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On The Optimal Two-Block H^{∞} Problem

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***Abstract*— This paper provides the duality structure of the optimal two-block H^{∞} problem. The dual description leads naturally to a numerical solution based on convex programming for LTI (including infinite dimensional) systems. Alignment conditions are obtained and show that the optimal solution is flat in general, and unique in the SISO case. It is also proved that under specific conditions a well-known Hankel-Toepltiz operator achieves its norm on the discrete spectrum, therefore generalizing a similar result obtained formerly for finite-dimensional (rational) systems. The norm of this Hankel-Toeplitz operator corresponds to the optimal two-block H^{∞} performance.**

NOTATION

$\mathbb{R}$, $\mathbb{C}$ stand for the field of real and complex numbers respectively . $< \cdot\ ,\ \cdot >$ denotes either the inner or duality product depending on the context. I denotes the identity map. If B is a Banach space then $B^{\star}$ denotes its dual space. For an n-vector $\zeta \in \mathbb{C}_n$, where $\mathbb{C}_n$ denotes the n-dimensional complex space, $|\zeta|$ is the Euclidean norm. $\mathbb{C}_{n\times n}$ is the space of $n \times n$ matrices A, where $|A|$ is the largest singular value of A. $\mathbb{C}_{2n}$ denotes the complex Banach space of $2n$-vectors ζ, $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$; ζ_1, $\zeta_2 \in \mathbb{C}_n$ with the norm

$$|\zeta| = \sqrt{|\zeta_1|^2 + |\zeta_2|^2} \tag{1}$$

Let $\mathbb{C}_{2n\times n}$ denote the complex Banach space of $2n \times n$ matrices A,

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},\ \ A_1,\ A_2 \in \mathbb{C}_{n\times n}$$

with the following norm

$$\|A\| \ \ := \ \ \sqrt{|A_1|^2 + |A_2|^2} \tag{2}$$

Let $STr(A_1) := Tr(A_1^{\star}A_1)^{\frac{1}{2}} = \sum_{j=1}^{n} \sigma_j(A_1)$, where $\sigma_j(A_1)$ is the i-th singular value of A_1, and $Tr(A)$ denotes the trace of A. $STr(A_1)$ is known as the trace-class norm of A_1. The dual space of $\mathbb{C}_{2n\times n}$, denoted $\mathbb{C}^{\star}_{2n\times n}$, is the space of matrices under the norm

$$\|A\|_{\star} := \sqrt{STr(A_1)^2 + STr(A_2)^2} \tag{3}$$

The symbol $\mathfrak{D}$ denotes the unit disk of the complex plane, $\mathfrak{D} = \{z \in \mathbb{C} :\ \ |z| < 1\}$. $\partial\mathfrak{D}$ denotes the boundary of $\mathfrak{D}$, $\partial\mathfrak{D} = \{z \in \mathbb{C} :\ \ |z| = 1\}$. If E is a subset of $\partial\mathfrak{D}$, then E^c denotes the complement of E in $\partial\mathfrak{D}$. m denotes the normalized Lebesgue measure on the unit circle $\partial\mathfrak{D}$, $m(\partial\mathfrak{D}) = 1$. m a.e. is the label used for "Lebesgue almost everywhere". For a matrix or vector-valued function F on the unit circle, $|F|$ is the real-valued function defined on the unit circle by $|F|(e^{i\theta}) = |F(e^{i\theta})|$, $\theta \in [0, 2\pi)$. If X denotes a finite dimensional complex Banach space, $L^p(X)$, $1 \leq p \leq \infty$, stands for the Lebesgue-Bochner space of p-th power absolutely integrable X-valued functions on $\partial\mathfrak{D}$ under the norm

$$\|f\|^p_{L^p(X)} := \int_{[0,2\pi)} \|f(e^{i\theta})\|^p_X dm, \ \text{ for } \ 1 \leq p < \infty \tag{4}$$

$$\|f\|_{L^{\infty}(X)} := \text{ess} \sup_{\theta\in[0,2\pi)} \|f(e^{i\theta})\|_X, \ \text{ for } \ p = \infty \tag{5}$$

where $f \in L^p(X)$, and $\|\cdot\|_X$ denotes the norm on X. $\operatorname{ess\,sup}$ denotes the essential supremum. If $f \in L^p(X)$, $1 \leq p \leq \infty$, the k-th Fourier coefficient is defined by $\hat{f}_k := \int_{\partial\mathfrak{D}} f(z)z^{-k}dm$, which define the well known Fourier series representation of f. $H^p(X)$, $1 \leq p \leq \infty$, is the Hardy space of X-valued analytic functions on the unit disk $\mathfrak{D}$, viewed as a closed subspace of $L^p(X)$. In fact these spaces can be realized as

$$H^p(X) = \{f \in L^p(X) : \hat{f}_k = 0 \ \text{ if } \ k < 0\} \tag{6}$$

The space $H^1_o(X)$ is defined as

$$\{f \in H^1(X), \text{ such that } \int_0^{2\pi} f(e^{i\theta})dm = 0\}$$

Finally, $\mathcal{C}(X)$ denotes the space of continuous X-valued functions defined on $\partial\mathfrak{D}$. $\Re(A)$ denote the real part of A. $l.i.m$ denotes limit in the quadratic mean.

