H_∞ Control of Descriptor Systems in a Differential Inclusion Setting

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***Abstract*— Based on a differential inclusion approach, the notion of quadratic H_∞ performance is introduced for a rather general class of nonlinear descriptor systems. The approach is specialized to the performance analysis and controller synthesis of descriptor systems with norm bounded uncertainties. In this case quadratic H_∞ performance can be characterized by means of parameterized linear matrix inequalities (LMIs).**

I. INTRODUCTION

The term "descriptor variable" was introduced in [16] as the *natural* variables to describe a given system. Cartesian coordinates, for example, may be considered as natural variables for mechanical systems. Formally, descriptor systems (sometimes also termed differential-algebraic equation systems or DAE systems, singular systems, or semi-state systems) refer to system descriptions of the form

$$\mathbf{0} = \boldsymbol{f}(\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t), t) \tag{1}$$

where $\boldsymbol{f} : I\!R^{n_x} \times I\!R^{n_x} \times I\!R^{n_u} \times I\!R^{n_w} \times I\!R \rightarrow I\!R^{n_x}$ denotes some possibly non-linear vector-valued function. The arguments of this function are the *descriptor vector* $\boldsymbol{x}(t) \in I\!R^{n_x}$ and its derivative $\dot{\boldsymbol{x}}(t) \in I\!R^{n_x}$, the control input vector $\boldsymbol{u}(t) \in I\!R^{n_u}$ and additional inputs $\boldsymbol{w}(t) \in I\!R^{n_w}$ which may at this point be interpreted as disturbances. Finally, for time-varying systems, also the time t explicitly occurs as an input variable to the function $\boldsymbol{f}$.

In contrast to the standard state-space description $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t), t)$, Eq. (1) admits an implicit dependency on $\dot{\boldsymbol{x}}(t)$ and especially the notation (1) includes the possibility that certain components of $\dot{\boldsymbol{x}}$ do not explicitly appear in the system description. In this case the corresponding component of the descriptor vector $\boldsymbol{x}(t)$ is termed *algebraic* variable (the other variables are called *differential* variables).

Descriptor system descriptions cover a broad class of systems not only of theoretical interest but also of great practical significance. Models of chemical processes, for example, typically consist of differential equations describing the dynamical balances of mass and energy while additional algebraic relations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. [15], [11]. In mechanics, descriptor system descriptions result from holonomic and non-holonomic constraints [24]. Especially they are encountered in multi-body and mechatronical system descriptions [19], [9]. Also in electronics [13] and even in economics [17] descriptor systems are encountered.

This paper considers the analysis of descriptor systems and controller synthesis for control problems given in descriptor form in a differential inclusion setup [3]. *Analysis*, in this context, means the detection of a certain property (e.g. stability, or an H_∞ norm bound) for a given descriptor system. More specifically, we aim at quadratic stability [4] and quadratic H_∞ norm boundedness, i.e. properties that are established by means of quadratic Lyapunov functions. *Synthesis*, on the other hand, is concerned with changing the system behavior by means of descriptor or output feedback (static and dynamical respectively) such that certain properties are achieved for the closed loop. The properties considered for synthesis are essentially the same as those considered for descriptor system analysis. In analogy to standard state-space theory "descriptor feedback" means that the descriptor vector is fed back. Consequently "output feedback" considers the situation (in case of linear descriptor systems) where only a linear combination of the descriptor variables is available for feedback.

The paper is structured as follows: for convenience a short introduction into linear descriptor system theory is given. Then the notion of quadratic stability and quadratical H_∞ performance is introduced for descriptor systems given in a differential inclusion setting. In the next section the approach is specialized to a pratical important subproblem, i.e. descriptor systems with norm bounded uncertainties, and the corresponding analysis problem is solved. This result is used to develop a solution for the synthesis problem, namely to find a controller for an uncertain open loop descriptor system such that the closed loop is quadratically H_∞ performant for all possible realizations of the assumed uncertainty description.

II. LINEAR DESCRIPTOR SYSTEMS

The material in this section summarizes the needed material from [7]. We consider linear, time-invariant descriptor systems

$$\begin{aligned} E\dot{\boldsymbol{\xi}}(t) &= A\boldsymbol{\xi}(t) + B\boldsymbol{w}(t),\ \ t \geq 0,\ \boldsymbol{\xi}(0^-) = \boldsymbol{\xi}_0^- \\ \boldsymbol{z}(t) &= C\boldsymbol{\xi}(t) + D\boldsymbol{w}(t). \end{aligned} \tag{2}$$

with constant system matrices $E, A \in I\!R^{n_\xi \times n_\xi}$, $B \in I\!R^{n_\xi \times n_w}$, $C \in I\!R^{n_z \times n_\xi}$, and $D \in I\!R^{n_z \times n_w}$ and $n_\xi \geq \operatorname{rank}(E) =: r$. $\boldsymbol{\xi}(t) \in I\!R^{n_\xi}$ denotes the descriptor variables,

$\boldsymbol{w}(t) \in \mathbb{R}^{n_w}$ the input variables, and $\boldsymbol{z}(t) \in \mathbb{R}^{n_z}$ the output variables.

As a shorthand notation for system (2) we often write (E, A, B, C, D) (or (E, A, B, C) if $D = 0$).

Definition 1: Two systems $(E,\ A,\ B,\ C,\ D)$ and $(\tilde{E},\ \tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D})$ are said to be (system) equivalent, denoted by $(E, A, B, C, D) \sim (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, if there exist non-singular transformation matrices $L, R \in \mathbb{R}^{n_\xi \times n_\xi}$ such that the equations $\tilde{E} = LER$, $\tilde{A} = LAR$, $\tilde{B} = LB$, $\tilde{C} = CR$, $\tilde{D} = D$ hold true (i.e. the two systems have the same input-output behavior).

