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Use of Lambert W Function to Stability Analysis of Time-Delay Systems

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***Abstract*— The Lambert W function has been used in an extremely wide variety of applications, including the stability analysis of fractional-order as well as integer-order time-delay systems. In this paper, we re-examine an application of using the Lambert W function through actually constructing the root distributions of the derived TCEs of some chosen orders. It is found that the rightmost root of the original TCE is not necessarily a principal branch Lambert W function solution, and that a derived TCE obtained by taking the nth power of the original TCE introduces superfluous roots to the system. With these observations, some deficiencies displayed in the literature are pointed out. Moreover, we clarify the correct use of Lambert W function to stability analysis of a class of time-delay systems. This will actually enlarge the application scope of the Lambert W function, which is becoming a standard library function for various commercial symbolic software packages, to time-delay systems.**

I. INTRODUCTION

Time-delay systems are often described by delay-differential equations (DDEs) [1]. A linear or linearized time-invariant system with a single delay has in general a transcendental characteristic equation (TCE) of the form $A(s) + B(s)e^{-\tau s} = 0$, where τ is the delay time. Due to the presence of the exponential function $e^{-\tau s}$, this equation has an infinite number of roots, which makes the analytical stability analysis of a time-delay system extremely difficult. Up to now, no simple and general algebraic criterion, like the Routh-Hurwitz criterion for delay-free systems, has been presented in the literature for testing the root distribution of a TCE with respect to the imaginary axis of the complex plane. Usually, the stability analysis of time-delay systems relies on graphical methods, e.g., Nyquist criterion [2] or D-partition technique [3], [4] .

Recently, an approach of using the Lambert W function [5] has been presented by Chen & Moore [6], [7] to obtain stability bound for a class of time-delay systems having the TCE

$$(s + \alpha)^{n/m} + K_p e^{-\tau s} = 0 \qquad (1)$$

where n and m are positive integers and $\alpha, \tau > 0, K_p$ are real numbers. Essentially, Chen and Moore's approach is an extension of the old result of Wright [8] and the recent result of Asl and Ulsoy [9]. It is based on casting TCE (1) in the form

$$(as + b)e^{cs} + d = 0 \qquad (2)$$

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whose roots can be represented in terms of the Lambert W function as

$$s = \frac{1}{c}W(-\frac{dc}{a}e^{bc/a}) - \frac{b}{a} \qquad (3)$$

where $W(z)$ satisfies the functional equation

$$W(z)e^{W(z)} = z \qquad (4)$$

Since the above equation always has an infinite number of solutions, so $W(z)$ is a multivalued function and the different possible solutions are denoted by $W_k(z)$ for $k = 0, \pm1, \pm2$, etc. Without noting the multivalued characteristic of the Lambert W function, Chen and Moore used (3) with only the principal branch solution $W = W_0$ and with only a real argument to construct the stability bound in the parameter space for the time-delay systems whose TCEs are of the form (1). Unfortunately, they have not explained why just use the principal branch W function with a real argument in the stability analysis of time-delay systems.

As it can be noted, the cast of (1) into (2) is achieved through taking the mth power and nth root on the equation involved. One may naturally ask if every W function solution to the derived TCE satisfies also the original TCE. Due to the fact that the rightmost root of (1) play an important role in the stability analysis, one may also ask if the rightmost root in the set of W function solutions to the derived TCEs of a delay system belongs to the set of principal branch W function solutions. To answer these two questions, we re-examine in this paper the application of Lambert W function in stability analysis for the class of time-delay systems having the TCE (1). Through actually computing the W function solutions of the derived TCEs for the some specific systems, using the Lambert W function routine in the Symbolic software package Maple, it is found that the answers are negative. The observation from the case studies motivates us to present a correct use of the Lambert W function in solving stability analysis problems of time-delay systems.

II. THE LAMBERT W FUNCTION

The Lambert W function is the complex function which solves for w the equation:

$$we^w = z, \qquad w, z \in \mathbf{C} \qquad (5)$$

where $\mathbf{C}$ is the set of complex numbers. The complex function $W(z)$ satisfies the functional equation (5) for $z \in \mathbf{C}$. This function has acquired popularity only recently, due to advances in computational mathematics and its implementation in the mathematical library of the computer

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algebra program Maple [10]. Actually, Lambert W function received its name during the implementation of this function in Maple. The history, mathematical developments, and applications of the Lambert W function have been presented in [5], [11].

