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Feasibility of H_∞ design specifications: an interpolation method

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***Abstract*— Given a set of H_∞ design specifications, the issue is to check whether there exists a controller, whose order is free, which satisfies these specifications. The classical solution, which is based on Youla parametrisation and convex closed loop design, is not really satisfactory since it should use an infinite dimensional basis of filters, which cannot be done in practice. Let J^* the minimal value of the design objective over such an infinite dimensional basis of filters. A Nevanlinna Pick interpolation method is proposed here to compute lower and upper bounds of J^*, by solving the design problem on a finite set of frequencies. Finite-time convergence of the algorithm is proved.**

***Index Terms*— convex closed loop design, Youla parametrisation, H_∞ control, feasibility of design specifications, Nevanlinna Pick, interpolation.**

I. INTRODUCTION

Checking the feasibility of a set of design specifications is a major issue in an industrial context, since performance is often to be maximised. But what are the limits of performance ? In the same spirit, given a set of (ambitious) design specifications, does there exist a controller which satisfies this set ? A solution was proposed at the beginning of the 90's in the pionneering work of [3]. Using Youla parametrisation [14], [1] and an initial controller, the closed loop transfer matrix, which is a highly non-linear function of the feedback controller $K(s)$, is first put under the affine form $T_1(s) + T_2(s)Q(s)T_3(s)$, where the $T_i(s)$ are fixed while the Youla parameter $Q(s)$ is free. Let then $Q(s) = \sum_i \theta_i Q_i(s)$, where the basis of filters $Q_i(s)$ is fixed while the θ_i are the design parameters. Most nominal performance specifications and unstructured robustness ones can then be translated as convex constraints or minimisation objectives w.r.t. the θ_i.

The convexity of the optimisation problem is crucial since it enables to claim that if no solution is found, the constraints are not feasible: it cannot be argued that these constraints are just locally infeasible, but possibly globally feasible. Moreover, if the infinite dimensional basis of filters $Q_i(s)$ is chosen to cover the whole set of asymptotically stable transfer matrices, the parametrisation $T_1(s)+T_2(s)Q(s)T_3(s)$ covers the whole set of achievable closed loops [9]: given any stabilizing feedback controller $K(s)$, there exists a corresponding value of the Youla parameter which gives the same closed loop, and conversely.

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As a consequence it becomes possible to check whether there exists a controller, whose order is free, which satisfies a set of design specifications.

However a finite dimensional basis is always used in practice. If J^* denotes the minimal value of the design objective over a suitable infinite dimensional basis, the use of a finite dimensional basis just provides an upper bound of J^*. In the context of H_∞ design specifications we propose a first solution which is to compute a lower bound by solving the design problem on a finite set of frequencies: a finite-dimensional convex optimisation problem is to be solved with an LMI solver [10]. A frequency response $(Q(j\omega_i))_{i\in[1,\ N]}$ is obtained on a finite frequency gridding, which satisfies the design specifications at these points. An interpolation Nevanlinna-Pick constraint [8], [7] can be added so that the $Q(j\omega_i)$ necessarily correspond to a stable transfer matrix. This improves the quality of the lower bound.

If the gap between the lower and upper bounds is small enough, the result provided by the finite dimensional basis of filters is validated. Nevertheless the choice of the basis is not so easy in practice, it is usually necessary to guess the poles of the basis, so that we propose an alternative method to compute an upper bound: an interpolation algorithm computes a stable transfer matrix $Q(s)$, whose frequency response is $(Q(j\omega_i))_{i\in[1,\ N]}$. Noting that the design specifications are not necessarily met between the points of the frequency gridding, an upper bound of J^* is deduced. A way to reduce the gap between the bounds is to refine the frequency gridding. Finite-time convergence of the algorithm is proved.

The Nevanlinna-Pick condition, as well as other interpolation conditions have been widely used in identification [5], [6] and H_∞ control [4], [12], [2] theories: it was especially used to develop analytic solutions to the H_∞ control problem. Nevertheless, the issue was to minimize the H_∞ norm of a single MIMO transfer matrix, or more generally to shape a single MIMO transfer matrix. In the more general context of convex closed loop design our aim is to independently shape different MIMO transfer matrices, i.e. to miminize the H_∞ norm of one or several MIMO transfer matrices under H_∞ constraints on other MIMO transfer matrices. No analytic solution is now available, and this design problem is much more complex than the H_∞ one.