In contrast to non-descriptor linear systems, (2) may have no solution, one solution or even multiple solutions for the same initial condition and input [18]. The solutions in general exhibit impulsive behaviour (i.e. are *generalized* solutions [8]) even if the input $\boldsymbol{w}(\cdot)$ is continuous [7]. For our purposes it will be necessary to characterize these properties in some detail:

Definition 2: The system (E, A, B, C, D) and the associated matrix pencil $sE - A$ are said to be *regular* if the characteristic polynomial $p(s) := \det(sE - A)$ does not vanish identically in s. Otherwise it is called *singular*.

Obviously regularity is invariant under system equivalence. Furthermore a regular system guarantees a unique solution of (2). On the other hand a singular system (2) always admits multiple solutions for the unforced ($\boldsymbol{w}(\cdot) \equiv \mathbf{0}$) homogeneous initial value problem [18]. Finally for regular systems (2) the transfer matrix

$$G(s) := C(sE - A)^{-1}B + D \tag{3}$$

s defined. The question of impulsive solutions of regular systems is usually studied in terms of the Weierstrass canonical form (WCF) of (E, A, B, C, D):

Theorem 1: [10] Let (E, A, B, C, D) be regular. Then there exists an equivalent system $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \sim (E, A, B, C, D)$ with

$$\tilde{E} = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\xi - r} \end{bmatrix} \tag{4}$$

where $J \in \mathbb{R}^{(n_\xi - r) \times (n_\xi - r)}$, $N \in \mathbb{R}^{r \times r}$ are matrices in Jordan canonical form and N is nilpotent.

Definition 3: The index of nilpotence ν of N, i.e. $\nu := \min\{q | N^q = 0, q \in \mathbb{N}\}$ is said to be the *index* of the linear descriptor system (E, A, B, C, D). Systems with $\nu \geq 2$ are called *high* index descriptor systems.

If (2) is in WCF, i.e.

$$\begin{bmatrix} \dot{\boldsymbol{\xi}}_1(t) \\ N\dot{\boldsymbol{\xi}}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1(t) \\ \boldsymbol{\xi}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \boldsymbol{w}(t),$$
$$t \geq 0,\ \boldsymbol{\xi}_1(0^-) = \boldsymbol{\xi}_{10}^-,\ \boldsymbol{\xi}_2(0^-) = \boldsymbol{\xi}_{20}^- \tag{5}$$

then the part $\boldsymbol{\xi}_1$ of the descriptor vector $\boldsymbol{\xi}^{\mathrm{T}} = [\boldsymbol{\xi}_1^{\mathrm{T}}, \boldsymbol{\xi}_2^{\mathrm{T}}]$ is governed by an ordinary differential equation while

$$\boldsymbol{\xi}_2(t) = -\sum_{i=0}^{\nu-1} \delta^{(i)}(t) N^{i+1} \boldsymbol{\xi}_{20}^- - \sum_{i=0}^{\nu-1} N^i \tilde{B}_2 \boldsymbol{w}^{(i)}(t),$$

solves the "algebraic part" in (5) (with $\delta(t)$ the Dirac delta and superscript (i) the ith distributional derivative). We conclude that descriptor systems will have no impulsive solutions (for all $\boldsymbol{w}(\cdot) \in L_2[0, \infty)$ and all initial conditions) iff their index is one.

Similar to non-descriptor systems the stability of *regular* descriptor systems can be studied by means of the pencil $sE - A$:

Theorem 2: [7] Let (E, A, B, C, D) be regular. The unforced ($\boldsymbol{w}(\cdot) \equiv \mathbf{0}$) system is asymptotically stable if and only if $\sigma(E, A) := \{s | s \in \mathbb{C}, \det(sE - A) = 0\} \subset \mathbb{C}^-$.

III. Quadratic Stability and Quadratic H_∞ Performance of Nonlinear Descriptor Systems

Consider the differential inclusion

$$E\dot{\boldsymbol{x}}(t) = A(\boldsymbol{x}(t), t)\boldsymbol{x}(t), \quad A(\boldsymbol{x}(t), t) \in \Omega \tag{6}$$

with the vector of descriptor variables $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$ and a constant matrix E which may be singular. Here $A(\boldsymbol{x}(t), t) \in \mathbb{R}^{n_x \times n_x}$ is a descriptor vector- and/or time-dependent matrix confined to range in some subset Ω of $\mathbb{R}^{n_x \times n_x}$. Later on, this dependency is further specified.

Additionally the extension of (6) to an input/output system

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A(\boldsymbol{x}(t), t)\boldsymbol{x}(t) + B(\boldsymbol{x}(t), t)\boldsymbol{w}(t) \\ \boldsymbol{z}(t) &= C(\boldsymbol{x}(t), t)\boldsymbol{x}(t) + D(\boldsymbol{x}(t), t)\boldsymbol{w}(t) \end{aligned} \tag{7}$$

in descriptor form is taken into account. For the latter system, additionally, external inputs/ outputs $\boldsymbol{w}(k) \in \mathbb{R}^{n_w}$ and $\boldsymbol{z}(t) \in \mathbb{R}^{n_z}$ respectively are considered. These inputs/outputs are connected to the system via the matrices $B(\boldsymbol{x}(t), t) \in \mathbb{R}^{n_x \times n_w}$, $C(\boldsymbol{x}(t), t) \in \mathbb{R}^{n_z \times n_x}$, and $D(\boldsymbol{x}(t), t) \in \mathbb{R}^{n_z \times n_w}$ with

$$\begin{bmatrix} A(\boldsymbol{x}(t), t) & B(\boldsymbol{x}(t), t) \\ C(\boldsymbol{x}(t), t) & D(\boldsymbol{x}(t), t) \end{bmatrix} \in \Omega \subset \mathbb{R}^{(n_x + n_z) \times (n_x + n_w)}. \tag{8}$$