The Lambert function $W(z)$ is multivalued and as such it has many branches [5], [12]. The different possible branches are denoted by $W_k(z)$ for any $k = 0, \pm 1, \pm 2$, etc. According to the implementation in Maple [5], the curve which separates the principal branch $W_0(z)$ from the branches $W_1(z)$ and $W_{-1}(z)$ is

$$\{-\eta \cot \eta + \mathbf{i}\eta : -\pi < \eta < \pi\}, \qquad \mathbf{i} = \sqrt{-1} \quad (6)$$

The curve separating $W_1(z)$ and $W_{-1}(z)$ is simply $(-\infty, -1]$. Finally, the curves separating the remaining branches are

$$\{-\eta \cot \eta + \mathbf{i}\eta : 2k\pi < \pm\eta < (2k+1)\pi\} \quad k = 1, 2, \ldots \quad (7)$$

Since the images of boundary curves under the mapping $z = we^w$ with $w = W(z)$ are the branch cuts in the z-plane, each boundary curve belongs to the region below it. Such an assignment conforms to the rule of counter-clockwise continuity around the branch points.

The branch $W_0(z)$ is called the principal branch of W. It contains the real line $[-1, +\infty)$ in its range and has a double branch point at $z = -e^{-1}$ corresponding to $w = -1$, which it shares with both $W_1(z)$ and $W_{-1}(z)$. It is noted that $W_{-1}(z)$ is real for real $z \in [-e^{-1}, 0)$. Thus $W_0(z)$ and $W_{-1}(z)$ are the only branches of W that take on real values. The computer algebra system Maple has had an arbitrary precision implementation of real-valued branches of W for many years, and since Release 2 has had an arbitrary precision implementation of all branches [10].

III. Integer-Order Time-Delay Systems

Consider the proportional feedback control system shown in Fig. 1 with the time-delay plant

$$G_p(s) = \frac{e^{-\tau s}}{(s+\alpha)^n} \quad (8)$$

where n is an positive integer and τ the delay time. The closed-loop transfer function of this control system is readily obtained as

$$G_{CL}(s) = \frac{K_p e^{-\tau s}}{(s+\alpha)^n + K_p e^{-\tau s}} \quad (9)$$

The closed-loop TCE can be alternatively written as

$$(s+\alpha)^n e^{\tau s} = -K_p \quad (10)$$

For $n = 1$, Asl and Ulsoy [9] obtained roots s_k of the above characteristic equation in terms of Lambert W function as follows:

$$s_k = \frac{1}{\tau} W_k(-K_p \tau e^{\tau\alpha}) - \alpha, \quad k = 0, \pm 1, \pm 2, \ldots \quad (11)$$

They also used these formulas to construct the stability domain in the parameter space and to compute time response.

Recently, Chen and Moore [6] has extended the above approach to construct the stability bound on the negative controller gain K_p for $n = 2$. With $n = 2$ and K_p being replaced by $-K_p$, they first took the square root for both sides of (10) to yield

$$(s+\alpha)e^{(\tau/2)s} = \pm\sqrt{K_p} \quad (12)$$

then represented the roots of the above-derived TCE in terms of the Lambert W function as

$$s = \frac{2}{\tau} W\left(\frac{\tau}{2} e^{(\tau/2)\alpha}(\pm\sqrt{K_p})\right) - \alpha \quad (13)$$

and finally concluded the stability condition for all possible τ, α and K_p as

$$\frac{2}{\tau} W\left(\frac{\tau}{2} e^{(\tau/2)\alpha}(\pm\sqrt{K_p})\right) - \alpha \le 0 \quad (14)$$

Moreover, they further remarked that the stability bound of the closed-loop system (9) with a positive integer order n is given by

$$\frac{n}{\tau} W\left(\frac{\tau}{n} e^{(\tau/n)\alpha} \sqrt[n]{K_p}\right) - \alpha \le 0 \quad (15)$$