The paper is organised as follows. Section 2.1 briefly describes Youla parametrisation. The problem is then stated in section 2.2. Lower bounds of J^* are proposed in section

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3, while the upper bound is described in section 4. Section 5 summarizes the results in an algorithm, whose finite-time convergence is proved. Section 6 describes the application to a realistic large transport aircraft. Concluding remarks end the paper.

II. PRELIMINARIES

A. Youla parametrisation

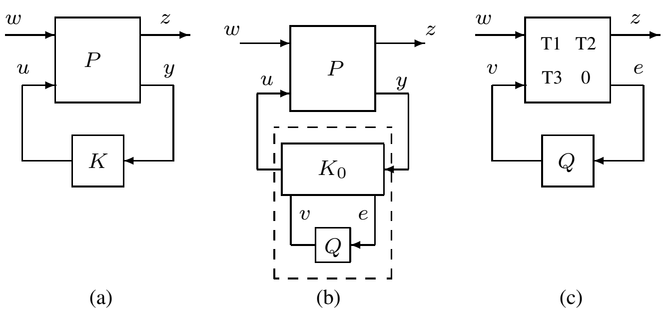

Fig. 1. The design problem (a) and Youla parametrisation (b,c).

Consider the standard design problem of figure 1.a, where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ is an augmented plant. The closed loop transfer matrix $F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$ is a highly nonlinear function of controller K. Suppose an initial stabilizing controller K_0, whose order is at least equal to the order of P_{22}, is available. Additional inputs and outputs v and e are introduced in K_0 (see figure 1.b), with the key constraint that the transfer matrix between v and e is zero: see figure 1.c. A solution to achieve this property is to put K_0 under the form of an observed state feedback controller [1]. When connecting then a free stable transfer matrix Q to these additional inputs and outputs, $F_l(P,K)$ can be rewritten as $T_1 + T_2QT_3$, where fixed transfer matrices T_i depend on P and K_0, while Q is the design parameter.

B. Problem statement

To simplify the exposition the following problem is solved, with just one H_∞ constraint and one H_∞ minimisation objective. Let J^* the minimal value of γ under the constraints on the frequency interval $[\underline{\omega},\ \overline{\omega}]$:

$$\overline{\sigma}(T_1(j\omega) + T_2(j\omega)Q(j\omega)T_3(j\omega)) \leq \gamma \quad (1)$$
$$\overline{\sigma}(T_4(j\omega) + T_5(j\omega)Q(j\omega)T_6(j\omega)) \leq 1 \quad (2)$$

and under the α shifted H_∞ constraint ($\alpha > 0$):

$$\sup_{\Re(s) \geq -\alpha} \overline{\sigma}(Q(s)) \leq M \quad (3)$$

J^* is chosen as $+\infty$ if constraints (2,3) are not feasible. $T_1 + T_2QT_3$ and $T_4 + T_5QT_6$ represent parts of the large MIMO transfer matrix $F_l(P,K) = T_1 + T_2QT_3$ on figure 1. The extension of our results to several H_∞ constraints and minimisation objectives is straightforward.

Remark: constraint (3) implies that $Q(s)$ has a minimal degree of stability α, and the H_∞ norm of $Q(s)$ is less than M. Note that a frequency dependent template $m(s)$ could be introduced instead of a constant value M [5].

III. TWO LOWER BOUNDS

A simple lower bound is proposed in the first subsection. An interpolation constraint is then added to obtain an improved lower bound. The use of the Nevanlinna Pick condition requires additional constraints on the degree of stability and shifted H_∞ norm of $Q(s)$.

A. A first simple lower bound

The issue is to obtain a frequency response $(Q(j\omega_i))_{i\in[1,\ N]}$ on a finite frequency gridding, which satisfies the design specifications at these points.

Proposition 3.1: Let $(\omega_i)_{i\in[1,\ N]}$ a fixed frequency gridding of $[\underline{\omega},\ \overline{\omega}]$. let J_1 the minimal value of γ under the constraints ($\forall i \in [1, N]$):

$$\overline{\sigma}(T_1(j\omega_i) + T_2(j\omega_i)Q(j\omega_i)T_3(j\omega_i)) \leq \gamma \quad (4)$$
$$\overline{\sigma}(T_4(j\omega_i) + T_5(j\omega_i)Q(j\omega_i)T_6(j\omega_i)) \leq 1 \quad (5)$$
$$\overline{\sigma}(Q(j\omega_i)) \leq M \quad (6)$$

If constraints (5,6) are not feasible at a frequency ω_i the initial design problem (1,2,3) is not feasible. Otherwise J_1 is a lower bound of J^*.