The idea of considering the differential inclusion setup is to guarantee a certain property (i.e. stability, H_∞ norm boundedness,...) not only for one system description, but for all systems within a certain set. Knowing for a specific system at hand that it lies within such a set of descriptions is therefore, even without being able to specify all system parameters, sufficient in order to establish certain system properties. The degree of conservatism of such an approach necessarily increases with an increasing set Ω in (6) or (7). However, at this point no restrictions of the set Ω at all are taken into account. This viewpoint together with the idea of looking for properties which can be established by means of quadratic Lyapunov functions turns out to be a sound basis for defining meaningful properties for the system descriptions (6), (7). These properties (defined in the sequel) become numerically feasible if the set Ω is restricted in a structured manor (Section IV).

For non-descriptor systems *quadratic stabilization* [4], i.e. stabilization such that stability can be proven by means of a quadratic Lyapunov function, was introduced in order to deal with uncertain *linear time-varying* (LTV) systems. Later on, this idea was extended to also deal with H_∞ constraints (*quadratic H_∞ performance*) for LTV systems [1] and also for a class of nonlinear systems [22].

In the same fashion in this section *quadratic admissibility* and *quadratic H_∞-performance* are introduced for the nonlinear descriptor system (6), (7) respectively.

Definition 4: The descriptor system (6) is said to be *quadratically admissible* if there exists a constant matrix $X \in I\!R^{n_x \times n_x}$ and a real number ϵ, $\epsilon > 0$, such that the matrix inequalities

$$E^{\mathrm{T}} X = X^{\mathrm{T}} E \geq 0, \quad A^{\mathrm{T}}(\boldsymbol{x}, t) X + X^{\mathrm{T}} A(\boldsymbol{x}, t) + \epsilon I < 0 \tag{9}$$

hold true for all $\boldsymbol{x}$ and for all times $t \geq 0$.

Definition 5: For a given real number γ with $\gamma > 0$, a descriptor system (7) with $\|D(\boldsymbol{x}, t)\| < \gamma$ for all $\boldsymbol{x}$ and for all t is said to have a *quadratical H_∞ performance* less than γ if there exists a constant matrix $X \in I\!R^{n_x \times n_x}$ and a real number ϵ, $\epsilon > 0$, such that the matrix inequalities

$$E^{\mathrm{T}} X = X^{\mathrm{T}} E \geq 0$$
$$\begin{bmatrix} A^{\mathrm{T}}(\boldsymbol{x}, t) X + X^{\mathrm{T}} A(\boldsymbol{x}, t) & X^{\mathrm{T}} B(\boldsymbol{x}, t) & C^{\mathrm{T}}(\boldsymbol{x}, t) \\ B^{\mathrm{T}}(\boldsymbol{x}, t) X & -\gamma I & D^{\mathrm{T}}(\boldsymbol{x}, t) \\ C(\boldsymbol{x}, t) & D(\boldsymbol{x}, t) & -\gamma I \end{bmatrix} + \epsilon I < 0 \tag{10}$$

hold true for all $\boldsymbol{x}$ and all times $t \geq 0$.
These definitions require pointwise (i.e. for every $\boldsymbol{x}$, t) the conditions which are necessary and sufficient for linear time invariant (LTI) descriptor systems [21]. Formally the only difference is the term ϵI. Such a perturbation of strict LMIs for LTI systems is always possible [23]. However, in the nonlinear case considered here, such a perturbation argument is no more valid. Therefore the perturbation (needed for asymptotical stability) is explicitly enforced.

Nevertheless, it is necessary to check if these definitions make sense for the systems (6) or (7), since it is well known [6] that such a “frozen-time” approach in general does not automatically transfer the properties of the frozen-time linear systems to the overall nonlinear system.

The properties in question are the index one property, stability, and H_∞ norm-boundedness. The index one property here is understood in the sense that the algebraic equations in (6) respectively (7) can be inverted pointwise (for other meaningful index definitions for nonlinear descriptor systems see [5]). In fact for (9) respectively (10) being valid, the arguments in the proofs for admissibility (stability plus index one) and H_∞ norm-boundedness [21], just can be repeated pointwise since they directly reveal the system to be index one without the detour of regularity.

With the Lyapunov type function $\mathcal{V}(\cdot)$, $\mathcal{V}(\boldsymbol{x}) \ =^{\mathrm{T}} \boldsymbol{x}^{\mathrm{T}} E X \boldsymbol{x}$ it is immediate that the total time derivative of $\mathcal{V}(\cdot)$ is uniformly bounded away from zero by means of the $\mathcal{K}_\infty$ function α, $\alpha(\boldsymbol{x}) := \epsilon^{\mathrm{T}} \boldsymbol{x}\boldsymbol{x}$. Thus, one has global uniform asymptotic stability [14] and H_∞ norm boundedness respectively by standard Lyapunov arguments. In this context note that the definition of quadratic H_∞ norm-boundedness due to the (1,1) entry in the matrix inequality (10) already includes the requirement of quadratic admissibility.

The Definitions 4 and 5 arise from the idea to transfer the linear analysis approach to the rather general nonlinear system (6) and (7) respectively. The drawback of this approach is that actually testing the definitions is impossible since they impose an infinite number of LMI conditions on (6) respectively on (7). Therefore in the following two sections additional system specifications, i.e. a nonlinear uncertainty description, are considered. With these additional structural restrictions it then becomes possible to derive feasible test procedures from the Definitions 4 and 5.