I. INTRODUCTION

The standard two-block H^{∞} problem received a large attention in the control community for two decades (see for example, [1], [2], [27], [3] and references therein.) For finite dimensional LTI systems state space techniques such as [4] proved to be quiet successful. However, for infinite dimensional LTI systems such techniques require solving operator Riccati equations, which remains a problem from the computational point of view [12], [13]. Important applications of infinite dimensional systems include parallel computation with communication time delays [5], [6].

In the frequency domain Jonckheere and Verma showed that the problem of optimizing the H^{∞} mixed-sensitivity is equivalent to characterizing the spectrum of a "Hankel-Toeplitz" operator [7]. In particular, they proved that for rational plants under certain conditions, the H^{∞} performance is reached at an isolated eigenvalue with finite multiplicity. More recently Georgiou and Smith using normalized coprime factorizations showed that the problem of optimizing the radius of stability in the gap metric is equivalent to a special version of the standard two-block H^{∞} problem [8].

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In particular they were able to provide explicit formulas for the optimal radius of stability and the optimal controller in terms of a Hankel operator and its maximal vectors and corresponding eigenvalues when the problem data are continuous. In this paper we generalize some of their results to the standard two-block H^∞ problem by applying the duality theory developed in [17], [18], [19], but under *weaker assumptions* and with providing simple observations to characterize dual and predual spaces instead of the lengthy arguments used there to compute similar spaces. In particular, we show that for MIMO (including infinite-dimensional) systems the optimum is flat and the Hankel-Toeplitz operator discussed earlier achieves its norm on the discrete spectrum therefore answering a question posed in [7]. Zames and Mitter presented a method of computing spectrums, eigenvalues and eigenvectors for general systems subject to continuous weightings [9]. Another attempt to study the norm of the Hankel-Toeplitz is proposed in [11] and [14] for a special class of infinite dimensional systems. It should be noted that the standard two-block H^∞ problem provides a "good" approximate solution for the optimal robust disturbance attenuation problem (ORDAP) in the case of "almost" complementary weightings W and V, i.e., $\|W^\star V\|_\infty = \epsilon << 1$ [15], [16].

For simplicity we consider linear time-invariant stable plants. The unstable case can be settled similarly using coprime factorizations.

Consider the feedback control problem of Figure 1, where a known stable plant P is subject to unknown disturbances $d1$ and $d2$, respectively, at the output and the controller C input (for e.g., sensor noise). The disturbances act through stable filters W and V. The objective is to design a stabilizing controller C, which optimally suppresses the effect of these disturbances on the plant output. This problem is known as the optimal mixed-sensitivity or the optimal two-block H^∞ problem [2], [3], [7], [27]. In this paper, without loss

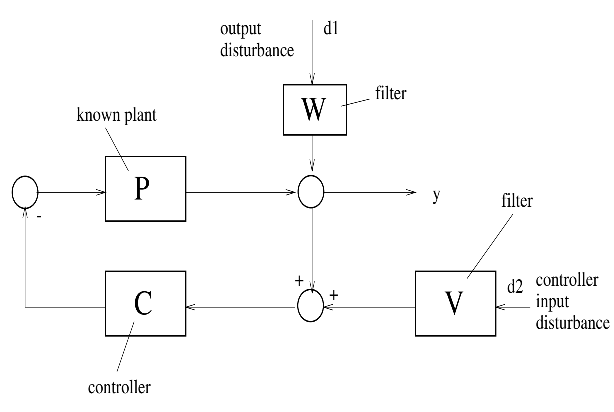

Fig. 1. Feedback Control in Presence of two Sources of Disturbances

of generality the performance index under consideration after introducing the Youla parameter Q, (i.e. $Q = C(I + PC)^{-1}$), is [2], [16], [17], [27]

$$\beta = \inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \|(|W(I-PQ)|^2 + |VPQ|^2)^{\frac{1}{2}}\|_\infty \quad (7)$$

where $P \in H^\infty(\mathbb{C}_{n\times n})$, W and V are outer weighting functions in $H^\infty(\mathbb{C}_{n\times n})$. Problem (7) corresponds to the mixed sensitivity problem, but the theory developed here holds almost *verbatim* for any two-block H^∞ problem including the robust stabilization in the gap metric problem described in [8]. Following [15], [16], [17], [18], [19], we assume throughout (**A1**) $(W^\star W + V^\star V)(e^{i\theta}) > \epsilon I$, $\forall \theta \in [0, 2\pi)$, for some $\epsilon > 0$. Let $WP = U_i\tilde{W}$ and $VP = V_i\tilde{V}$ be inner-outer factorizations of WP and VP respectively, and Λ the outer spectral factor of $(U_i\tilde{W})^\star U_i\tilde{W} + (V_i\tilde{V})^\star V_i\tilde{V}$. Then by letting $R_1 := U_i\tilde{W}\Lambda^{-1}$ and $R_2 := V_i\tilde{V}\Lambda^{-1}$, $R := (R_1^T,\ R_2^T)^T$ is inner, i.e., $R^\star R = I$ a.e. This problem is thus equivalent to