Remark: constraints (4,5,6) can be solved with an LMI solver, since $\overline{\sigma}(M) \leq \alpha$ if and only if $\begin{bmatrix} \alpha I & M \\ M^* & \alpha I \end{bmatrix} > 0$. The computational requirement is low since the problem is independently solved at each frequency.

B. A second lower bound

The following lemma is known as the very classical Pick condition [8].

Lemma 3.2: Let $(s_i)_{i\in[1,\ N]}$, N complex points satisfying $\Re(s_i) > 0$. Let $(W_i)_{i\in[1,\ N]}$, N matrices of $\mathbb{C}^{n_o \times n_i}$ satisfying $\overline{\sigma}(W_i) \leq 1$ for $i \in [1,\ N]$. There exists a transfer matrix $H(s)$:

1) which is analytic in the Right Half Plane $\Re(s) \geq 0$.
2) which satisfies $H(s_i) = W_i$ for $i \in [1,\ N]$.
3) which satisfies $\sup_{\Re(s)\geq 0} \overline{\sigma}(H(s)) \leq 1$.

if and only if the following Pick matrix P is positive definite:

$$P = \left[\frac{I_{n_o} - W_k W_l^*}{s_k + \bar{s}_l}\right]_{1\leq k,l\leq N} \quad (7)$$

The above lemma cannot directly be applied to our problem, since the points s_i should strictly belong to the Right Half Plane, whereas $s_i = j\omega_i$ in section III-A. The following lemma explains how to transform our problem.

Lemma 3.3: There exists a transfer matrix $Q_r(s)$:

1) which is analytic in the Half Plane $\Re(s) \geq -\alpha$.
2) which satisfies $Q_r(j\omega_i) = Q(j\omega_i)$ for $i \in [1,\ N]$, where $Q(j\omega_i)$ satisfy constraints (4,5,6).
3) which satisfies:

$$\sup_{\Re(s)\geq -\alpha} \overline{\sigma}(Q_r(s)) \leq M \quad (8)$$

if and only if there exists a transfer matrix $\hat{Q}_r(s)$:

1) which is analytic in the Right Half Plane $\Re(s) \geq 0$.
2) which satisfies $\hat{Q}_r(j\omega_i + \alpha) = \frac{Q(j\omega_i)}{M}$.
3) which satisfies:

$$\sup_{\Re(s)\geq 0} \overline{\sigma}(\hat{Q}_r(s)) \leq 1 \quad (9)$$

With reference to Lemmas 3.2 and 3.3 the interpolation constraint, which is obtained by applying Lemma 3.2 to $\hat{Q}_r(s)$, is thus:

$$P = \left[\frac{I_{n_o} - \frac{Q(j\omega_k)Q^*(j\omega_l)}{M^2}}{j(\omega_k - \omega_l) + 2\alpha} \right]_{1\leq k,l\leq N} \geq 0 \quad (10)$$

where n_o (resp. n_i) represents the number of outputs (resp. inputs) of $Q(s)$. This interpolation constraint equivalently implies the existence of $\hat{Q}_r(s)$ and $Q_r(s)$.

Proof of Lemma 3.3: we first prove that the existence of $Q_r(s)$ implies the existence of $\hat{Q}_r(s)$. let $\hat{Q}_r(s) = \frac{Q_r(s-\alpha)}{M}$. Then $\hat{Q}_r(j\omega_i + \alpha) = \frac{Q_r(j\omega_i)}{M} = \frac{Q(j\omega_i)}{M}$. Moreover $Q_r(s)$ analytic in the Half Plane $\Re(s) \geq -\alpha$ implies $\hat{Q}_r(s)$ analytic in the Right Half Plane $\Re(s) \geq 0$. Finally (8) implies (9).

In the same way we prove that the existence of $\hat{Q}_r(s)$ implies the existence of $Q_r(s)$. Let $Q_r(s) = M\hat{Q}_r(s+\alpha)$. Then $Q_r(j\omega_i) = M\hat{Q}_r(j\omega_i + \alpha) = Q(j\omega_i)$. Moreover $\hat{Q}_r(s)$ analytic in the Right Half Plane $\Re(s) \geq 0$ implies $Q_r(s)$ analytic in the Half Plane $\Re(s) \geq -\alpha$. Finally (9) implies (8).