IV. Analysis of Descriptor Systems with Norm Bounded Uncertainties

The aim of this section is to provide a feasible characterization of quadratic H_∞ performance of a class of uncertain nonlinear descriptor systems. Here only the quadratic H_∞ performance problem is considered since the problem to characterize quadratic admissibility is included as a subproblem. The system matrices $A(\boldsymbol{x}, t)$, $B(\boldsymbol{x}, t)$, $C(\boldsymbol{x}, t)$, $D(\boldsymbol{x}, t)$ in the setup from Eq. (7) are assumed to be uncertain, nonlinear, and possibly fast time-varying in an unknown fashion. However, it is assumed that they are Lebesgue measurable and bounded by a compact set. To be more specific

$$\begin{aligned} A(\boldsymbol{x}, t) &= A_N + \Delta A(\boldsymbol{x}, t), \quad B(\boldsymbol{x}, t) = B_N + \Delta B(\boldsymbol{x}, t), \\ C(\boldsymbol{x}, t) &= C_N + \Delta C(\boldsymbol{x}, t), \quad D(\boldsymbol{x}, t) = D_N + \Delta D(\boldsymbol{x}, t) \end{aligned} \tag{11}$$

are considered where the constant matrices A_N, B_N, C_N, D_N denote nominal values of $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, $C(\cdot,\cdot)$, $D(\cdot,\cdot)$ and $\Delta A(\cdot,\cdot)$, $\Delta B(\cdot,\cdot)$, $\Delta C(\cdot,\cdot)$, $\Delta D(\cdot,\cdot)$ with

$$\begin{bmatrix} \Delta A(\boldsymbol{x}, t) & \Delta B(\boldsymbol{x}, t) \\ \Delta C(\boldsymbol{x}, t) & \Delta D(\boldsymbol{x}, t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(\boldsymbol{x}, t) \begin{bmatrix} K_1 & K_2 \end{bmatrix}, \tag{12}$$
$$\|F(\boldsymbol{x}, t)\| \leq 1, \quad F(\boldsymbol{x}, t) \in I\!R^{n_H \times n_K}$$

contain the uncertainties (treated as being possibly nonlinear). Here $H_1 \in I\!R^{n_x \times n_H}$, $H_2 \in I\!R^{n_z \times n_H}$, $K_1 \in I\!R^{n_K \times n_x}$, $K_2 I\!R^{n_K \times n_w}$ are given constant matrices accounting for the compact set where the uncertain system matrices may range. Due to (12) the nonlinear uncertainties are all lumped into perturbation terms. Since the system description is linear in the perturbations, actually a linear parameter-varying descriptor analysis problem is solved in the sequel. As a consequence one has to expect conservative results in those cases where an explicit description (11), (12) of the descriptor system subject to analysis is known. With the additional restriction from the uncertainty description (11) and (12), it is now possible to derive a feasible characterization of quadratic H_∞ performance for the system description (7).

Theorem 3: The system (7) with an uncertainty description (11), (12) and matrices D, K_2, H_2 such that

$$\exists\tilde{\lambda}>0:\quad \begin{bmatrix}-\gamma I & 0 & D_N & \tilde{\lambda}H_2\\ 0 & -\tilde{\lambda}I & K_2 & 0\\ D_N^{\mathrm{T}} & K_2^{\mathrm{T}} & -\gamma I & 0\\ \tilde{\lambda}H_2^{\mathrm{T}} & 0 & 0 & -\tilde{\lambda}I\end{bmatrix}<0 \qquad (13)$$

has a quadratic H_∞ performance less than $\gamma>0$ if and only if there exists a matrix $X\in I\!R^{n_x\times n_x}$ and a real number $\lambda>0$ such that the matrix inequalities

$$E^{\mathrm{T}}X = X^{\mathrm{T}}E\geq 0, \qquad (14)$$

$$\begin{bmatrix}A_N^{\mathrm{T}}X+X^{\mathrm{T}}A_N & X^{\mathrm{T}}B_N & \sqrt{\gamma\lambda}\,X^{\mathrm{T}}H_1 & C_N^{\mathrm{T}} & \sqrt{\frac{\gamma}{\lambda}}K_1^{\mathrm{T}}\\ B_N^{\mathrm{T}}X & -\gamma I & 0 & D_N^{\mathrm{T}} & \sqrt{\frac{\gamma}{\lambda}}K_2^{\mathrm{T}}\\ \sqrt{\gamma\lambda}\,H_1^{\mathrm{T}}X & 0 & -\gamma I & \sqrt{\gamma\lambda}\,H_2^{\mathrm{T}} & 0\\ C_N & D_N & \sqrt{\gamma\lambda}\,H_2 & -\gamma I & 0\\ \sqrt{\frac{\gamma}{\lambda}}K_1 & \sqrt{\frac{\gamma}{\lambda}}K_2 & 0 & 0 & -\gamma I\end{bmatrix}<0 \qquad (15)$$

hold true.

Proof: Define

$$Y_b := \begin{bmatrix}A_N^{\mathrm{T}}X+X^{\mathrm{T}}A_N & X^{\mathrm{T}}B_N & C_N^{\mathrm{T}}\\ B_N^{\mathrm{T}}X & -\gamma I & D_N^{\mathrm{T}}\\ C_N & D_N & -\gamma I\end{bmatrix},\quad \boldsymbol{\xi}:=\begin{bmatrix}\boldsymbol{\xi_1}\\ \boldsymbol{\xi_2}\\ \boldsymbol{\xi_3}\end{bmatrix}$$

with $\boldsymbol{\xi}$ being an arbitrary real vector corresponding to the block partition of Y_b. Evaluating Definition 5 of quadratic H_∞ norm-boundedness for the system (7) together with the uncertainty description (11), (12) leads to $\exists\epsilon>0:$

$$\boldsymbol{\xi}^{\mathrm{T}}Y_b\boldsymbol{\xi}+2\left(\boldsymbol{\xi_1}^{\mathrm{T}}X^{\mathrm{T}}H_1+\boldsymbol{\xi_3}^{\mathrm{T}}H_2\right)F(t)\left(K_1\boldsymbol{\xi_1}+K_2\boldsymbol{\xi_2}\right)$$
$$+\epsilon\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}<0, \forall F(t):\ \sigma_{\max}\left(F(t)\right)\leq 1,\quad \forall\boldsymbol{\xi}\neq\mathbf{0} \qquad (16)$$

