|x^u(\tau; x, t)\|_2^2 + |u(\tau)|_R^2\right] d\tau + F(x(t+T)) \tag{20}$$

where $q > 0$, R is a strictly positive definite matrix, $x^u(\tau) = x(\tau; x, t)$ denotes the solution of Eq.9, due to control u, with initial state x at time t, and $F(\cdot)$ denotes the terminal penalty. The minimizing control $u^0(\cdot) \in S$ is then applied to the system over the interval $[k\Delta, (k+1)\Delta]$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M(x) := u^0(t; x, t) \tag{21}$$

Remark 1: It is well known that the control law defined by Eqs.18-21 is not necessarily stabilizing (even for the finite-dimensional system) [13]. For finite-dimensional systems, the issue of closed-loop stability is usually addressed by means of imposing suitable penalties and constraints on the state at the end of the optimization horizon (e.g., see [13] for surveys of different approaches). For the simulation example presented here, and for the choice of MPC parameters and initial conditions, the closed–loop system under MPC was found to be stabilizing; we therefore do not impose stability constraints in the optimization problem, but focus on the task of state constraint satisfaction.

One possible way to formulate the constrained nonlinear MPC problem is to design it on the basis of the full system of Eq.14. The control action is then obtained by solving the following optimization problem:

$$\min_u \int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + q_f\|x_f(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau \tag{22}$$

$$\begin{aligned} s.t.\ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau) + \mathcal{F}_s(x_s(\tau), x_f(\tau)) + \mathcal{B}_s u(\tau) \\ \dot{x}_f(\tau) &= \mathcal{A}_f x_f(\tau) + \mathcal{F}_f(x_s(\tau), x_f(\tau)) + \mathcal{B}_f u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \chi^{min} &\leq (r, x(\tau)) \leq \chi^{max}, \tau \in [t, t+T] \end{aligned} \tag{23}$$

where q_s, $q_f > 0$ and R is a positive definite matrix. The above formulation includes penalties on both the slow and fast states and uses models that describe their evolution for prediction purposes. The infinite dimensional nature of the controller, however, renders it unsuitable for the purpose of online implementation. We now present and compare nonlinear MPC formulations that differ in the way the state constraints are enforced and in the construction of the performance functional in the optimization problem.

Low-order predictive control formulations. In this formulation, the predictive controller is designed on the basis of the low-order, finite-dimensional slow subsystem describing the evolution of the x_s states (the fast subsystem is neglected). Specifically, the nonlinear MPC law is obtained by solving, in a receding horizon fashion, the following optimization problem:

$$\min_u \int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau \tag{24}$$

$$\begin{aligned} s.t.\ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau) + \mathcal{F}_s(x_s(\tau)) + \mathcal{B}_s u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \chi^{min} &\leq (r, x_s(\tau)) \leq \chi^{max}, \tau \in [t, t+T] \end{aligned} \tag{25}$$

To simplify the presentation of the results, we will work with the amplitudes of the eigenmodes of the PDE of Eq.1. Specifically, using Galerkin's method, we derive the following high-order ODE system that describes the temporal evolution of the amplitudes of the first l eigenmodes:

$$\begin{aligned}\dot{a}_s(t) &= A_s a_s(t) + F_s(a_s(t), a_f(t)) + B_s u(t) \\ \dot{a}_f(t) &= A_f a_f(t) + F_f(a_s(t), a_f(t)) + B_f u(t)\end{aligned} \quad (26)$$

where $a_s(t) = [a_1(t)\ a_2(t)]'$, $a_f(t) = [a_3(t)\ \cdots\ a_l(t)]'$, $a_i(t) \in \mathbb{R}$ is the modal amplitude of the i-th eigenmode, the notation a_s' denotes the transpose of a_s, $u(t) = [u_1(t)\quad u_2(t)]'$, the matrices A_s and A_f are diagonal matrices, given by $A_s = diag\{\lambda_i\}$, for $i = 1, 2$ and $A_f = diag\{\lambda_i\}$, for $i = 3, \cdots, l$. B_s and B_f are a 2×2 and $(l-2) \times m$ matrices, respectively whose (i,j)-th element is given by $B_{ij} = (b_j(z), \phi_i(z))$. Note that $\bar{x}(z,t) = \sum_{i=1}^{l} a_i(t)\phi_i(z)$, $x_s(t) = a_1(t)\phi_1 + a_2(t)\phi_2$, $x_f(t) = \sum_{i=3}^{l} a_i(t)\phi_i$ and that $(x_s(t), \phi_i) = a_i(\phi_i, \phi_i)$. Using these projections, the state constraints of Eq.3 can be expressed as constraints on the modal amplitudes as follows:

$$\chi^{min} \leq \sum_{i=1}^{2} a_i(t)\phi_i(z_c) + \sum_{i=3}^{l} a_i(t)\phi_i(z_c) \leq \chi^{max} \quad (27)$$

The MPC formulation of Eq.31, when written in terms of the amplitudes of the eigenmodes takes the following form:

$$\min_{u} \int_{t}^{t+T} \left[q_s |a_s(\tau)|^2 + |u(\tau)|_R^2\right] d\tau \quad (28)$$

$$\begin{aligned} s.t.\ \ \dot{a}_s(\tau) &= A_s a_s(\tau) + F_s(a_s) + B_s u(\tau) \\ u_{min} &\leq u_i(\tau) \leq u_{max},\ i = 1, 2 \\ \chi^{min} &\leq C_s a_s(\tau) \leq \chi^{max}, \tau \in [t, t+T] \end{aligned} \quad (29)$$

where $C_s = [\phi_1(z_c)\ \ \phi_2(z_c)]$ is a row vector. We now proceed with the implementation of the predictive control formulation of Eqs.28-29 and choose $q_s = 1000$, $R = rI$, with $r = 0.01$, $T = 0.0011$ and $l = 30$. The resulting program is solved using the MATLAB subroutine fmincon. The control action is then implemented on the 30-th order model of Eq.26 (higher-order approximation of the PDE system led to identical results). For an initial condition $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.07sin(3z)$, the closed-loop state and manipulated input profiles under the MPC controller of Eqs.28-29 are shown in Fig.2 and Figs.7-8 (solid lines), respectively. It is clear that the controller successfully stabilizes the state at the zero steady-state. However, by examining Fig.6 (solid line), we observe that the state at $z_c = 1.156$ violates the lower constraint for some time. The violation of the state constraint is a consequence of neglecting the contribution of the a_f states to the full state of the PDE in the MPC formulation.

Remark 2: Note that while the controller is designed only on the basis of the slow modes, the stabilization of the slow modes of the system leads to the stabilization of the infinite dimensional system, since the remaining fast modes are open–loop stable (for a similar result in the context of linear parabolic PDE systems, see [8]).

For linear parabolic PDEs, low order predictive controller formulations can be derived, which, upon being feasible, guarantee stabilization and state constraint satisfaction of

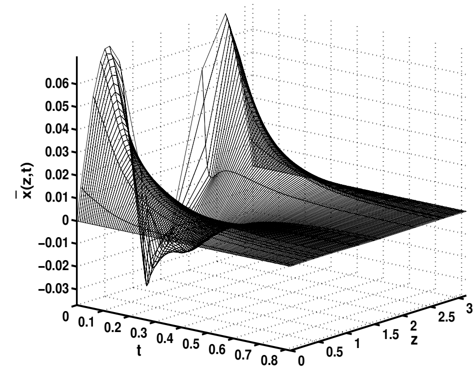

Fig. 2. Closed-loop state profile under the MPC formulation of Eqs.28-29 with initial condition initial condition $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.07sin(3z)$.

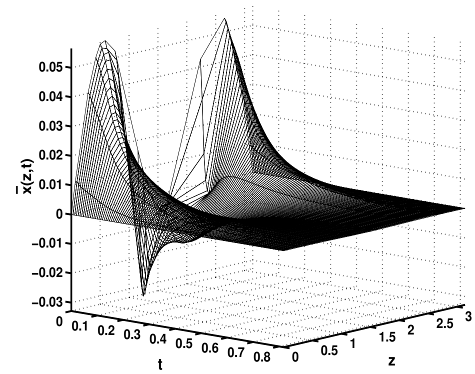

Fig. 3. Closed-loop state profile under the MPC formulation of Eqs.30-31 with initial condition $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.05sin(3z)$ and $\mathcal{S}^{max} = 1.9$ and $\mathcal{S}^{min} = -0.034$.