Before making comments on the result of Chen and Moore [6], we first consider the Lambert W function solutions to (10) for an arbitrary positive integer n. Taking the nth root for both sides of (10) gives the following n derived TCEs:

$$(s_l+\alpha)e^{(\tau/n)s_l} = \begin{cases} \sqrt[n]{|K_p|}\exp\left(\mathbf{i}\frac{(2l+1)\pi}{n}\right), & \text{if } K_p \ge 0 \\ \sqrt[n]{|K_p|}\exp\left(\mathbf{i}\frac{2l\pi}{n}\right), & \text{if } K_p < 0 \end{cases} \quad (16)$$

In terms of the Lambert W function, the roots of these TCEs can be written as

$$s_{l,k} = \begin{cases} \frac{n}{\tau} W_k\left(\frac{\tau}{n} e^{(\tau/n)\alpha} \sqrt[n]{|K_p|}\exp\left(\mathbf{i}\frac{(2l+1)\pi}{n}\right)\right) - \alpha & \text{if } K_p \ge 0 \\ \frac{n}{\tau} W_k\left(\frac{\tau}{n} e^{(\tau/n)\alpha} \sqrt[n]{|K_p|}\exp\left(\mathbf{i}\frac{2l\pi}{n}\right)\right) - \alpha & \text{if } K_p < 0 \end{cases} \quad (17)$$

where $k = 0, \pm 1, \pm 2, \ldots$.

Using the Lambert W function of Maple, we plot in Figs. 2-5 the root distributions of $s_{l,k}$ for $(n, K_p) = (2, -2), (2, 2), (3, -2)$ and $(3, 2)$, respectively, where the parameters (α, τ) is set to $(1, 1)$. It is noted that in these figures the roots marked with a solid circle satisfy the original TCE (10) and they are the poles of the closed-loop system. From Figs. 3-6, we have the following observations: (i) all the roots $s_{l,k}$ of the derived TCEs in (16) are the poles of the closed-loop system (9); (ii) the rightmost root(s) of the system can be real root or complex conjugate and the principal branch solution $s_{0,0}$ of the derived TCE in (16) with $l = 0$ is a rightmost root; (iii) the roots of the n derived TCE locate in an interlaced manner in either upper or lower root branch.

Now, some comments on the results displayed in equations (12)-(16) for constructing stability bound have certain deficiencies are in order. First, according to equations (14) and (15) and Figure 1 shown in [6], it seems that Chen and Moore have made the following two presumptions: (i) the arguments of the W functions in (14) and (15) are real; (ii) the principal branch Lambert W function of the first derived TCE, i.e., the equation with $l = 0$ in (16), also ensures real value. The disadvantage of presumption (i) is that it restricts the gain K_p to be negative because, according to (17), a positive K_p leads to a complex argument for the W function. Negative gain is seldom used in a practical feedback control system since a positive feedback often tends to destabilize the system. Next, we note from Figs. 2 and 4 that the principal branch solutions $s_{0,0}$ and $s_{1,0}$ for $n = 2$, and $s_{0,0}$ and $s_{2,0}$ for $n = 3$ are complex values whose imaginary parts do not vanish. Moreover, as reviewed in Section II, the value of the principal branch W function can be complex even if its argument is real. Hence, the inequality (15), which is a consequence of presumption (ii), becomes sometimes meaningless when the principal branch solution $s_{0,0}$ is a complex value. In fact, the generally correct equation for describing the boundary of the stability region in the parameter space should read as

$$\Re\{s_{0,0}\} = \begin{cases} \frac{n}{\tau}W_0(\frac{\tau}{n}e^{(\tau/n)\alpha}\sqrt[n]{|K_p|}\exp(\mathbf{i}\frac{\pi}{n})) - \alpha = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } K_p \geq 0 \\ \frac{n}{\tau}W_0(\frac{\tau}{n}e^{(\tau/n)\alpha}\sqrt[n]{|K_p|} - \alpha = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } K_p < 0 \end{cases} \tag{18}$$

where $\Re\{\cdot\}$ denotes the operator of taking real part of the indicated quantity. Finally, it should be noted that in [6] the stability region in the $\tau - K_p$ plane is constructed through first making grids on the parameter plane, then evaluating the rightmost root $s_{0,0}$ with the Lambert W library function of Maple for each grid point, and finally obtaining the stable region by identifying the area of those grid points at each of which the rightmost root of the TCE has a negative real part. Obviously, this brute-force approach to constructing stability region is by no means efficient and it cannot produce an analytic stability bound.

