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Robust Stability and $\mathcal{H}_\infty$ Performance Analysis of Interval-Dependent Time Delay Systems

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***Abstract*— Robust stability and performance analysis techniques are presented for time-delay systems consisting of the feedback interconnection of a linear time-delay system, with a bounded and casual linear operator. This linear operator features all the parameter and/or dynamical uncertainties excluding the delay elements. The delays considered are non-commensurate, time-invariant but uncertain, residing within a bounded interval excluding zero. A (sufficient) delay-free comparison system is formed by replacing the delay elements with parameter-dependent filters, satisfying certain properties. It is shown that robust stability of this finite dimensional comparison system, guarantees stability of the original time-delay system. A (necessary) comparison system is also provided. It is shown that the worst case $\mathcal{H}_\infty$ performance of the sufficient (necessary) comparison system provides an upper (a lower) bound for the $\mathcal{H}_\infty$ performance of the time-delay system. An LMI formulation is given for calculating an upper bound for the worst-case $\mathcal{H}_\infty$ performance of the time-delay system.**

***Index Terms*— Time-delay; robust stability; $\mathcal{H}_\infty$ performance.**

I. Introduction

Presence of time-delay can severely complicate both theoretical and practical aspects of analysis and design of control systems. In this paper, we first address the problem of robust stability analysis of time-delay systems subject to parameter or dynamical uncertainties. Second, we provide criteria for analysis of H_∞ performance of the nominal (without dynamical uncertainty) time-delay system.

We examine *interval-dependent* time-delay systems for which, the delays are known to reside within bounded intervals excluding zero, i.e. $\left[\underline{\tau_k}, \overline{\tau_k}\right]$, $k = 1...N$. For many engineering systems of practical importance, the time-delay is known to reside in an interval excluding zero. Moreover, the time-delay system may be stable for some value of delay τ but not for some $\widetilde{\tau} < \tau$ [1], [24],[27]. Therefore, an analysis criterion which does not exploit stability of the delay-free system must be formulated. Few researchers have examined this problem in the past [12],[9],[23]. The results of [23] may not be extended to robust stability and $\mathcal{H}_\infty$ performance analysis. The result of [12] requires an *a priori* assumption regarding the stability of the time-delay system for some nominal delay value $\tau_{0_k} \in \left[\underline{\tau_k}, \overline{\tau_k}\right]$ and also may be very conservative. Unfortunately, the result of [9] suffers from both drawbacks of [12] and [23]. (See [28] for a more detailed discussion regarding the technical difficulties and the benefits of relaxing the assumption of nominal stability for some $\tau_{0_k} \in \left[\underline{\tau_k}, \overline{\tau_k}\right]$).

Previous research efforts have also considered the use of Integral Quadratic Constraints (IQCs) for the analysis of linear and nonlinear/uncertain time-delay systems in the delay-dependent case $\tau \in [0, \overline{\tau}]$. See for instance [10], [11], [18], [4]. While by nature, the IQC-based results may be employed for the robust stability and $\mathcal{H}_\infty$ performance analysis problem, extension of these methods to the interval-dependent problem is not straightforward. (See [28], Chapter 6). Moreover, in these papers, IQCs are used to capture the delay elements as well as the nonlinearities and/or uncertainties (pure IQC analysis). It was demonstrated in [31] that the manner in which these type of IQCs cover the delay value set, usually results in a high degree of conservatism in stability analysis. See also [26] for comparison. In this paper, we advocate our previously established *comparison system* framework ([14],[15],[28]) for the problem of robust stability and performance analysis. In this framework, the delay elements are tackled via parameter-dependent filters, tightly covering the value set of the delay elements.

For the problem formulation, we consider the family of systems formed by the feedback interconnection of a linear delay-differential system and a linear time-invariant, causal and bounded operator, which contains any and all uncertain components of the system, excluding the delay elements. We replace the delay elements with parameter-dependent filters, and develop necessary and sufficient (uncertain) parameter-dependent comparison systems. It can also be shown that similar to the linear case with no dynamical uncertainty ([14],[15]), if the comparison systems are developed in a particular form from Padé approximations, then the degree-of-conservatism of the sufficient comparison system is independent of the system data and can be made arbitrarily small. Due to space limitations we do not present this result in this paper and instead refer the reader to [28]). Robust stability of the sufficient comparison system can be examined via either a mixed-μ technique or via the IQC Theorem [18]. Finally, upper and lower bounds for the *worst-case* $\mathcal{H}_\infty$ *performance* of the (nominal) time-delay system are derived from the comparison systems.