the infinite dimensional system (see, [8]). The key idea of the approach is to tighten the constraints in the optimization problem, and derive bounds on the initial conditions for the fast states, such that the contribution of the fast states to the state of the PDE does not result in violation of the state constraints. The inherent coupling between the fast and slow subsystems through the terms $F_s(x_s, x_f)$, $F_f(x_s, x_f)$, however, significantly complicates the derivation of similar results in the nonlinear setting. Nevertheless, the same idea can be utilized in the context of nonlinear parabolic PDEs, by implementing the reduced order MPC formulation with "shrunk" state constraints and by reducing the initial condition for the fast modes. Specifically, the nonlinear MPC law is obtained by solving, in a receding horizon fashion, the following optimization problem:

$$\min_{u} \int_{t}^{t+T} \left[q_s \|x_s(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau \quad (30)$$

$$\begin{aligned} s.t.\ \ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau) + \mathcal{F}_s(x_s(\tau)) + \mathcal{B}_s u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \mathcal{S}^{min} &\leq (r, x_s(\tau)) \leq \mathcal{S}^{max}, \tau \in [t, t+T] \end{aligned} \quad (31)$$

where $S^{min} \geq \chi^{min} - \alpha_1$ and $S^{max} \leq \chi^{max} + \alpha_2$, where $\alpha_1,\ \alpha_2$ are positive real numbers. The MPC formulation of Eq.30-31, when written in terms of the amplitudes of the eigenmodes takes the following form:

$$\min_{u}\int_{t}^{t+T}\left[q_s|a_s(\tau)|^2+|u(\tau)|_R^2\right]d\tau \qquad (32)$$

$$\begin{aligned} s.t.\ \dot{a}_s(\tau) &= A_s a_s(\tau)+F_s(a_s)+B_s u(\tau)\\ u_{min} &\le u_i(\tau) \le u_{max},\ i=1,2 \qquad (33)\\ \mathcal{S}^{min} &\le C_s a_s(\tau)\le \mathcal{S}^{max},\tau\in[t,t+T] \end{aligned}$$

where $C_s = [\phi_1(z_c)\ \phi_2(z_c)]$ is a row vector. The closed-loop state and manipulated input profiles under the MPC controller of Eqs.32-33 and with the initial condition $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.05sin(3z)$ and $\mathcal{S}^{max} = 1.9$, $\mathcal{S}^{min} = -0.034$ are shown in Fig.3 and Fig.6 (dashed line), and in Figs.7-8 (dashed lines). It is clear that the controller successfully stabilizes the state at the zero steady-state. Also, by examining Fig.6 (dashed line), we observe that for the choice of initial condition of the fast modes and the α_1, α_2 used in the example, the state at $z_c = 1.156$ does not violate the state constraints. Note that, since the control action is computed based on the slow states – the initial conditions for which are the same as in the previous scenario – the controller implements control action as before (the solid and the dashed lines are not discernible in Figs.7-8).

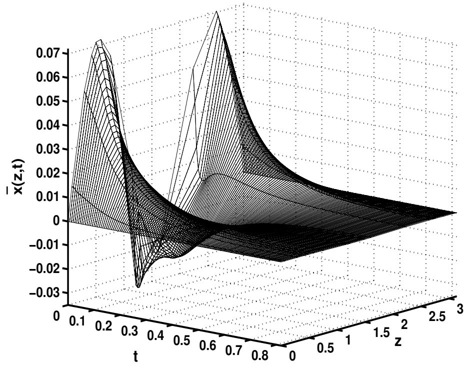

Fig. 4. Closed-loop state profile under the MPC formulation of Eqs.36-37 with initial $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.07sin(3z)$.

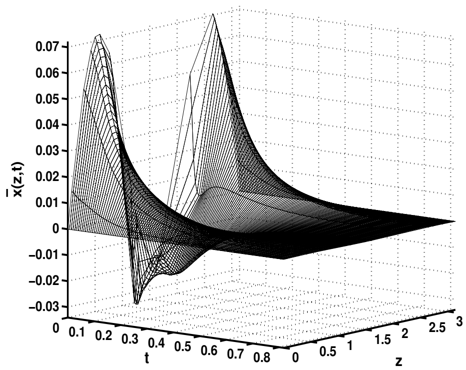

Fig. 5. Closed-loop state profile under the MPC formulation of Eqs.38-39 with initial condition $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.07sin(3z)$ and $\mathcal{S}^{max} = 1.9$ and $\mathcal{S}^{min} = -0.034$.