Before leaving the section we note that the stability boundary consists of those points in the parameter space at which the principal branch Lambert function W_0 in (18) has a real part α. In a two-dimensional parameter plane, say, $K_p - \tau$ plane, (18) defines one-dimensional manifolds or curves, which can be traced approximately but efficiently with an integer-labeling pivot procedure [13]. Using this cited path-following algorithm, we have constructed in Fig. 6 the stability domain in the $K_p - \tau$ parameter plane for the system with $n = 2$ and $\alpha = 1$. It is seen from this figure that the stability domain covers both positive and negative regions of K_p. However, the stability domain shown in Fig. 1 of [6] only lies in negative K_p half plane, which suggests a positive feedback has to be used in the control system in of Fig. 1.

IV. Fractional-Order Time-Delay Systems

Many real-world physical systems are well characterized by fractional-order differential equations [14], i.e., equations involving noninteger-order derivatives. Moreover, with the success in the synthesis of real noninteger differentiator and the emergence of new electrical circuit element called "fractance" [15] fractional-order controllers [16], including fractional-order PID controllers [17], have been designed and applied to control a variety of dynamical processes of noninteger orders. The latter development has motivated the study of stability analysis for fractional-order control systems with or without time delays [18], [19]. In this line of research, Chen and Moore [7] have recently applied the Lambert W function to construct the stability bound for a class of fractional-order systems. In this section, we shall revisit the problem of stability analysis for fractional-order systems using the Lambert W function and present our findings.

Consider the feedback control system shown in Fig.1 with a fractional-order plant:

$$G_p(s) = \frac{e^{-\tau s}}{(s+\alpha)^{n/m}} \tag{19}$$

It is readily shown that the closed-loop system has the TCE in (1), which can be cast as

$$(s+\alpha)^{n/m}e^{\tau s} = -K_p = \begin{cases} |K_p|e^{\mathbf{i}\pi}, & K_p \geq 0 \\ |K_p|, & K_p < 0 \end{cases} \tag{20}$$

Taking first the mth power and then the nth root on both sides of the above equation, we have

$$(s_l+\alpha)e^{(\tau/n)s_l} = \begin{cases} \sqrt[n]{|K_p|^m}\exp(\mathbf{i}\frac{(2l+m)\pi}{n}), \\ \qquad\qquad\qquad\qquad \text{if } K_p \geq 0 \\ \sqrt[n]{|K_p|^m}\exp(\mathbf{i}\frac{2l\pi}{n}), \\ \qquad\qquad\qquad\qquad \text{if } K_p < 0 \end{cases} \tag{21}$$

where $l = 0, 1, \ldots, n-1$. According to (2) and (3), the roots of the derived TCEs in (21) are given by

$$s_{l,k} = \begin{cases} \frac{n}{m\tau}W_k(\frac{m\tau}{n}e^{(m\tau/n)\alpha}\sqrt[n]{|K_p|^m}\exp(\mathbf{i}\frac{(2l+m)\pi}{n})) \\ \qquad\qquad\qquad\qquad -\alpha, \quad \text{if } K_p \geq 0 \\ \frac{n}{m\tau}W_k(\frac{m\tau}{n}e^{(m\tau/n)\alpha}\sqrt[n]{|K_p|^m}\exp(\mathbf{i}\frac{2l\pi}{n})) - \alpha \\ \qquad\qquad\qquad\qquad \text{if } K_p < 0 \end{cases} \tag{22}$$

To have an insight into the relationship between the roots of the derived TCEs and the poles of the system, we actually compute the roots $s_{l,k}$ for systems of orders $n/m = 1/2, 1/3$, and $2/3$ using the Lambert W function implemented in Maple. First, consider the system of order $n/m = 1/2$. We show in Figures 7 and 8, where $\alpha = 0.5, \tau = 1.5$, the root distributions for $K_p = 1.5$ and $K_p = -1.5$, respectively. Since $n = 1$, there is only one derived TCE, which is simply obtained by taking square on both sides of the original TCE (1), and its roots are denoted by s_k in these two figures. The roots s_k that satisfy the original TCE (1) are marked with a solid circle. It is observed from