$$\beta = \inf_{Q \in H^\infty(\mathbb{C}_{n\times n})} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - RQ \right\|_\infty \quad (8)$$

where $R \in H^\infty(\mathbb{C}_{2n\times n})$ is inner, and W outer in $H^\infty(\mathbb{C}_{n\times n})$. Recall that $H^\infty(\mathbb{C}_{2n\times n})$ is the Banach space consisting of pairs of bounded analytic $2n\times n$ matrix-valued functions on the unit disc $\mathfrak{D}$, under the norm

$$\begin{aligned} \|K\|_\infty &= ess \sup_{\theta\in[0,2\pi)} \left(|K_1(e^{i\theta})|^2 + |K_2(e^{i\theta})|^2\right)^{\frac{1}{2}} \\ K &= (K_1^T,\ K_2^T)^T \end{aligned} \quad (9)$$

Expression (8) is the shortest distance from $\begin{pmatrix} W \\ 0 \end{pmatrix}$ to the subspace $\mathbb{S} = RH^\infty(\mathbb{C}_{n\times n})$ in the $H^\infty(\mathbb{C}_{2n\times n})$-norm.

II. Duality Structure of the Problem

Define $L^\infty(\mathbb{C}_{2n\times n})$ to be the Banach space of bounded matrix-valued functions on the unit circle $\partial\mathfrak{D}$ under the norm (9). $L^1(\mathbb{C}^\star_{2n\times n})$ is the Banach space of $\mathbb{C}_{2n\times n}$-valued integrable functions defined on $\partial\mathfrak{D}$ with the norm

$$\begin{aligned} \|G\|_{L^1(\mathbb{C}^\star_{2n\times n})} &= \int_0^{2\pi} \left((STrG_1(e^{i\theta}))^2 + \right. \\ &\left. (STrG_2(e^{i\theta}))^2\right)^{\frac{1}{2}} dm \end{aligned}$$

Recall that STr is the trace-class norm for $n\times n$-matrices, its dual norm is the largest singular value $|\cdot|$. It turns out that if A_1 and A_2 are matrices in $\mathbb{C}_{n\times n}$ then the dual norm of $\left(|A_1|^2 + |A_2|^2\right)^{\frac{1}{2}}$ is $\left(STr(A_1)^2 + STr(A_2)^2\right)^{\frac{1}{2}}$ [22].

Note that from the theory of vector-valued L^p-spaces the dual of $L^1(\mathbb{C}^\star_{2n\times n})$ is isometrically isomorphic to $L^\infty(\mathbb{C}_{2n\times n})$ [21], since $\mathbb{C}_{2n\times n}$ is the dual space of $\mathbb{C}^\star_{2n\times n}$ [22], and vise-versa since these spaces are finite dimensional. It is important to note that these simple observations avoid the lengthy proofs provided in [16] to characterize similar dual and predual spaces.

To every functional ϕ on $L^1(\mathbb{C}^\star_{2n\times n})$ there corresponds a vector function K_ϕ related to ϕ through the following bilinear form

$$\begin{aligned}\phi(G) &= < K_\phi, G > \\ &= \int_0^{2\pi} Tr\{K_1^\star G_1 + K_2^\star G_2\}(e^{i\theta})dm \quad (10)\end{aligned}$$

$$\text{and } \|\phi\| = \|K_\phi\|_\infty\ , \quad K_\phi = (K_1^T,\ K_2^T)^T \quad (11)$$

The same argument used in [17], [18] yields the preannihilator of $\mathbb{S}$ as

$$^\perp\mathbb{S} = (I - RR^\star)L^1(\mathbb{C}^\star_{2n\times n}) \oplus R\overline{H}^1_o(\mathbb{C}_{n\times n})/\mathbb{X} \quad (12)$$

where

$$\begin{aligned}\mathbb{X} &= \big((I - RR^\star)L^1(\mathbb{C}^\star_{2n\times n}) \oplus R\overline{H}^1_o(\mathbb{C}_{n\times n})\big) \\ &\bigcap \ \overline{H}^1_o(\mathbb{C}^\star_{2n\times n}) \quad (13)\end{aligned}$$

Hence the following existence Theorem which is a Corollary to Theorem 2 p. 121 in [24].

Theorem 1: Under assumption (A1) there exists at least one optimal $Q_o \in H^\infty(\mathbb{C}_{n\times n})$ such that

$$\begin{aligned}&\inf_{Q\in H^\infty(\mathbb{C}_{n\times n})} \|(|W - R_1Q|^2 + |R_2Q|^2)^{\frac{1}{2}}\|_\infty = \\ &\qquad \|(|W - R_1Q_o|^2 + |R_2Q_o|^2)^{\frac{1}{2}}\|_\infty \\ &= \sup_{\substack{\|\,[f]\,\|_{^\perp\mathbb{S}} \leq 1 \\ [f]\in ^\perp\mathbb{S}}} \left|\int_0^{2\pi} Tr\{(W^\star, 0)f\}(e^{i\theta})dm\right| \quad (14)\end{aligned}$$

In the following section qualitative properties of the optimum are provided.