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A. Preliminaries

Notation 1: The notation used is standard.

Definition 1: Given a continuous function $g(q) : [0,\infty) \to \mathcal{D}$ where $\mathcal{D} = \{z \in \mathbb{C} |\ |z| = 1\}$, letting Γ_r be the path created by mapping the interval $q \in [0, r]$ via $g(q)$ to $\mathcal{D}$, we define a *continuous argument* (phase) function for the value $g(r)$ as $\mathrm{Arg}(g(r)) = \arg(g(r)) + 2\pi n(\Gamma_r, 0)$ where $\arg(z) \in (-2\pi, 0]$ is the unique argument of $z \in \mathbb{C}$, $z \neq 0$ and $n(\Gamma_r, 0)$ is the winding number of path Γ about a.

Definition 2: [18] The feedback system

$$\begin{aligned} v &= Gw + f \\ w &= \Delta v + e \end{aligned} \tag{1}$$

is said to be well-posed if the map from $(v, w) \mapsto (e, f)$ has a causal inverse on $\mathbb{L}_{2e}^{l_1+m_2}[0,\infty) \times \mathbb{L}_{2e}^{m_1+m_2}[0,\infty)$.

Definition 3: [18] The feedback system (1) is said to be stable with finite-gain if it is well posed and there exists a constant $C > 0$, such that for any solution of (1),

$$\int_0^T \left(|v|^2 + |w|^2\right) dt \leq C \int_0^T \left(|f|^2 + |e|^2\right) dt, \quad \forall T \geq 0. \tag{2}$$

II. Problem Statement

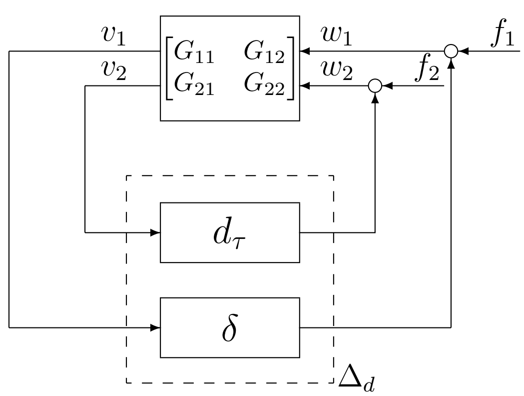

Fig. 1. Uncertain time-delay system

Problem 1: Examine robust stability of the feedback system Σ_d^u illustrated in Figure (1),

$$\begin{aligned} v &= Gw + f \\ w &= \Delta_d v \end{aligned} \tag{3}$$

where $\Delta_d = \begin{bmatrix} \delta & 0 \\ 0 & d_\tau \end{bmatrix}$, $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$, $w = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T$, and $f = \begin{bmatrix} f_1^T & f_2^T \end{bmatrix}^T$. The operator δ is an uncertain, linear, bounded, and causal operator on $\mathbb{L}_{2e}^{l_1}[0,\infty)$ to $\mathbb{L}_{2e}^{m_1}[0,\infty)$. $d_\tau := \mathrm{diag}\{d_{\tau_k} : v_k(t) \to v_k(t - \tau_k)\}$, $k = 1, ..., N$ is the diagonal (structured) delay operator where each time-delay τ_k belongs to an interval $[\underline{\tau_k}, \overline{\tau_k}] \subset (0, \infty)$. Using the state space notation, the feedback system (3) is denoted as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w_1(t) + B_2 w_2(t) \\ v_1(t) &= C_1 x(t) + D_{11} w_1(t) \\ v_2(t) &= C_2 x(t) \\ w_1(t) &= \delta v_1(t) + f_1(t) \\ w_2(t) &= d_\tau v_2(t) + f_2(t), \qquad \tau \in \prod_{k=1}^{k=N} [\underline{\tau_k}, \overline{\tau_k}]. \end{aligned} \tag{4}$$