Figures 7 and 8 that the roots $s_{\pm 1}, s_{\pm 3}, \ldots$, for $K_p = 1.5$ while the roots $s_0, s_{\pm 2}, s_{\pm 4}, \ldots$, for $K_p = 1.5$ do not satisfy the original TCE. Also observed is that the true roots and superfluous roots interlace. These observations imply that taking power of 2 on both sides of the original TCE (1) gives rise to a derived TCE which introduces superfluous roots to the original one. To verify this implication, we then consider the system of order $1/3$. In this case, the derived TCE is obtained by taking the power of 3 on both sides of the original TCE (1). Figures 9 and 10 show the root distributions for $K_p = 1$ and -1, respectively, whereas the parameters (α, τ) are set as $(0.5, 1)$. It is seen from these two figures that on each root branch, there is only one pole of the system among every three adjacent roots of the derived TCEs. The same observation can be made from Figures 11 and 12, where the system order is $2/3$ and the parameters (α, τ, K_p) are $(1, 1, 1.5)$ and $(1, 1, -1.5)$, respectively. It is noted that there are two TCEs for the system of oder $2/3$.

With the observations on root distributions shown in Figures 7-12, we may reasonably infer that for a TCE derived from the original TCE by taking an integer power of $m > 1$, there is only one true root for the original TCE in the m-element root set $\{s_{l,k\pm j}\}_{j=0}^{m-1}, k \geq 0$. Without recognizing the generation of false poles using the Lambert W function representation, direct use of the roots (22) will result in an incorrect result. Moreover, as it can be seen from Figures 7, 9, and 11, the rightmost root(s) of the derived TCEs are not necessarily the roots of the original TCE. Hence, stability bound for the fractional-order time-delay system (2) cannot be constructed directly using (18). However, to be safe, the boundary of stability domain in the space of parameters α, and τ, K_p, can be constructed with the formula:

$$\max_{\substack{l=\overline{0,n-1}, k=\overline{0,m-1} \\ \Delta(s_{l,k})=0}} \Re\{s_{l,k}\} = 0 \tag{23}$$

where $\Delta(s) = 0$ denotes the original TCE (1).

Finally, it is noted that Chen and Moore [7] have applied the Lambert W function approach to construct the stability bound for fractional-order time-delay control system with the TCE

$$(s+\alpha)^r - K_p e^{-\tau s} = 0 \tag{24}$$

where r is positive rational number. They arrived at the stability condition

$$\tfrac{r}{\tau} W(\tfrac{r}{\tau} e^{(\tau/r)\alpha} (K_p)^{1/r}) - \alpha \leq 0 \tag{25}$$

Due to the facts that the same presumptions as those stated in the previous section for integer-order delayed systems were made, and that the unawareness of superfluous roots generated from derived TCEs, their results are by no means generally correct for constructing stability bounds for fractional-order systems. Indeed, the stability bound shown in Fig. 2 of [7] for $\alpha = 0.5$ and $r = 1/3$ is not complete since the surface corresponding to the Lambert W function solutions with nonzero imaginary parts is missed. We present the correct one in Fig. 13, in which the vertical coordinate $\Re\{s(k_p, \tau)\}$ represents the real part of the rightmost root of the original TCE. Note that the missing surface in Fig. 2 of [7] might be due to an inappropriate use of Maple's plot function.[1] However, even if the Maple's plot function is invoked correctly, the use of principal branch W function solutions as the rightmost roots of the original TCE also gives rise to an erroneous plot since the missing solution surface locates in an area of the $\tau - K_p$ plane within which the rightmost root of the original is the W_1 branch rather than the principal branch W_0.

V. Conclusions

In this note, the application of Lambert W function to the stability analysis of time-delay systems is re-examined. We have the following key observations obtained from the case study of fractional-order systems: (i) the operation of taking power on a transcendental characteristic equation gives rise to a derived transcendental characteristic equation which introduces superfluous roots to the original TCE; (ii) the rightmost root of the derived TCEs is not necessarily the principal branch Lambert W function solution of a derived TCE. With these observations, we have clarified the correct use of Lambert W function to stability analysis of a class of time-delay systems. Moreover, some unclear points or deficiencies displayed in the literature are pointed out. The critical points clarified in the paper will actually enlarge the application scope of the Lambert W function, which is becoming a standard library function for various commercial symbolic software packages, to time-delay systems.