III. Allpass Property and Alignment in the Dual

Let $\mathcal{C}(\mathbb{C}_{2n\times n})$ denote the space of $\mathbb{C}_{2n\times n}$-valued functions which are continuous on $\partial\mathfrak{D}$ under the sup-norm (9). The dual space of $\mathcal{C}(\mathbb{C}_{2n\times n})$ is isometrically isomorphic to the space $M(\mathbb{C}^\star_{2n\times n})$ of bounded $\mathbb{C}^\star_{2n\times n}$- valued measures under the norm for $\nu = (\nu_1^T,\ \nu_2^T)^T$ as follows

$$\begin{aligned}\|\nu\|_{M(\mathbb{C}^\star_{2n\times n})} &= \int_{[0,2\pi)} \big((STrG_{\nu,1}(e^{i\theta}))^2 + \\ &\qquad (STrG_{\nu,2}(e^{i\theta}))^2\big)^{\frac{1}{2}} d_\theta\omega_\nu(\theta)\end{aligned}$$

where ω_ν is the total variation on $[0, 2\pi)$ of all entries of ν, and $G_{\nu,r} \in L^1(\mathbb{C}_{n\times n}, \omega_\nu)$, $r = 1, 2$. If $\phi \in \mathcal{C}(\mathbb{C}_{2n\times n})^\star$, then the isometric isomorphic is given by the bilinear mapping

$$\begin{aligned}\overline{\phi}(K) &= <\overline{\nu, K}>= \int_{[0,2\pi)} \{Tr(K_1^\star G_{\nu_1}) + \\ &\qquad Tr(K_2^\star G_{\nu_2})\} d_\theta\omega_\nu(\theta) \\ \|\phi\| &= \|\nu\|_{M(\mathbb{C}^\star_{2n\times n})} \quad (15)\end{aligned}$$

Define the subspace $\mathbb{S}_c = \mathbb{S}\cap\mathcal{C}(\mathbb{C}_{2n\times n})$, then the annihilator of $\mathbb{S}_c$ is given by

$$\begin{aligned}\mathbb{S}_c^\perp = \Big\{&\nu \in M(\mathbb{C}^\star_{2n\times n}) :\ d\nu(\theta) = (I - RR^\star)d\nu'(\theta_1) + \\ &RGd\theta_1,\ \ \nu' \in M(\mathbb{C}^\star_{2n\times n}), G \in \overline{H}^1_o(\mathbb{C}_{n\times n})\Big\} \ / \ \mathbb{Y} \quad (16)\end{aligned}$$

where

$$\begin{aligned}\mathbb{Y} = \Big\{&\nu \in M(\mathbb{C}^\star_{2n\times n}) :\ d\nu(\theta) = (I - RR^\star)d\nu'(\theta_1) \\ &+RGd\theta_1,\ \ \nu' \in M(\mathbb{C}^\star_{2n\times n}), G \in \overline{H}^1_o(\mathbb{C}_{n\times n})\Big\} \\ &\bigcap\ \overline{H}^1_o(\mathbb{C}^\star_{2n\times n}) \quad (17)\end{aligned}$$

Under assumption
(**A2**) W is continuous on the unit circle,
the following Lemma establishes that the distance from $(W^T, 0^T)^T \in \mathcal{C}(\mathbb{C}_{2n\times n})$ to $\mathbb{S}_c$ is the same as to $\mathbb{S}$. Note that assumption (A2) is weaker than a corresponding assumption in [15], [17], [18], [19].

Lemma 1: Under assumptions (A1) and (A2) the following hold

$$\begin{aligned}&\inf_{X\in\mathbb{S}_c} \left\|\begin{pmatrix} W \\ 0\end{pmatrix} - X\right\|_\infty = \left\|\begin{pmatrix} W \\ 0\end{pmatrix} - RQ_o\right\|_\infty \\ &= \max_{\substack{\|\,[\nu]\,\|_{\mathbb{S}_c^\perp} \leq 1 \\ [\nu]\in\mathbb{S}_c^\perp}} \left|\int_{[0,2\pi)} Tr(W^\star, 0)d\nu(\theta)\right| \quad (18)\end{aligned}$$

Proof. Let $0 < r < 1$, be the scaling of the unit disk, and $F_r(z) = F(rz)$, and

$$Y := \begin{pmatrix} W \\ 0\end{pmatrix}, \quad X := RQ_o$$

and note that $X_r \in S_c$, and Y is continuous on the unit circle by assumption (A2), then

$$\|Y - X_r\|_\infty \leq \|Y - Y_r\|_\infty + \|Y_r - X_r\|_\infty$$

Note that $\|Y_r - X_r\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})}$ is bounded above by $\|Y - X\|_\infty$, since $\big(|(Y_1 - X_1)(e^{i\theta})|^2 + |(Y_2 - X_2)(e^{i\theta})|^2\big)^{\frac{1}{2}}$ is subharmonic and satisfies the maximum principle. By continuity $\|Y - Y_r\|_{H^\infty(\tilde{\mathbb{C}}_{2n\times n})} \to 0$ as $r \to 1$. Hence

$$\begin{aligned}\min_{X\in S_c} \|Y - X\|_\infty &\leq \lim_{r\to1} \|Y - X_r\|_\infty \\ &\leq \|Y - X\|_\infty\end{aligned}$$

The reverse inequality is clear since $S_c \subset S$. The third equality follows from Theorem 1 (page 121, [24]), and the Lemma is proved.
When the open unit disc analyticity is removed

$$\beta_{oo} = \inf_{Q\in\mathcal{C}(\mathbb{C}_{n\times n})} \|(|W - R_1Q|^2 + |R_2Q|^2)^{\frac{1}{2}}\|_\infty \quad (19)$$

Then a similar result to Theorem 2 [16] follows.