(pre- and post-multiplication of (10) with $\boldsymbol{\xi}^{\mathrm{T}}$, $\boldsymbol{\xi}$ respectively, for $\boldsymbol{\xi}\neq\mathbf{0}$).
Define $f:\left\{F|F\in I\!R^{n_H\times n_K},\sigma_{\max}(F)\leq 1\right\}\rightarrow I\!R$ by

$$f:\quad F\mapsto\left(\boldsymbol{\xi_1}^{\mathrm{T}}X^{\mathrm{T}}H_1+\boldsymbol{\xi_3}^{\mathrm{T}}H_2\right)F\left(K_1\boldsymbol{\xi_1}+K_2\boldsymbol{\xi_2}\right).$$

This function attains its maximum for

$$F=\frac{\left(H_1^{\mathrm{T}}X\boldsymbol{\xi_1}+H_2^{\mathrm{T}}\boldsymbol{\xi_3}\right)\left(K_1\boldsymbol{\xi_1}+K_2\boldsymbol{\xi_2}\right)^{\mathrm{T}}}{\|H_1^{\mathrm{T}}X\boldsymbol{\xi_1}+H_2^{\mathrm{T}}\boldsymbol{\xi_3}\|\,\|K_1\boldsymbol{\xi_1}+K_2\boldsymbol{\xi_2}\|}.$$

Then (16) holds true, if and only if it is valid for this maximizing F, i.e. if and only if

$$\boldsymbol{\xi}^{\mathrm{T}}(Y_b+\epsilon I)\boldsymbol{\xi}+2\|H_1^{\mathrm{T}}X\boldsymbol{\xi_1}+H_2^{\mathrm{T}}\boldsymbol{\xi_3}\|\,\|K_1\boldsymbol{\xi_1}+K_2\boldsymbol{\xi_2}\|<0. \qquad (17)$$

With X_a, Z_c defined as

$$X_a:=\begin{bmatrix}X^{\mathrm{T}}H_1H_1^{\mathrm{T}}X & 0 & X^{\mathrm{T}}H_1H_2^{\mathrm{T}}\\ 0 & 0 & 0\\ H_2H_1^{\mathrm{T}}X & 0 & H_2H_2^{\mathrm{T}}\end{bmatrix},$$

$$Z_c:=\begin{bmatrix}K_1^{\mathrm{T}}K_1 & K_1^{\mathrm{T}}K_2 & 0\\ K_2^{\mathrm{T}}K_1 & K_2^{\mathrm{T}}K_2 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Eq. (17) can be rewritten as $\left(\boldsymbol{\xi}^{\mathrm{T}}(Y_b+\epsilon I)\boldsymbol{\xi}\right)^2 - 4\ \boldsymbol{\xi}^{\mathrm{T}}X_a\boldsymbol{\xi}\ \boldsymbol{\xi}^{\mathrm{T}}Z_c\boldsymbol{\xi}>0$with $Y_b<0$, $X_a,Z_c\geq 0$. This is equivalent [20] to

$$\exists\lambda>0:\ \lambda X_a+(Y_b+\epsilon I)+\frac{1}{\lambda}Z_c<0. \qquad (18)$$

Since (18) is a strict matrix inequality, one also has $\lambda X_a+Y_b+\frac{1}{\lambda}Z_c<0$ with the usual perturbation argument, or explicitly

$$\begin{bmatrix}A_N^{\mathrm{T}}X+X^{\mathrm{T}}A_N & X^{\mathrm{T}}B_N & C_N^{\mathrm{T}}\\ B_N^{\mathrm{T}}X & -\gamma I & D_N^{\mathrm{T}}\\ C_N & D_N & -\gamma I\end{bmatrix}+$$
$$\begin{bmatrix}\frac{1}{\sqrt{\lambda}}K_1^{\mathrm{T}} & \sqrt{\lambda}X^{\mathrm{T}}H_1\\ \frac{1}{\sqrt{\lambda}}K_2^{\mathrm{T}} & 0\\ 0 & \sqrt{\lambda}H_2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{\lambda}}K_1 & \frac{1}{\sqrt{\lambda}}K_2 & 0\\ \sqrt{\lambda}H_1^{\mathrm{T}}X & 0 & \sqrt{\lambda}H_2^{\mathrm{T}}\end{bmatrix}<0.$$

Application of Schur's Lemma yields

$$\begin{bmatrix}A_N^{\mathrm{T}}X+X^{\mathrm{T}}A_N & X^{\mathrm{T}}B_N & C_N^{\mathrm{T}} & \frac{1}{\sqrt{\lambda}}K_1^{\mathrm{T}} & \sqrt{\lambda}X^{\mathrm{T}}H_1\\ B_N^{\mathrm{T}}X & -\gamma I & D_N^{\mathrm{T}} & \frac{1}{\sqrt{\lambda}}K_2^{\mathrm{T}} & 0\\ C_N & D_N & -\gamma I & 0 & \sqrt{\lambda}H_2\\ \frac{1}{\sqrt{\lambda}}K_1 & \frac{1}{\sqrt{\lambda}}K_2 & 0 & -I & 0\\ \sqrt{\lambda}H_1^{\mathrm{T}}X & 0 & \sqrt{\lambda}H_2^{\mathrm{T}} & 0 & -I\end{bmatrix}<0.$$

Scaling of the last two rows and columns by $\sqrt{\gamma}$ and rearrangement of the rows and columns finally leads to (15). ∎

Remark 1: The premise (13) in Theorem 3 corresponds to the condition $\|D\|<\gamma$ in the LTI case and can be derived analogously to the preceding proof. If the condition does not hold, it can be enforced by scaling of the external input $\boldsymbol{w}$.