Higher-order predictive control formulation. In order to account for the evolution of the fast states in the optimization problem, we consider the following MPC formulation with the objective function and constraints given by:

$$\min_{u}\int_{t}^{t+T}\left[q_s\|x_s(\tau)\|_2^2+|u(\tau)|_R^2\right]d\tau \qquad (34)$$

$$\begin{aligned} s.t.\ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau)+\mathcal{F}_s(x_s(\tau),x_f(\tau))+B_s u(\tau)\\ \dot{x}_f(\tau) &= \mathcal{A}_f x_s(\tau)+\mathcal{F}_f(x_s(\tau),x_f(\tau))+B_f u(\tau)\\ u_{min} &\le u_i(\tau) \le u_{max},\ i=1,2\\ \chi^{min} &\le (r,x_s(\tau)+x_f(\tau)) \le \chi^{max} \end{aligned} \qquad (35)$$

where $\tau \in [t, t+T]$. Note that even though the fast modes appear explicitly in the state constraints equation, they do not appear in the cost functional, keeping the computational requirement relatively low. The MPC formulation above,

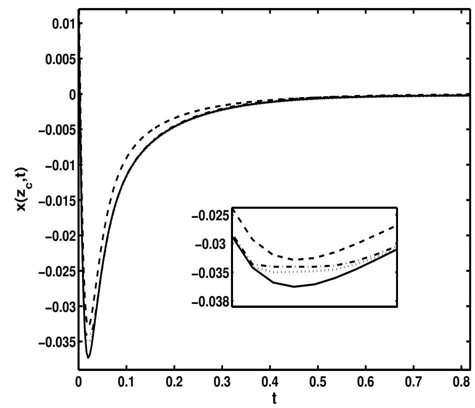

Fig. 6. Closed-loop state profile at $z_c = 1.156$ under the MPC formulation of Eqs.28-29 (solid), under the MPC formulation of Eqs.32-33 (dashed), under the MPC formulation of Eqs.36-37 (dotted), and under the MPC formulation of Eqs.38-39 (dashed-dotted).

when written using modal amplitudes, takes the following form:

$$\min_{u}\int_{t}^{t+T}\left[q_s|a_s(\tau)|^2+|u(\tau)|_R^2\right]d\tau \qquad (36)$$

$$\begin{aligned} s.t.\ \dot{a}_s(\tau) &= A_s a_s(\tau)+F_s(a_s(\tau),a_f(\tau))+B_s u(\tau)\\ \dot{a}_f(\tau) &= A_f a_s(\tau)+F_f(a_s(\tau),a_f(\tau))+B_f u(\tau)\\ u_{min} &\le u_i(\tau) \le u_{max},\ i=1,2\\ \chi^{min} &\le C_s a_s(\tau)+C_f a_f(\tau) \le \chi^{max} \end{aligned} \qquad (37)$$

where $\tau \in [t, t+T]$, $C_f = [\phi_3(z_c)\ \cdots\ \phi_{30}(z_c)]$ is a row vector and the MPC tuning parameters have the same values used in the previous formulation.

Starting from the initial condition, $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.07sin(3z)$, Fig.4 and Fig.6 (dotted lines) show the closed–loop evolution of the states under the MPC formulation of Eqs.36-37. The controller successfully stabilizes the state profile at the zero steady-state and the state constraints are satisfied for all times. The corresponding manipulated input profiles are given in Figs.7-8.

Remark 3: Note that even though the optimization problem is nonconvex, and the solution obtained may only represent a local minimum, it does not detrimentally affect the task of state constraint satisfaction, because state constraints are posed as explicit constraints in the optimization problem. Even if a solution is not a global minimum (which, in general it will not be), the feasibility of the constraints in the optimization problem ensure that upon implementation of this control action, the state constraints will be satisfied for the infinite dimensional system.