Theorem 2: Under assumptions (A1) and (A2), if $\beta > \beta_{oo}$ then

i. Any optimal $Q_o \in H^\infty(\mathbb{C}_{n\times n})$ satisfies the allpass condition

$$\big(|(W - R_1Q)(e^{i\theta})|^2 + |R_2Q_o(e^{i\theta})|^2\big)^{\frac{1}{2}} = \beta \quad (20)$$

ii. If $\{Q_n\}_{n=1}^\infty$ is any sequence in $H^\infty(\mathbb{C}_{n\times n})$ such that

$$\lim_{n\to\infty} \|(|W - R_1Q_n|^2 + |R_2Q_n|^2)^{\frac{1}{2}}\|_\infty = \beta \quad (21)$$

then

$$l.i.m_{n\to\infty}(|W-R_1Q_n(e^{i\theta})|^2+|R_2Q_n(e^{i\theta})|^2)^{\frac{1}{2}}=\beta$$

The condition $\beta > \beta_{oo}$ is sharp for both conclusions in the sense that if $\beta = \beta_{oo}$, then there exist W, V and P for which (20) and (22) are false.

Proof. Follows by the same argument used in the proof of Theorem 2 [16]. The counter example given after Theorem 2 in [16] can be used to show that condition $\beta > \beta_{oo}$ is sharp.

Remark: The same argument used in [19], [20] shows that the extremal measure ν_o in (18) is absolutely continuous with respect to the Lebesgue measure. More precisely, there exists a vector function $F_m \in L^1(\mathbb{C}^\star_{2n\times n})$ such that $d\omega_o = F_m dm$, and hence $d\nu_o = F_m G dm$. Therefore the supremum in (14) is achieved by the coset $[f_o] = [F_m G]$. There also exists a vector function $F_o \in [f_o]$, such that $\|[f_o]\|_{\perp\mathbb{S}} = \|F_o\|_{L^1(\mathbb{C}^\star_{2n\times n})} = 1$. It should be noted that the duality theory developed here for the standard two-block H^∞ problem fits into the convex programming algorithm along the lines of [16], and therefore provides another numerical solution different from the usual well known ϵ iterations (see for, e.g., [7]).

Duality theory leads naturally to a dual pair of numerical solutions, which converge to the optimal β from opposite directions, and has the merit of producing estimates of β within known tolerances *without any restriction on system dimensionality*. That is, in principle, the numerical solution applies also to infinite dimensional systems.

Theorem 3: Under assumptions (A1) and (A2) and $\beta > \beta_{oo}$, so that flatness holds, then $F_o = \begin{pmatrix} F_{o1} \\ F_{o2} \end{pmatrix} \in [F] \in^\perp \mathbb{S}$, $\|[F]\|_{\perp\mathbb{S}} = 1$, is an extremal kernel for $[F]$, and Q_o is optimal if and only if

$$\begin{aligned}&Tr\{((W^\star,0)+Q_o^\star R^\star)\begin{pmatrix} F_{o1} \\ F_{o2} \end{pmatrix}\}(e^{i\theta}) = \\ &\big(|(W-R_1Q_o)(e^{i\theta})|^2+|R_2Q_o(e^{i\theta})|^2\big)^{\frac{1}{2}}\times \\ &\quad\Big((STrF_{o1}(e^{i\theta}))^2+(STrF_{o2}(e^{i\theta}))^2\Big)^{\frac{1}{2}} \qquad (22)\end{aligned}$$

The optimal $\begin{pmatrix} W-R_1Q_o \\ R_2Q_o \end{pmatrix}$ is then a dual extremal function for $\min_{f\in\mathbb{X}} \|F+f\|_{L^1(\mathbb{C}^\star_{2n\times n})} = 1$.