The characterization of H_∞ norm-boundedness for the descriptor system (7) with uncertainty description (11), (12) requires the solution of the bilinear matrix inequalities (14), (15) in the variables λ, X. Such a bilinear matrix inequality in general can be solved with the methods from [12]. However, since $\lambda>0$, a simple line search coupled with the solution of LMIs (for a fixed λ the matrix inequalities (14), (15) become LMIs) is more appropriate. Recently, also new numerical tools especially tailored for such problems have been developed [2].

The parameter λ can be interpreted as the condensed influence of the considered uncertainty description (11), (12). However, since the "size" of uncertainty is normalized in (12), it is not possible to directly link λ with the idea of an uncertainty measure. Therefore it is also not possible to derive some a priori guess on the value of λ in order to further facilitate the BMI problem (14), (15).
A straightforward consequence of Theorem 3 is stated in the following corollary. This result relates the H_∞ norm-boundedness of the nonlinear descriptor system (7) with the uncertainty description (11), (12) to an LTI descriptor system and will be the starting point for the corresponding controller synthesis problem in the next section.

Corollary 1: Consider the descriptor system (7) together with an uncertainty description (11), (12), and let $\lambda'>0$ be a real number such that (13) holds true with $\tilde{\lambda} = \lambda'$. Then this descriptor system is quadratical H_∞ norm-bounded if and only if there exists a $\lambda > 0$ with $\tilde{\lambda} := \lambda$ satisfying (13) and such that the linear time-invariant descriptor system

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A_N\boldsymbol{x}(t) + \begin{bmatrix} B_N & \sqrt{\gamma\lambda}H_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}(t) \\ \hat{\boldsymbol{w}}(t) \end{bmatrix} \\ \hat{\boldsymbol{z}}(t) &= \begin{bmatrix} C_N \\ \sqrt{\frac{\gamma}{\lambda}}K_1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} D_N & \sqrt{\gamma\lambda}H_2 \\ \sqrt{\frac{\gamma}{\lambda}}K_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}(t) \\ \hat{\boldsymbol{w}}(t) \end{bmatrix} \end{aligned} \quad (19)$$

is an admissible system with H_∞ norm less than γ, $\gamma > 0$.

Proof: Congruence transformation of (13) (for $\tilde{\lambda} := \lambda$) with $\operatorname{diag}(I, \sqrt{\frac{\gamma}{\lambda}}, I, \sqrt{\frac{\gamma}{\lambda}})$ leads to

$$\left[\begin{array}{cc|cc} -\gamma I & 0 & D_N & \sqrt{\gamma\lambda}H_2 \\ 0 & -\gamma I & \sqrt{\frac{\gamma}{\lambda}}K_2 & 0 \\ \hline D_N^{\mathrm{T}} & \sqrt{\frac{\gamma}{\lambda}}K_2^{\mathrm{T}} & -\gamma I & 0 \\ \sqrt{\gamma\lambda}H_2^{\mathrm{T}} & 0 & 0 & -\gamma I \end{array}\right] < 0.$$

Therefore the existence of $\lambda > 0$ is necessary and sufficient for $D := \begin{bmatrix} D_N & \sqrt{\gamma\lambda}H_2 \\ \sqrt{\frac{\gamma}{\lambda}}K_2 & 0 \end{bmatrix}$ to be norm-bounded by $\gamma > 0$. Application of the generalized bounded real lemma for LTI descriptor systems [21] to (19) then renders the inequalities (14), (15). The claim finally follows from Theorem 3. ■

Remark 2: Note that the LTI system (19) has different inputs and outputs than the system (7). In fact (19) can be interpreted as the nominal descriptor system (7), (11), (12) with $F(\cdot) \equiv 0$ augmented by a disturbance model which accounts for the uncertainty.

As before the feed-through matrix of the considered descriptor system has to be restricted by $\|D\| < \gamma$ in order to get a necessary and sufficient result. This leads to the somewhat twisted formulation of the premise in Corollary 1.

V. Synthesis of Descriptor Systems with Norm Bounded Uncertainties

The idea of this section is to reformulate the control problem for a descriptor system with norm bounded uncertainties such that the synthesis problem is reduced to a synthesis problem for an LTI descriptor system. Then the results from the preceding section can be used to actually compute a controller.

As in the analysis setting the approach taken here aims at quadratic H_∞ performance for the closed loop system. However, other performance criteria which can be expressed by means of quadratic Lyapunov functions can be treated in the same fashion.

In the differential inclusion framework introduced in Section III the synthesis problem is described by a generalized plant

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A(\boldsymbol{x}(t),t)\boldsymbol{x}(t) + B_1(\boldsymbol{x}(t),t)\boldsymbol{w}(t) + B_2\boldsymbol{u}(t) \\ \boldsymbol{z}(t) &= C_1(\boldsymbol{x}(t),t)\boldsymbol{x}(t) \\ \boldsymbol{y}(t) &= C_2(\boldsymbol{x}(t),t)\boldsymbol{x}(t)\ . \end{aligned} \quad (20)$$

Here $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$ denotes the control inputs and $\boldsymbol{y}(t) \in \mathbb{R}^{n_y}$ the measured outputs. For the other quantities the same notation as in Section IV applies.