VI. ACKNOWLEDGMENTS

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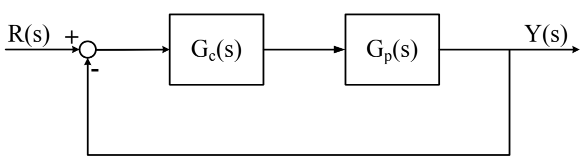

Fig. 1. A typical feedback control system.

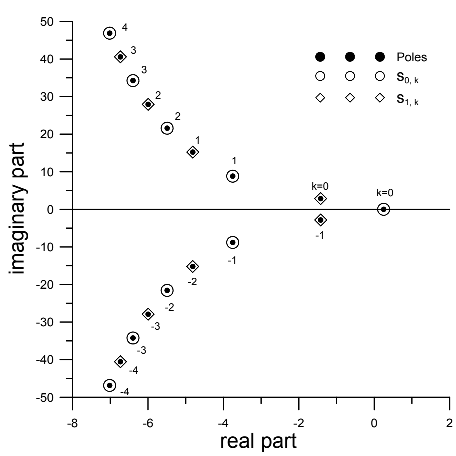

Fig. 2. Root distribution of the derived TCEs for (n,K_p,α,τ)=(2,-2,1,1).

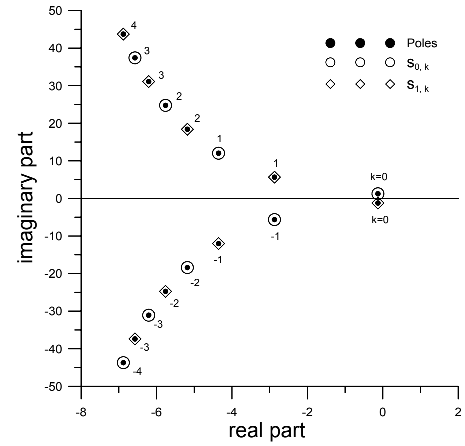

Fig. 3. Root distribution of the derived TCEs for (n,K_p,α,τ)=(2,2,1,1).

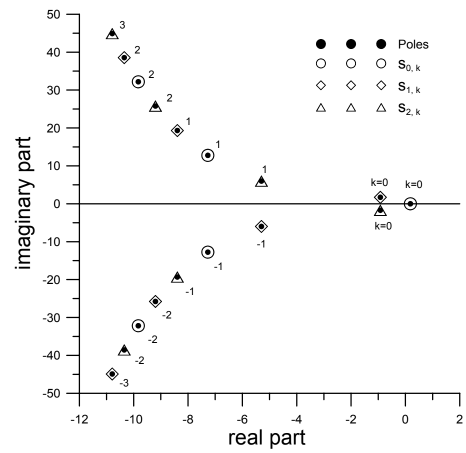

Fig. 4. Root distribution of the derived TCEs for (n,K_p,α,τ)=(3,-2,1,1).

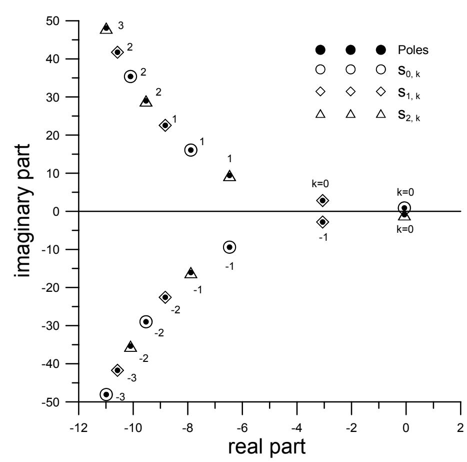

Fig. 5. Root distribution of the derived TCEs for (n,K_p,α,τ)=(3,2,1,1).