Proposition 3.4: Let $(\omega_i)_{i\in[1,\ N]}$ a fixed frequency gridding of $[\underline{\omega},\ \overline{\omega}]$. let J_2 the minimal value of γ under the constraints (4,5,6) and under the additional interpolation constraint (10). If constraints (5,6,10) are not feasible the initial design problem (1,2,3) is not feasible. Otherwise, J_2 is a lower bound of J^*.

Remarks:
(i) With reference to proposition 3.1 $J_2 \geq J_1$ since the optimisation problem is more constrained.
(ii) The computational requirement can be much higher than the one in proposition 3.1, since it's no more possible to independently solve the problem at each frequency.

As a final point the interpolation constraint (10) must be put under an LMI form to use an LMI solver.

Lemma 3.5: Let $R_o = \left[\frac{I_{n_o}}{j\omega_k - j\omega_l + 2\alpha}\right]_{1\leq k,l\leq N}$, $R_i = \left[\frac{I_{n_i}}{j\omega_k - j\omega_l + 2\alpha}\right]_{1\leq k,l\leq N}$ and $\tilde{Q} = blockdiag(\frac{Q(j\omega_1)}{M}, \ldots, \frac{Q(j\omega_N)}{M})$. Then $P > 0$ if and only if $\begin{bmatrix} R_o & \tilde{Q} \\ \tilde{Q}^* & R_i^{-1} \end{bmatrix} > 0$

To prove this lemma simply apply the Schur complement to $P = R_o - \tilde{Q}R_i\tilde{Q}^*$, where $R_o > 0$.

IV. AN UPPER BOUND

The issue is to compute a transfer matrix $Q_r(s)$ that interpolates the frequency reponse $(Q(j\omega_i))_{i\in[1,\ N]}$ given by the computation of the lower bound J_2. We will first compute a state-space representation of $\hat{Q}_r(s)$. Then, using the equivalence given by lemma 3.3 a representation of $Q_r(s)$ is deduced. Several algorithms are available to compute solutions to the classical Pick interpolation problem. We focus in this paper on the solution proposed by [7], since the direct computation of a state-space representation is expected to be better conditioned than the computation of a polynomial one, i.e. the computation of the coefficients of a high order transfer matrix.

Proposition 4.1: Let :

1) $(s_i)_{i\in[1,\ N]}$, N complex points satisfying $\Re(s_i) > 0$
2) $(W_i)_{i\in[1,\ N]}$, N matrices of $\mathbb{C}^{n_o\times n_i}$ satisfying $\overline{\sigma}(W_i) \leq 1$ for $i \in [1,\ N]$
3) $s_i = \overline{s_{i-N}}$ and $W_i = \overline{W_{i-N}}$ for $i \in [N+1,\ 2N]$
4) $G \in \mathbb{C}^{n_o\times n_i}$ a fixed complex matrix satisfying $\overline{\sigma}(G) < 1$
5) The Pick matrix $P = \left[\frac{I_{n_o} - W_k W_l^*}{s_k + \bar{s}_l}\right]_{1\leq k,l\leq 2N}$

If P is positive definite the transfer matrix $H(s)$ described by the real state-space representation (A, B, C, D) below is analytic in the Right Half Plane, its H_∞ norm is less than 1 and $H(s_i) = W_i$ is satisfied for $i \in [1,\ 2N]$.

$$A = \begin{bmatrix} \Re(s_1)I_{n_i} & \Im(s_1)I_{n_i} & \cdots & 0 & 0 \\ -\Im(s_1)I_{n_i} & \Re(s_1)I_{n_i} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Re(s_N)I_{n_i} & \Im(s_N)I_{n_i} \\ 0 & 0 & \cdots & -\Im(s_N)I_{n_i} & \Re(s_N)I_{n_i} \end{bmatrix}$$
$$- \begin{bmatrix} 0_{n_i} \\ I_{n_i} \\ \vdots \\ 0_{n_i} \\ I_{n_i} \end{bmatrix} \begin{bmatrix} -2\Im(K^{(1)})^T \\ 2\Re(K^{(1)})^T \\ \vdots \\ -2\Im(K^{(N)})^T \\ 2\Re(K^{(N)})^T \end{bmatrix}^T$$