We will use Σ_d^u to refer to the system (3) (or indifferently (4)). The superscript 'u' is chosen to emphasize the uncertainty associated with the delay-differential system. In this paper, δ is a finite-dimensional, linear time-invariant (FDLTI) *internally stable uncertain dynamical system* in $\underline{\mathbb{U}} \subset \mathbb{U}$, where $\mathbb{U}$ is the structured uncertainty set defined by

$$\begin{aligned} \mathbb{U} &:= \{diag\{\Delta_r, \Delta_c\} | \Delta_r \in \mathbb{R}_u, \Delta_c \in \mathbb{C}_u\}, \\ \mathbb{R}_u &:= \left\{diag\left\{r_1 I_{k_1^r}, \cdots, r_{l_r} I_{k_{l_r}^r}\right\} | r_i \in \mathbb{R}, |r_i| \leq 1\right\}, \\ \mathbb{C}_u &:= \left\{diag\{\Delta_1, \cdots, \Delta_{l_u}\} | \Delta_i \in \mathbb{C}^{n_i \times n_i}, \overline{\sigma}(\Delta_i) \leq 1\right\}. \end{aligned}$$

For clarity and convenience in notation, we present the results for the single delay case only. All the theories presented here, are readily extendable to the multiple-delay case. Extensions may be carried out in a straightforward fashion (see [28], [15]).

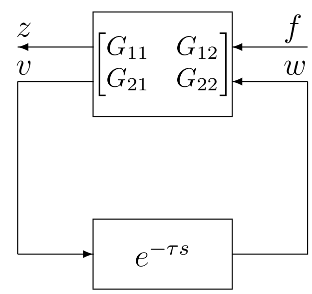

Fig. 2. The LFT setup for $\mathcal{H}_\infty$ performance analysis.

Problem 2: Consider the feedback system Σ_d illustrated in Figure (2),

$$\begin{aligned} \begin{bmatrix} z \\ v \end{bmatrix} &= G \begin{bmatrix} f \\ w \end{bmatrix} \\ w &= e^{-\tau s} v \end{aligned}$$

Assume that stability of Σ_d on $[\underline{\tau}, \overline{\tau}]$ has been established. It is desired to find the *worst case* $\mathcal{H}_\infty$ *performance* γ_d^* of the system Σ_d, defined as

$$\gamma_d^* := \max_{\tau \in [\underline{\tau}, \overline{\tau}]} \left\| T_{zf}^{(d)}(s, \tau) \right\|_\infty \tag{5}$$

where, $T_{zf}^{(d)}(s, \tau)$ is the transfer function from the input f

to the output z. That is,

$$\begin{aligned}T_{zf}^{(d)}(s,\tau)\\&= G_{11}(s) + G_{12}(s)e^{-\tau s}\left(I_{m_2} - G_{22}(s)e^{-\tau s}\right)^{-1}G_{21}(s)\\&\triangleq \mathcal{F}\left(G, e^{-\tau s}I_{m_2}\right)\end{aligned}$$

Indeed, the worst case $\mathcal{H}_\infty$ performance analysis problem is equivalent to the problem of finding $\gamma_d^* > 0$, subject to

$$\begin{aligned}\|z\| &< \gamma_d^*\|f\|,\ \forall\tau\in[\underline{\tau},\overline{\tau}],\ \forall f\in\mathbb{L}_{2e}^{m_1}[0,\infty)\\ z &= \mathcal{F}\left(G, e^{-\tau s}I_{m_2}\right)f\end{aligned}$$

III. Sufficient Stability Criterion

A. Filter properties

Consider a parameter-dependent, rational polynomial transfer function $h_o(\theta,s)$, with θ belonging to Θ, a bounded set of real numbers, and $h_o(\theta,s)$ having the following properties:
be

P_O-1. $h_o(\theta,s)$ is Hurwitz for all $\theta\in\Theta$.
P_O-2. The *value set* of $h_o(\theta,s)$, $\theta\in\Theta$, covers that of $e^{-\tau s}, \tau\in[\underline{\tau},\overline{\tau}]$, i.e.

$$\Omega_d(\omega)\subseteq\Omega_o(\omega),\qquad\forall\omega\geq 0$$

where

$$\begin{aligned}\Omega_o(\omega) &= \{c\in\mathbb{C}\,|c = h_o(\theta,j\omega),\qquad \theta\in\Theta\}\\ \Omega_d(\omega) &= \left\{c\in\mathbb{C}\,|c = e^{-j\tau\omega},\qquad \tau\in[\underline{\tau},\overline{\tau}]\right\}\end{aligned}$$

P_O-3. There exists a $\widetilde{\theta}\in\Theta$ such that $h_o\left(\widetilde{\theta},j\omega\right)\in\Omega_d(\omega),\forall\omega\geq 0$.