Proof. "Only if" by assumption $\exists F_o \in^\perp \mathbb{S}$, $\|F_o\|_{L^1(\mathbb{C}^\star_{2n\times n})} = 1$, and $Q_o \in H^\infty(\mathbb{C}_{n\times n})$ such that the following alignment condition holds

$$\begin{aligned}&\|(|W-R_1Q_o|^2+|R_2Q_o|^2)^{\frac{1}{2}}\|_\infty \|F_o\|_{L^1(\mathbb{C}^\star_{2n\times n})} = \\ &\qquad \int_0^{2\pi} Tr\{\big((W^\star,0)+Q_o^\star R^\star\big)F_o\}(e^{i\theta})dm \quad (23)\end{aligned}$$

but the integrand

$$\begin{aligned}&Tr\left\{\big((W^\star,0)+Q_o^\star R^\star\big)F_o\right\}(e^{i\theta}) \le \qquad (24)\\ &(|W-R_1Q_o|^2+|R_2Q_o|^2)^{\frac{1}{2}} \times \\ &\big((STrF_{o1})^2+(STrF_{o2})^2\big)^{\frac{1}{2}} \\ &\text{a.e. by Cauchy} - \text{Schwarz inequality} \qquad (25)\\ &\le \mu_o\big((STr(F_{o1})^2+(STr(F_{o2})^2\big)^{\frac{1}{2}}(e^{i\theta}),\ m \text{ a.e.}\end{aligned}$$

integrating implies equality must hold throughout. This combined with flatness imply $\big((STr(F_{o1})^2 + (STr(F_{o2})^2\big)^{\frac{1}{2}}(e^{i\theta}) = 1,\ \ m$a.e..

"If" suppose that (22) holds, integrating it yields

$$\begin{aligned}&\mu_o\|F_o\|_{L^1(\mathbb{C}^\star_{2n\times n})} = \\ &\int_0^{2\pi} Tr\{\big((W^\star,0)+Q_o^\star R^\star\big)F_o\}(e^{i\theta})dm = \\ &\int_0^{2\pi} Tr\{\big((W^\star,0)+Q_o^\star R^\star\big)F\}(e^{i\theta})dm\ ,\ \forall F\in[F_o] \\ &= \int_0^{2\pi} Tr\{\big((W^\star,0)+Q^\star R^\star\big)F\}(e^{i\theta})dm,\ \forall Q \\ &\le \|(|W-R_1Q|^2+|R_2Q|^2)^{\frac{1}{2}}\|_\infty \|[F]\|_{\perp\mathbb{S}},\ \forall Q \\ \le\ &\inf_{Q\in H^\infty(\mathbb{C}_{n\times n})} \|(|W-R_1Q|^2+|R_2Q|^2)^{\frac{1}{2}}\|_\infty \|\,[F]\,\|_{\perp\mathbb{S}} \\ \le\ &\mu_o\|F_o\|_{L^1(\mathbb{C}^\star_{2n\times n})}\end{aligned}$$

hence equality must hold throughout and Q_o is optimal.

Remark: In the SISO case, (22) reduces to

$$\begin{aligned}&W^\star F_{o1}(e^{i\theta}) + (R_1^\star F_{o1} + R_2^\star F_{o2})Q_o^\star = \qquad (26)\\ &(|(W-R_1Q_o)(e^{i\theta})|^2+|R_2Q_o(e^{i\theta})|^2)^{\frac{1}{2}} \\ &(|F_{o1}(e^{i\theta})|^2+|F_{o2}(e^{i\theta})|^2)^{\frac{1}{2}} = \mu_o\end{aligned}$$

which implies almost everywhere

$$arg(W-R_1Q_o) = arg(F_{o1})\ ,\ arg(R_2Q_o) = arg(F_{o2})$$

since $(|F_{o1}(e^{i\theta})|^2 + |F_{o2}(e^{i\theta})|^2)(e^{i\theta}) = 1,\ $ m a.e. the set $E = \{\theta : F_{o1}(e^{i\theta}) = F_{o2}(e^{i\theta}) = 0\}$ has Lebesgue measure 0. However integrating (25) and (26), and since equality must hold (in the Cauchy-Schwarz inequality), we get

$$\begin{aligned}|(W-R_1Q_o)(e^{i\theta})| &= \gamma|F_{o1}(e^{i\theta})|,\ \ m \text{ a.e.} \\ |(R_2Q_o)(e^{i\theta})| &= \gamma|F_{o2}(e^{i\theta})|,\ \ m \text{ a.e.}\end{aligned}$$

for some positive constant γ. This shows that F_{o1} and F_{o1} cannot vanish on a set of positive measure unless $Q_o \equiv 0$. But this would give a non-flat solution for $|W|$ non-constant. Expression (26) determines Q_o uniquely.

IV. On the Norm of a Hankel-Toepltiz Operator

Let again Π be the orthogonal projection onto the closed subspace $H^2(\mathbb{C}_{2n}) \ominus RH^2(\mathbb{C}_n)$ of $H^2(\mathbb{C}_{2n})$, where $H^2(\mathbb{C}_{2n})$ is understood to be the Hardy space of $\mathbb{C}_{2n}$-valued functions defined on $\mathfrak{D}$, under the Hilbert space norm

$$\begin{aligned}\|F\|^2_{H^2(\mathbb{C}_{2n})} &= \int_0^{2\pi}\sum_{j=1}^{2n}|f_j(e^{i\theta})|^2dm, \\ F &= (f_1,\ f_2,\ \cdots,\ f_{2n})^T \qquad (27)\end{aligned}$$

Define the operator Ξ_2 by

$$\begin{aligned} \Xi_2 : H^2(\mathbb{C}_n) &\longrightarrow H^2(\mathbb{C}_{2n}) \ominus RH^2(\mathbb{C}_n) \\ \Xi_2 f = \Pi \begin{pmatrix} W \\ 0 \end{pmatrix} f, &\qquad f \in H^2(\mathbb{C}_n) \end{aligned} \tag{28}$$

We obtain a Theorem similar to Theorem 3 [19], [20].