$$B = \sqrt{2}\begin{bmatrix} 0_{n_i} & I_{n_i} & \cdots & 0_{n_i} & I_{n_i} \end{bmatrix}^T$$
$$C = \sqrt{2}\begin{bmatrix} -\Im((W_1 - GI_{n_i})K^{(1)})^T \\ \Re((W_1 - GI_{n_i})K^{(1)})^T \\ \vdots \\ -\Im((W_N - GI_{n_i})K^{(N)})^T \\ \Re((W_N - GI_{n_i})K^{(N)})^T \end{bmatrix}^T$$
$$D = G$$

and

$$K = P^{-1}\left[\begin{array}{c} I_{n_i} - W_1^* G \\ \vdots \\ I_{n_i} - W_{2N}^* G \end{array}\right] = \left[\begin{array}{c} K^{(1)} \\ \vdots \\ K^{(2N)} \end{array}\right]$$

Remarks:
(i) The use of this result implies to modify the interpolation constraint in the computation of the lower bound J_2, since positive and negative frequencies ω_i and $-\omega_i$ are now to be considered in this constraint.
(ii) The state-space representation of $Q_r(s)$ is simply deduced from the one of $\hat{Q}_r(s)$ (computed with the above proposition) using $Q_r(s) = M\hat{Q}_r(s+\alpha)$.

V. CONVERGENCE OF THE ALGORITHM

Feasibility of the design specifications is first checked, before minimising the criterion. Convergence is proved.

A. A feasibility algorithm

The issue is to compute λ^*, the minimal value of λ under the constraints on the interval $[\underline{\omega},\ \overline{\omega}]$:

$$\overline{\sigma}(T_4(j\omega) + T_5(j\omega)Q(j\omega)T_6(j\omega)) \leq \lambda \qquad (11)$$

and under constraint (3). If λ^* is less than 1 the design problem (1,2,3) is feasible. The following algorithm is used:

1) An initial frequency gridding $(\omega_i)_{i\in[1,\ N]}$ is defined, possibly on the basis of the frequency response of the open loop plant or of the initial closed loop (i.e. corresponding to the initial controller). The maximal allowable α shifted H_∞ norm M of $Q(s)$, as well as its minimal degree of stability α are fixed. Let also ϵ a small strictly positive scalar value.
2) Solve the LMI problem of minimising λ under the constraints:
$$\begin{array}{rcl} \overline{\sigma}(T_4(j\omega_i) + T_5(j\omega_i)Q(j\omega_i)T_6(j\omega_i)) & \leq & \lambda \\ \overline{\sigma}(Q(j\omega_i)) & \leq & M \end{array}$$
on the frequency gridding, with the interpolation constraint (10). This LMI problem is always feasible, since $Q(j\omega_i) = 0$ is a solution. Let λ_{LB}^* the associated minimal value of λ. $\lambda_{LB}^* \leq \lambda^*$.
3) Compute a transfer matrix $Q_r(s)$ which corresponds to the frequency response $(Q(j\omega_i))_{i\in[1,\ N]}$ obtained at step 2. The interpolation algorithm ensures that the α shifted H_∞ norm of $Q_r(s)$ is less than M, and that $Q_r(s)$ satisfies the minimal degree of stability constraint. Let λ_{UB}^* the minimal value of λ satisfying (11) on the interval $[\underline{\omega},\ \overline{\omega}]$. $\lambda^* \leq \lambda_{UB}^*$.
4) If the gap between the bounds of λ^* is less than ϵ STOP, λ^* is computed with a satisfactory accuracy. Otherwise compute the frequency corresponding to the peak value of λ_{UB}^*, include it in the frequency gridding and go back to step 2.

Remark: if the issue is just to check the feasibility of the design specifications the algorithm can be stopped as soon as $\lambda_{LB}^* > 1$, which means infeasibility, or $\lambda_{UB}^* \leq 1$, which means feasibility.

Finite-time convergence of the algorithm is now proved. To this aim the issue is essentially to bound the variation of $T_4(j\omega) + T_5(j\omega)Q_r(j\omega)T_6(j\omega)$ between the points of the frequency gridding. This is done using the minimal degree of stability and maximal H_∞ norm assumptions on $Q_r(s)$, as explained in the following Lemma. As a preliminary let $\bar{B}H_\infty(\mathbb{C}_{\alpha^+}, M)$ the set of transfer matrices that are analytic in $\Re(s) \geq -\alpha$ and whose α shifted H_∞ norm is less than M.