Possible input dependencies on $\boldsymbol{u}(t)$ and $\boldsymbol{w}(t)$ in the equations for $\boldsymbol{z}(t)$, $\boldsymbol{y}(t)$ in (20) can be eliminated by augmenting the descriptor vector. This is assumed to be done here, since for this case the elaborated handling of the constraint $\|D_{cl}\| < \gamma$ (with D_{cl} describing the feed-through from $\boldsymbol{w}$ to $\boldsymbol{z}$ in the closed loop), which has to be ensured for H_∞ characterization, is not necessary.

For the descriptor system (20) system parameter variations of the form

$$\begin{aligned} &\begin{bmatrix} A(\boldsymbol{x}(t),t) & B_1(\boldsymbol{x}(t),t) & C_1(\boldsymbol{x}(t),t)^{\mathrm{T}} & C_2(\boldsymbol{x}(t),t)^{\mathrm{T}} \end{bmatrix} \\ &\qquad \in \Omega \subset \mathbb{R}^{(n_x \times (n_x+n_w+n_z+n_y)} \end{aligned} \quad (21)$$

are considered. Here the variations in the system matrices in the generalized plant (20) are interpreted as uncertainties which can be captured by

$$\begin{aligned} A(\boldsymbol{x},t) = A + \Delta A(\boldsymbol{x},t), \quad C_1(\boldsymbol{x},t) = C_1 + \Delta C_1(\boldsymbol{x},t), \\ C_2(\boldsymbol{x},t) = C_2 + \Delta C_2(\boldsymbol{x},t) \end{aligned} \quad (22)$$

with constant matrices A, C_1, C_2 denoting the nominal part of $A(\boldsymbol{x},t)$, $C_1(\boldsymbol{x},t)$, $C_2(\boldsymbol{x},t)$ and

$$\begin{bmatrix} \Delta A(\boldsymbol{x},t) \\ \Delta C_1(\boldsymbol{x},t) \\ \Delta C_2(\boldsymbol{x},t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} F(\boldsymbol{x},t)K_1, \quad (23)$$

with $\|F(\boldsymbol{x},t)\| \leq 1$, $F(\boldsymbol{x},t) \in \mathbb{R}^{n_H \times n_K}$ describing the uncertainty structure such that the nonlinearities are captured by it. Here $H_1 \in \mathbb{R}^{n_x \times n_H}$, $H_2 \in \mathbb{R}^{n_z \times n_H}$, $H_3 \in \mathbb{R}^{n_y \times n_H}$ $K_1 \in \mathbb{R}^{n_K \times n_x}$ are given constant matrices accounting for the compact set where the uncertain system matrices may range. For $B_1(\boldsymbol{x},t)$ no uncertainty is taken into account, i.e. $B_1(\boldsymbol{x},t) = B_1$. This assumption actually is not necessary here but it simplifies the exposition.

Problem 1: Quadratic descriptor H_∞ control problem with norm-bounded uncertainties: Find a controller

$$K: \quad \begin{aligned} E\dot{\boldsymbol{\xi}}(t) &= A_K\boldsymbol{\xi}(t) + B_K\boldsymbol{y}(t) \\ \boldsymbol{u}(t) &= C_K\boldsymbol{\xi}(t) + D_K\boldsymbol{y}(t) \end{aligned} \quad (24)$$

in descriptor form in descriptor form such that the closed loop system (24), (20), (22), (23), i.e.

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{bmatrix} &= A_{cl} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\xi}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \boldsymbol{w}(t) \\ \boldsymbol{z}(t) &= \begin{bmatrix} C_1 + \Delta C_1(\boldsymbol{x},t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\xi}(t) \end{bmatrix} \end{aligned} \quad (25)$$

with

$$A_{cl} = \begin{bmatrix} A + \Delta A(\boldsymbol{x},t) + B_2D_K\left(C_2 + \Delta C_2(\boldsymbol{x},t)\right) & B_2C_K \\ B_KC_2 + B_K\Delta C_2(\boldsymbol{x},t) & A_K \end{bmatrix}$$

has quadratic H_∞ performance γ for some given real number $\gamma > 0$.

A solution in the sense that the problem is related to an already solved control problem is given in the following theorem.

Theorem 4: Consider the descriptor system (20) together with an uncertainty description (22), (23) capturing the system nonlinearities. There exists a controller (24) with $E_K = E$ such that the closed loop descriptor system (25) has quadratic H_∞ performance γ if and only if there exists a real number $\lambda > 0$ such that the same controller applied to the LTI descriptor system

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + \begin{bmatrix} B_1 & \sqrt{\gamma\lambda}H_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}(t) \\ \hat{\boldsymbol{w}}(t) \end{bmatrix} + B_2\boldsymbol{u}(t) \\ \hat{\boldsymbol{z}}(t) &= \begin{bmatrix} C_1 \\ \sqrt{\frac{\gamma}{\lambda}}K_1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & \sqrt{\gamma\lambda}H_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}(t) \\ \hat{\boldsymbol{w}}(t) \end{bmatrix} \\ \boldsymbol{y}(t) &= C_2\boldsymbol{x}(t) + \begin{bmatrix} 0 & \sqrt{\gamma\lambda}H_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}(t) \\ \hat{\boldsymbol{w}}(t) \end{bmatrix} \end{aligned} \tag{26}$$

leads to an admissible closed loop system with H_∞ norm less than γ.