High-order predictive control formulation based on two-time-scale approximation. As evidenced by the examples shown before, accounting for the evolution of the fast modes is important for the purpose of satisfying state constraints. The computational complexity associated with accounting for the fast modes could be eased by approximating the dynamics of the fast modes, while retaining the nonlinear dynamics of the slow modes (so as to not adversely effect the task of stabilization). One possible way of approximation is to neglect the nonlinearity in the equations describing the evolution of the fast modes. This is because the term A_f behaves like $1/\epsilon$, where ϵ is a small parameter, and therefore, A_f is much larger than F_f and thus F_f can be neglected from the equation (see [5] for more discussion and analysis of this approximation). Using this approximation, and shrinking the state constraints in the optimization to account for the error induced due to this approximation, the predictive control formulation takes the following form:

$$\min_{u} \int_{t}^{t+T} \left[q_s |a_s(\tau)|^2 + |u(\tau)|_R^2 \right] d\tau \qquad (38)$$

$$\begin{array}{rcl} s.t.\ \dot{a}_s(\tau) & = & A_s a_s(\tau) + F_s(a_s(\tau), a_f(\tau)) + B_s u(\tau) \\ \dot{a}_f(\tau) & = & A_f a_f(\tau) + B_f u(\tau) \\ u_{min} & \leq & u_i(\tau) \leq u_{max}, \ \ i = 1,2 \\ \mathcal{S}^{min} & \leq & C_s a_s(\tau) + C_f a_f(\tau) \leq \mathcal{S}^{max} \end{array} \qquad (39)$$

where $\tau \in [t, t+T]$, $C_f = [\phi_3(z_c) \ \cdots \ \phi_{30}(z_c)]$ is a row vector and the MPC tuning parameters have the same values used in the previous formulation. For $\bar{x}(z,0) = 0.04sin(z) + 0.0005sin(2z) + 0.07sin(3z)$, $\mathcal{S}^{max} = 1.9$ and $\mathcal{S}^{min} = -0.034$, Figs.5-8 (dash–dotted lines) show the evolution of the closed–loop state and manipulated input profiles under the MPC formulation of Eqs.38-39. It can be seen that the controller successfully stabilizes the state profile at the zero steady-state and that the state constraints are satisfied for all time.

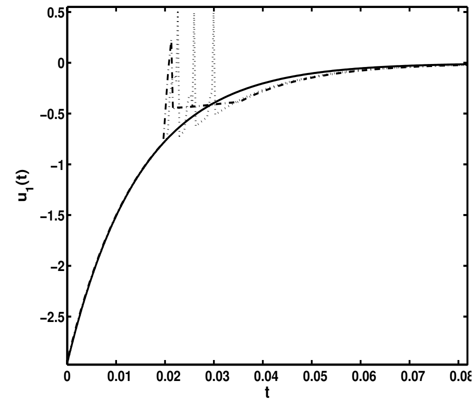

Fig. 7. Manipulated input profiles for the first control actuator applied at $z_{a_1} = \pi/3$ under the MPC formulation of Eqs.28-29 (solid), under the MPC formulation of Eqs.32-33 (dashed), under the MPC formulation of Eqs.36-37 (dotted), and under the MPC formulation of Eqs.38-39 (dashed-dotted).

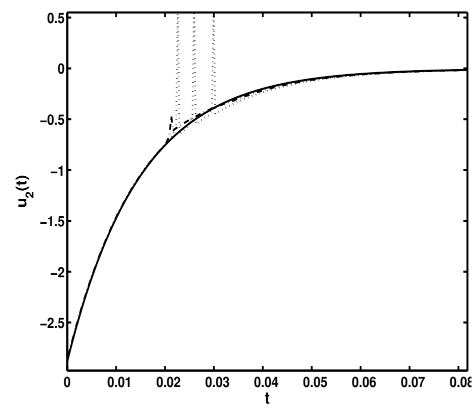

Fig. 8. Manipulated input profiles for the second control actuator applied at $z_{a_2} = 2\pi/3$ under the MPC formulation of Eqs.28-29 (solid), under the MPC formulation of Eqs.32-33 (dashed), under the MPC formulation of Eqs.36-37 (dotted), and under the MPC formulation of Eqs.38-39 (dashed-dotted).

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