Lemma 5.1: Let :

1) $\beta = min(\alpha, \lambda)$ where α is the minimal degree of stability of $Q_r(s)$ and λ the degree of stability of the initial closed loop $\left(\begin{array}{cc} T_4(s) & T_5(s) \\ T_6(s) & 0 \end{array}\right)$
2) $L = \|T_4\|_{\infty,\beta} + M\|T_5\|_{\infty,\beta}\|T_6\|_{\infty,\beta}$, where $\|.\|_{\infty,\beta}$ is the β shifted H_∞ norm
3) $(\omega_i)_{i\in[1,\ N]}$ a fixed frequency gridding of $[\underline{\omega},\ \overline{\omega}]$
4) $(W_i)_{i\in[1,\ N]}$, a set of N matrices of $\mathbb{C}^{n_o\times n_i}$ satisfying both $\overline{\sigma}(W_i) \leq M$ for $i \in [1,\ N]$ and the Pick interpolation condition over $\bar{B}H_\infty(\mathbb{C}_{\alpha^+}, M)$
5) $\hat{W}(s)$ an interpolating function of $\bar{B}H_\infty(\mathbb{C}_{\alpha^+}, M)$, satisfying $\hat{W}(j\omega_i) = W_i\ \forall i \in [1,\ N]$.

Then $\forall\epsilon \in]0,\ 2L[$:

$$\sup_{\omega\in[\omega_k-\delta\,,\,\omega_k+\delta]} \bar{\sigma}(T_4(j\omega) + T_5(j\omega)\hat{W}(j\omega)T_6(j\omega) - T_4(j\omega_k) - T_5(j\omega_k)W_kT_6(j\omega_k)) \leq \epsilon$$

is guaranteed $\forall k \in [1,\ N]$ and $\delta = \frac{2\beta}{\sqrt{(\frac{2L}{\epsilon})^2-1}}$.

Proof: let $\epsilon \in]0\,,2L[$, $k \in [1,\ N]$. $h_k(s) = T_4(s) + T_5(s)\hat{W}(s)T_6(s) - T_4(j\omega_k) - T_5(j\omega_k)W_kT_6(j\omega_k)$ is a function of $\bar{B}H_\infty(\mathbb{C}_{\beta^+}, 2L)$ that interpolates zero at $s = j\omega_k$. Thus, with reference to the Nevanlinna-Pick theory [5], $h_k(s)$ can be rewritten under the form $h_k(s) = \frac{s-j\omega_k}{s-j\omega_k+2\beta}g(s)$ with $g \in \bar{B}H_\infty(\mathbb{C}_{\beta^+}, 2L)$. Then :

$$\bar{\sigma}(h_k(j\omega)) \leq \frac{|\omega-\omega_k|}{\sqrt{|\omega-\omega_k|^2+4\beta^2}}\bar{\sigma}(g(j\omega))$$

Since $\frac{x}{\sqrt{x^2+4\beta^2}}$ is non-decreasing for x in $\mathbb{R}^+$:

$$\sup_{\omega\in[\omega_k-\delta\,,\,\omega_k+\delta]} \bar{\sigma}(h_k(j\omega)) \leq \frac{\delta}{\sqrt{4\beta^2+\delta^2}}2L$$

$\epsilon = \frac{\delta}{\sqrt{4\beta^2+\delta^2}}2L$ is equivalent to $\delta = \frac{2\beta}{\sqrt{(\frac{2L}{\epsilon})^2-1}}$.

Proposition 5.2 (Finite time convergence): First assume that the initial frequency gridding satisfies $\omega_1 = \underline{\omega}$, $\omega_N = \overline{\omega}$ and that all points ω_i are further than δ. Let $\lambda_{LB_N}^*$ and $\lambda_{UB_N}^*$ the lower and upper bounds computed by the algorithm with N points in the frequency gridding. For all $\epsilon \in (0, 2L)$ there exists $\hat{N} \in \mathbb{N}$ such that :

$$\forall N \geq \hat{N}\,,\ \lambda_{UB_N}^* - \lambda_{LB_N}^* \leq \epsilon$$

A possible value of $\hat{N}$ is $\hat{N} = E\left[(\overline{\omega} - \underline{\omega})\frac{\sqrt{\left(\frac{2L}{\epsilon}\right)^2 - 1}}{2\beta}\right] + 1$, where $E(x)$ is the integer part of a real x.