Theorem 4: Under assumptions (A1) and (A2), if $\beta > \beta_{oo}$ (i.e., flatness holds), then

i.
$$\beta = \|\Xi_2\| \tag{29}$$

ii. There exists a maximal vector $f \in H^2(\mathbb{C}_n)$ of $L^2(\mathbb{C}_n)$-norm 1 such that
$$\|\Xi_2 f\|_{L^2(\mathbb{C}_{2n})} = \|\Xi_2\| \tag{30}$$

Proof.

i. Follows either from the commutant lifting Theorem [23], or from slight changes to the proof of Theorem 3 [19], [20]. Note that flatness is not necessary for (29) to hold.

ii. Follows from a similar argument used in the proof of ii. in Theorem 3 [19], [20].

Theorem 1 implies existence of a vector function $\Psi \in H^\infty(\mathbb{C}_{2n\times n})$ such that

$$\|\Psi\|_\infty = \|\Xi_2\| = \beta \tag{31}$$

where $\Psi = \begin{pmatrix} W \\ 0 \end{pmatrix} - RQ_o$, for some $Q_o \in H^\infty(\mathbb{C}_{n\times n})$. Then

$$\begin{aligned} \|\Xi_2\|\|f\|_{L^2(\mathbb{C}_n)} &= \|\Xi_2 f\|_{L^2(\mathbb{C}_{2n})} = \|\Pi\Psi f\|_{L^2(\mathbb{C}_{2n})} \\ \leq \|\Psi f\|_{L^2(\mathbb{C}_{2n})} &\leq \|\Psi\|_\infty \|f\|_{L^2(\mathbb{C}_n)} = \|\Xi_2\|\|f\|_{L^2(\mathbb{C}_n)} \end{aligned}$$

since $\|\Pi\| \leq 1$. It follows that

$$\Psi f = \Xi_2 f, \quad m \text{ a.e.} \tag{32}$$

For SISO systems since the optimal Q_o is unique (32) implies $Q_o = R_1^\star W - R^\star \frac{\Xi f}{f}$ and the optimal controller C_o can be computed from $C_o = \frac{Q_o V^{-1}}{1 - Q_o V^{-1} P}$.

Under assumption (A1) there exists a square inner matrix $(R,\ R_\perp)$, where $R_\perp = (R_{1\perp}^T,\ R_{2\perp}^T)^T$, such that [25]

$$\begin{aligned} \|\Psi\|_\infty &= \left\| (R,\ R_\perp)^\star \left(\begin{pmatrix} W \\ 0 \end{pmatrix} - RQ_o \right) \right\|_\infty && (33) \\ &= \left\| \begin{pmatrix} R_1^\star W - Q_o \\ R_{1\perp}^\star W \end{pmatrix} \right\|_\infty && (34) \end{aligned}$$

$R_1^\star W$ and $R_{1\perp}^\star W$ belong to $L^\infty(\mathbb{C}_{n\times n})$. Using spectral factorizations of the entries of the matrix on the left-hand side of (34) one can show that there exist inner matrix functions $U,\ U' \in H^\infty(\mathbb{C}_{n\times n})$ (Theorem 1 [7], [26], Chapter 8 [27]) such that

$$\begin{aligned} G &:= UR_1^\star W \in H^\infty(\mathbb{C}_{n\times n}) \\ \Omega &:= U' R_{1\perp}^\star W \in H^\infty(\mathbb{C}_{n\times n}) \end{aligned}$$

Then we multiply the matrix on the right-hand side of (34) by the inner matrix $M_i := \begin{pmatrix} U & 0 \\ 0 & U' \end{pmatrix}$ and obtain

$$\beta = \|\Psi\|_\infty = \left\| \begin{pmatrix} G - UQ_o \\ \Omega \end{pmatrix} \right\|_\infty \tag{35}$$

where $G,\ \Omega \in H^\infty(\mathbb{C}_{n\times n})$ and $U \in H^\infty(\mathbb{C}_{n\times n})$ is inner. Our optimal performance index β has therefore the following "Hankel-Toeplitz" formulation [26], [7]

$$\beta^2 = \|\Gamma_{G^\star U}\Gamma_{U^\star G} + T_{\Omega^\star\Omega}\| \tag{36}$$

where $\Gamma_{U^\star G}$ and $T_{\Omega^\star\Omega}$ are respectively the Hankel and Toeplitz operators with symbols $U^\star G$ and $\Omega^\star\Omega$ (respectively), more explicitly if we let $\Theta = U^\star G$ to simplify the notation, then

$$\begin{aligned} \Gamma_\Theta :\ H^2(\mathbb{C}_n) &\longrightarrow H^2(\mathbb{C}_n)^\perp \\ \Gamma_\Theta &= P_- \Theta && (37) \\ T_{\Omega^\star\Omega} :\ H^2(\mathbb{C}_n) &\longrightarrow H^2(\mathbb{C}_n) \\ T_{\Omega^\star\Omega} &= P_+ \Omega^\star \Omega && (38) \end{aligned}$$