Proof: Consider the closed loop description (25). The uncertain matrices in this representation may be expressed as in (12), i.e.

$$\begin{bmatrix} \Delta A_{cl}(\boldsymbol{x},t) & \Delta B_{cl}(\boldsymbol{x},t) \\ \Delta C_{cl}(\boldsymbol{x},t) & \Delta D_{cl}(\boldsymbol{x},t) \end{bmatrix} := \tag{27}$$

$$\left[\begin{array}{cc|c} \Delta A(\boldsymbol{x},t) + B_2 D_K \Delta C_2(\boldsymbol{x},t) & 0 & 0 \\ B_K \Delta C_2(\boldsymbol{x},t) & 0 & 0 \\ \hline \Delta C_1(\boldsymbol{x},t) & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{c} H_1 + B_2 D_K H_3 \\ B_K H_3 \\ \hline H_2 \end{array}\right] F(\boldsymbol{x},t) \left[\begin{array}{cc|c} K_1 & 0 & 0 \end{array}\right],$$

$$\|F(\boldsymbol{x},t)\| \leq 1, \quad F(\boldsymbol{x},t) \in \mathbb{R}^{n_H \times n_K},$$

where the later equality is due to (23). The claim follows now by application of Corollary 1 to the closed loop (25) with uncertainty description (27) and comparison with the closed loop formed by (24), (26). ■

The preceding theorem shows that the synthesis problem for a descriptor control problem with norm-bounded uncertainties can be reduced to an LTI descriptor H_∞ control problem. However, this problem cannot be solved by standard LMI methods directly since it is parameterized with a scalar parameter $\lambda > 0$. As mentioned already in the analysis section, one method to overcome this problem is given by recent LMI based algorithms especially tailored for parameterized LMIs.

VI. Conclusion

The focus of this paper is a differential inclusion setting for nonlinear discriptor systems. The notion of quadratic admissibility and quadratic H_∞ performance is introduced for this class of systems. In general the setup is far to general to render constructive results. However, for practical relevant further restrictions of the system structure, a constructive procedure for analysis and controller synthesis can be derived. This is examplified for analysis and synthesis of descriptor systems with norm bounded uncertainties.

References

[1] P. Apkarian, P. Gahinet, and G. Becker. Self-scheduled H_∞ control of parameter-varying systems. In *American Control Conference, Baltimore, MD, June 29 - July 1*, pages 856–860, 1994.

[2] P. Apkarian and H. D. Tuan. Parameterized LMIs in control theory. *SIAM J. Contr. Optim.*, 38(4):1241–1264, 2000.

[3] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer Verlag, Berlin, 1984.

[4] B. R. Barmish. Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *J. Opt. Theory and Appl.*, 46(4):399–408, 1985.

[5] S. L. Campbell and E. Griepentrog. Solvability of general differential algebraic equations. *SIAM J. Scientific Computation*, 16:257–270, 1995.

[6] S. L. Campbell, R. Nikoukhah, and F. Delebecque. Nonlinear descriptor systems. In Paul M. Frank, editor, *Advances in Control*, pages 247–281, London, UK, 1999. Springer-Verlag.

[7] L. Dai. *Singular Control Systems*, volume 118 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, 1989.

[8] G. Doetsch. *Guide to the Application of the Laplace and Z-Transformations*. Von Nostrand Reinhold Company, London, 1971.

[9] D. Fürst. *Mathematische Modellbildung, regelungstechnische Analyse und Synthese mechatronischer Systeme in Deskriptorform*. Shaker Verlag, Aachen, 2000.

[10] F. R. Gantmacher. *Matrizenrechnung II*. Hochschulbücher für Mathematik, Band 37. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959.

[11] E. D. Gilles. Network theory for chemical processes. *Chem. Eng. Technol.*, 21(2):121–132, 1998.

[12] K.-C. Goh. *Robust control synthesis via bilinear matrix inequalities*. PhD thesis, University of Southern California, Los Angeles, 1995.

[13] M. Günther and U. Feldmann. The DAE-index in electric circuit simulation. *Math. Comput. Simulation*, 39:573–582, 1995.

[14] W. Hahn. *Stability of Motion*. Springer-Verlag, Berlin, 1967.

[15] A. Kumar and P. Daoutidis. *Control of Nonlinear Differential Algebraic Equation Systems*. Reserach notes in mathematics. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[16] D. G. Luenberger. Dynamic equations in descriptor form. *IEEE Trans. Automat. Contr.*, AC-22(3):312–321, 1977.

[17] D. J. Luenberger. Nonlinear descriptor systems. *J. Economics Dynamics and Control*, 1:219–242, 1979.

[18] V. L. Mehrmann. Existence, uniqueness, and stability of solutions to linear quadratic control problems. *Linear Algebra and its Applications*, 121:291–331, 1989.

[19] P. C. Müller. Linear quadratic optimal control of mechanical descriptor systems. In U. Helmke, R. Mennicken, and J. Sauer, editors, *Systems and Networks: Mathematical Theory and Applications. Proc. of Int. Symposium MTNS-93*, volume 2, pages 361–366. Akademie Verlag, 1994.

[20] I. R. Petersen and C. V. Hollot. A Riccati equation approach to the stabilization of uncertain linear systems. *Automatica*, 22(4):397–411, 1986.

[21] A. Rehm. *Control of Linear Descriptor Systems: A Matrix Inequality Approach*. Fortschr.-Ber. VDI Reihe 8 Nr. 1019, VDI Verlag, Düsseldorf, 2004.

[22] A. Rehm and F. Allgöwer. Self-scheduled nonlinear output feedback H_∞-control of a class of nonlinear systems. In *American Control Conference, Albuquerque, NM, June 6-10*, pages 386–390, 1997.

[23] C. W. Scherer. The Riccati inequality and state-space H_∞-optimal control. Ph.D. Thesis, Univ. Würzburg, 1990.

[24] R. Schüpphaus. *Regelungstechnische Analyse und Synthese von Mehrkörpersystemen in Deskriptorform*. Fortschr.-Ber. VDI Reihe 8 Nr. 478, VDI Verlag, Düsseldorf, 1995.