B. A minimisation algorithm

The issue is to compute an interval for J^*, the minimized objective of the initial design problem (1,2,3). We assume in this section that the problem is strictly feasible, i.e. there exists a small $\epsilon_1 > 0$ such that $\lambda^* \leq 1 - \epsilon_1$. Let also ϵ_2 the minimal tolerated gap between the lower and upper bounds of J^*. The algorithm, whose convergence is proved in the same way as in the previous subsection, is the following:

1) An initial frequency gridding is defined, either using the final frequency gridding of the feasibility phase or on the basis of the frequency response of the open loop plant or of the initial closed loop (i.e. corresponding to the initial controller). The maximal allowable α shifted H_∞ norm M of $Q(s)$, as well as its minimal degree of stability are fixed.
2) Solve the LMI problem of proposition 3.4. This LMI problem is necessarily feasible. A lower bound J^*_{LB} of J^* is obtained.
3) Solve here again the LMI problem of proposition 3.4, but with constraint (5) replaced by:

$$\overline{\sigma}\left(T_4(j\omega_i) + T_5(j\omega_i)Q(j\omega_i)T_6(j\omega_i)\right) \quad \leq \quad 1 - \epsilon_1$$

This LMI problem is necessarily feasible. Strictly speaking the minimal value of γ is not guaranteed to be a lower bound of J^* since the problem is slightly more constrained.
4) Compute a transfer matrix $Q_r(s)$ which corresponds to the frequency response $(Q(j\omega_i))_{i\in[1,\ N]}$ obtained at step 3.
5) If $Q_r(s)$ does not satisfy constraint (2) no upper bound of J^* can be computed. In this case compute the frequency where constraint (2) is the most violated, include it in the frequency gridding and go back to step 2.
6) If $Q_r(s)$ satisfies constraint (2) an upper bound of J^* is easily computed as the minimal value of γ satisfying (1). If the gap between the bounds is less than ϵ_2 STOP, J^* is computed with a satisfactory accuracy. Otherwise compute the frequency corresponding to the peak value of γ in (1), include it in the frequency gridding and go back to step 2.

VI. Application

We apply this method to the lateral flight control law design of a flexible civil transport aircraft. The presentation of this application and its requirements set were developped in [13]. We consider here a simplified problem.

A. Model description

The fly-by-wire enables any kind of control architecture. Below we give a conventional one for lateral control. The pilot's orders are transmitted by the side stick and pedals and respectively correspond to a roll rate objective and to a sideslip demand. The measurements consist of accelerations and angular information:

- p: roll rate
- r: yaw rate
- ϕ: bank angle
- Ny_{front}: lateral acceleration at the front of the aircraft
- Ny_{rear}: lateral acceleration at the rear of the aircraft
- Nz_{wing}: vertical acceleration on the wing

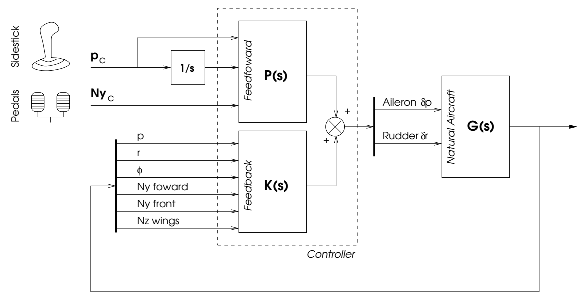

Fig. 2. Lateral law architecture

The pilot's orders and measurements are mixed through feedforward and feedback to produce orders for the ailerons (δ_p) and rudder (δ_r).

A complete aircraft model is required to compute laws. We thus consider a state-space representation built by coupling rigid and elastic body models. The evaluation methods are described in [11]. Note that the two models are coupled by connecting respective measurements outputs at the same structural point of the aircraft. Other methods for translating interactions between rigid and flexible bodies are known. Nevertheless, they are more complex and the one given here provides a good enough representation.