where $H^2(\mathbb{C}_n)^\perp$ is the orthogonal complement of $H^2(\mathbb{C}_n)$ in $L^2(\mathbb{C}_n)$, and P_-, P_+ respectively the negative and positive Riesz projections [26], [7], i.e.,

$$\begin{gathered} P_- :\ L^2(\mathbb{C}_n) \longrightarrow H^2(\mathbb{C}_n)^\perp \\ P_-\left(\textstyle\sum_{-\infty}^{\infty} a_n z^n\right) = \textstyle\sum_{-\infty}^{-1} a_n z^n, \quad a_n \in \mathbb{C}_n\ , n = 1, 2, \ldots \\ P_+ :\ L^2(\mathbb{C}_n) \longrightarrow H^2(\mathbb{C}_n) \\ P_+\left(\textstyle\sum_{-\infty}^{\infty} a_n z^n\right) = \textstyle\sum_{0}^{\infty} a_n z^n \end{gathered}$$

It is well known that $\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}$ has a spectrum with continuous and discrete parts [28], [7], [9]. The continuous part corresponds to the essential spectrum. Under assumption (A2), $R_1^\star W$ is continuous and the operator Γ_Θ is compact [28]. The spectrum of $\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}$ is formed by the essential spectrum plus isolated eigenvalues with finite multiplicity. We show that under assumptions of Theorem 3 [19], [20], the operator $\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}$ achieves its norm on the discrete spectrum, that is,

$$\begin{aligned} &\|\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}\| = && (39) \\ &\max\{\lambda^2 :\ \left(\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}\right)x = \lambda^2 x \in H^2(\mathbb{C}_n)\} \end{aligned}$$

and therefore generalizing the same result obtained for rational plants and weightings in [7] to infinite-dimensional plants.

Premultiplying (32) by $M_i(R,\ R_\perp)^\star$ we get

$$M_i(R,\ R_\perp)^\star \Psi f = M_i(R,\ R_\perp)^\star \Xi_2 f \tag{40}$$

and applying their respective adjoints yields

$$\begin{aligned} &\left(M_i(R,\ R_\perp)^\star\Psi\right)^\star M_i(R,\ R_\perp)^\star \Psi f = \\ &\quad \left(M_i(R,\ R_\perp)^\star\Xi_2\right)^\star M_i(R,\ R_\perp)^\star \Xi_2 f && (41) \end{aligned}$$

Since multiplication by inner matrices preserves the L^2 and ∞-norms, by passing to them from (35), (36) and (41) we obtain

$$\|\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega} f\|_{L^2} = \|\Psi f\|^2_{L^2} = \|\Xi_2 f\|^2_{L^2} = \|\Xi_2\|^2 \quad (42)$$

hence f is a maximal vector for $\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}$, in fact f is the eigenvector which corresponds to its maximal eigenvalue λ^2_{max}. Hence we proved the following Corollary to Theorem 4, which generalizes Theorem 6 in [7] obtained for finite-dimensional systems.

Corollary 1: Under assumptions (A1), (A2) and $\beta > \beta_{oo}$, the operator $\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}$ achieves its norm on its discrete spectrum

$$\sigma_d = \{\lambda^2 : \Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega} x = \lambda^2 x,\ \exists x \in H^2(\mathbb{C}_n)\} \quad (43)$$

that is

$$\|\Gamma_{\Theta^\star}\Gamma_\Theta + T_{\Omega^\star\Omega}\| = \lambda^2_{max},\ \ \lambda^2_{max} \in \sigma_d \quad (44)$$

A method of computing discrete spectrum and eigenvectors of mixed Hankel-Toeplitz operators for SISO infinite-dimensional systems subject to continuous weightings is presented in [9] though under *stronger assumptions*, and for MIMO systems in [11].

V. Conclusion

The recognition of the optimal two-block H^∞ problem as a distance minimization in a certain matrix valued H^∞ space enabled predual and dual representations to be obtained. These representations allow three insights into the problem. On an abstract level, alignment conditions relate the nearest element in the distance minimization to the maximal element in the dual optimization providing certain geometrical properties under weaker assumptions than [16], [17], [18], [19]. Some of these properties are that the optimal solution is flat and satisfies an extremal identity, also existence and uniqueness in the SISO case of optimal control laws were deduced. Simple observations were provided to determine dual and predual spaces instead of the lengthy arguments in [16].

A well known Hankel-Toeplitz operator, under specific conditions, is shown to achieve its norm on the discrete spectrum generalizing anterior results obtained for finite-dimensional plants, thus simplifying the computation of optimal performance β, and answering a question in [7]. On a more practical level, the predual and dual formulations allow the development of finite variable convex programming based algorithms to estimate numerically the optimum within known tolerance. These algorithms may be applied to infinite dimensional plants, such as the linear models developed in [5], [6] for load balancing in parallel computations in presence of communication time-delays. This is the subject of on-going research.

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