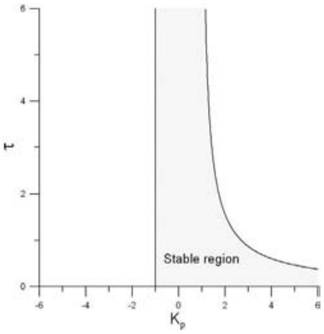

Fig. 6. Stable region in the K_p-τ plane for the system with n=2 and α=1.

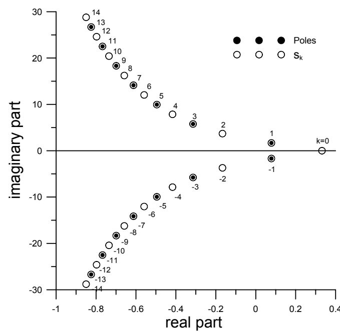

Fig. 7. Root distribution of the derived TCE for $(n/m$,K_p,α,$\tau)$=(1/2,1.5,0.5,1.5).

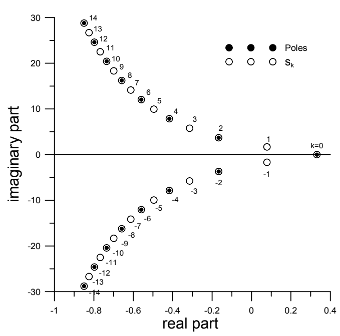

Fig. 8. Root distribution of the derived TCE for $(n/m$,K_p,α,$\tau)$=(1/2,-1.5,0.5,1.5).

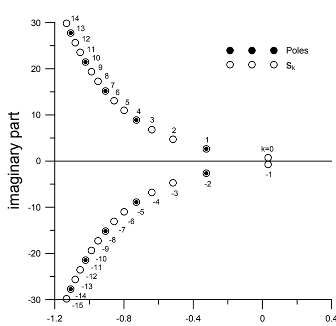

Fig. 9. Root distribution of the derived TCE for $(n/m$,K_p,α,$\tau)$=(1/3,1,0.5,1).

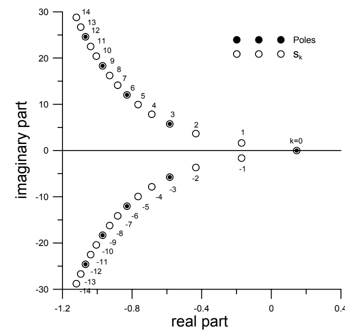

Fig. 10. Root distribution of the derived TCE for $(n/m$,K_p,α,$\tau)$=(1/3,-1,0.5,1).

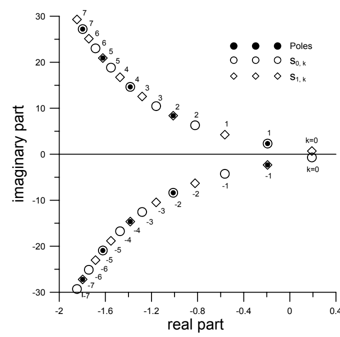

Fig. 11. Root distribution of the derived TCEs for $(n/m$,K_p,α,$\tau)$=(2/3,1.5,1,1).

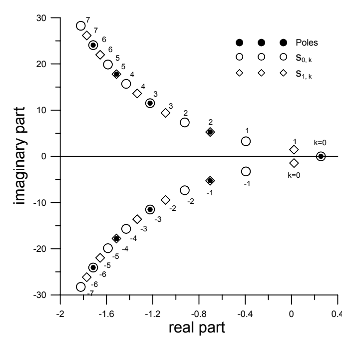

Fig. 12. Root distribution of the derived TCEs for $(n/m$,K_p,α,$\tau)$=(2/3,-1.5,1,1).

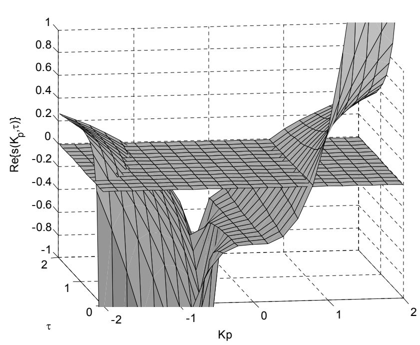

Fig. 13. The stability bound for system with n/m=1/3 and α=0.5.