B. The design problem

Our study deals with the optimization of the feedback part. The feedback mission is to ensure a robust stabilization of the aircraft and a good rejection of perturbations without saturating the actuators. We thus formalize the design problem as the minimisation of an objective under three constraints. The aim of the control law is to minimise the lateral accelerations (Ny) felt by passengers in turbulence (W_y). This objective is thus characterised by the transfer function $H_{W_y \to Ny}$. Requirements are :

1) Stabilize the aircraft. This requirement is ensured by the initial controller (an H_∞ one synthesized in [13]) and the Youla parametrisation itself.
2) Keep good robustness through the T margin (the complementary sensitivity function $KG(I-KG)^{-1}$). Here $\|T\|_\infty \leq 2$ is required.
3) Satisfy a roll-off constraint on the T transfer function to reject high frequency unmodelled dynamics and perturbations.

The last two requirements on the T transfer function are mixed through the frequency domain template $W_r(s)$. The issue is thus to minimize γ under the constraints:

$$\|H_{W_y \to Ny_{front}}\|_\infty \leq \gamma \qquad (12)$$
$$\|H_{W_y \to Ny_{rear}}\|_\infty \leq \gamma \qquad (13)$$
$$\|T.W_r^{-1}\|_\infty \leq 1 \qquad (14)$$

The frequency domain is restricted to $[0,\ 6\ Hz]$.

C. Results

The minimisation algorithm is applied with the following assumptions: the degree of stability is $\alpha = 1$ and the α-shifted H_∞ norm is $M = 10$. The tolerance on constraints is $\epsilon_1 = 5\%$ and the gap between the bounds of the minimisation objective is $\epsilon_2 = 2\%$.

The convergence of the minimisation algorithm is shown on figure 3. When starting with a 14 points frequency gridding 27 points are enough to achieve the desired accuracy. The computational time is rather high, around 12000 s, but remember the complexity of the initial design problem over an infinite dimensional basis of filters, and also the difficulty of our flexible aircraft application (with 9 bending modes).

Moreover the computed solution is quite satisfactory. The response $H_{W_y \to Ny_{front}}$ is presented on figure 4. The dashed line shows the initial closed loop ($Q(s) = 0$), the solid line the final solution, corresponding to the upper bound of the minimisation objective, and the dash-dotted line represents the lower bound (the bounds nearly coincide because of the gap that is lower than ϵ_2). The '$\circ$' finally represent the interpolated points. The reduction of the H_∞ norm of the initial closed loop (i.e. corresponding to $Q(s) = 0$) is around 65% on the front transfer function, which is more than we hoped. We finally check the α-stability and α-shifted H_∞ norm of the solution $Q(s)$ (these properties are ensured by the interpolation algorithm).

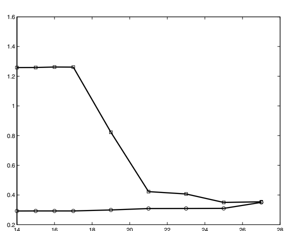

Fig. 3. Convergence of the minimisation algorithm.

VII. Conclusion

A method was proposed to check the feasibility of H_∞ design specifications, and more generally to compute the minimal value J^* of an H_∞ design objective under H_∞ constraints, using an infinite dimensional basis for the Youla parameter $Q(s)$. More precisely lower and upper bounds of J^* were computed using a finite frequency gridding and an interpolation Nevanlinna-Pick method.

When classically solving an H_∞ or convex closed loop design problem with the KYP lemma the computational burden is directly linked to the order of the open or closed loop model. This is not the case here, so that the method should be applicable to rather large dimension models, as testified by the applicative section. The limitation is rather the size of the frequency gridding, so that it is important to minimise this size, as done in our algorithm. More generally the applicative section enabled us to prove the applicability of our technique to a realistic, large dimension problem.

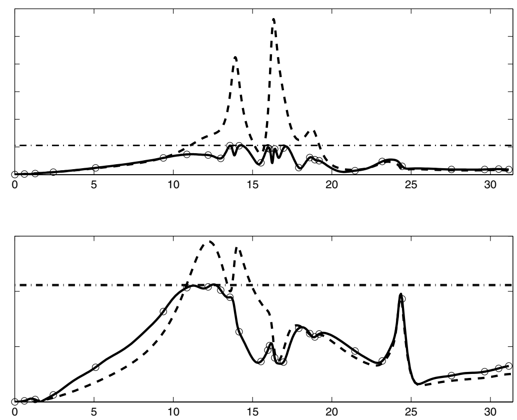

Fig. 4. Objective.

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