B. Sufficient comparison system

Consider the time-delay system Σ_d^u, defined in (3). Removing the delay element $e^{-\tau s}$ from the uncertain block Δ_d and replacing it with $h_o(\theta,s)$ results in the parameter-dependent, delay-free system Σ_o^u (Figure 3) :

$$\begin{aligned}v &= Gw + f\\ w &= \Delta_o v\end{aligned}\tag{6}$$

where

$$\Delta_o = \left[\begin{array}{cc}\delta & 0\\ 0 & h_o(\theta,s)\end{array}\right]\tag{7}$$

Note that Δ_o incorporates two uncertainties. One is the uncertainty associated with $h_o(\theta,s)$, and the other one is the uncertainty associated with $\delta\in\underline{\mathbb{U}}\subset\mathbb{U}$. In the sequel, we will demonstrate that the system Σ_o^u is a *sufficient comparison system* for the system Σ_d^u.

C. Main Result

Theorem 1: If the following condition is satisfied,

$\forall\theta\in\Theta$, the delay-free *comparison system* $\sum_o^u(\theta)$, defined in (6), is stable with finite gain for all $\delta\in\underline{\mathbb{U}}$.

Then, for every $\tau\in[\underline{\tau},\overline{\tau}]$, the time-delay system $\Sigma_d^u(\tau)$ is robustly stable with finite gain for all $\delta\in\underline{\mathbb{U}}$.

Proof: A proof of this Theorem is given in [28] and is omitted here for brevity. ■

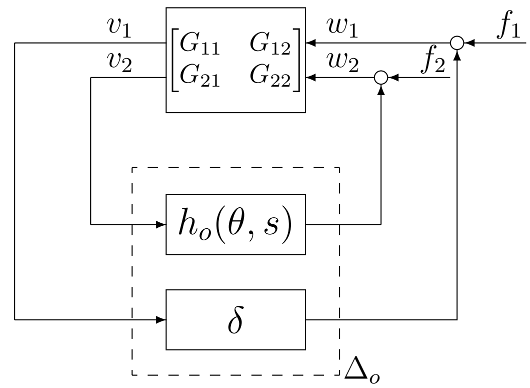

Fig. 3. The comparison system $\Sigma_o^u(\theta)$.

IV. Necessary Stability Criterion

A. Filter properties

Consider a parameter-dependent, rational polynomial transfer function $h_i(\theta,s)$ having the following properties:
be

P_I-1 $h_i(\theta,s)$ is Hurwitz for $\theta\in\Theta$.
P_I-2 The *value set* of $e^{-\tau s},\tau\in[\underline{\tau},\overline{\tau}]$, covers that of $h_i(\theta,s)$, $\theta\in\Theta$, i.e.,

$$\Omega_i(\omega)\subseteq\Omega_d(\omega),\forall\omega\geq 0$$

where

$$\Omega_i(\omega) := \{c\in\mathbb{C}\,|c = h_i(\theta,j\omega),\ \theta\in\Theta\}$$

B. Necessary Comparison System

Consider the time-delay system Σ_d^u, defined in (3). Removing the delay element $e^{-\tau s}$ from the uncertainty block Δ_d and replacing it with the parameter dependent filter $h_i(\theta,s)$ results in the delay-free system Σ_i^u :

$$\begin{aligned}v &= Gw + f\\ w &= \Delta_i v\end{aligned}\tag{8}$$

where

$$\Delta_i = \left[\begin{array}{cc}\delta & 0\\ 0 & h_i(\theta,s)\end{array}\right]\tag{9}$$

It is shown in the sequel that the delay-free system Σ_i^u is a *necessary comparison system* for the system Σ_d^u.

C. Main Result

Theorem 2: If the following condition is satisfied,

$\forall\tau\in[\underline{\tau},\overline{\tau}]$, the time-delay system $\Sigma_d^u(\tau)$ is stable with finite gain for all $\delta\in\underline{\mathbb{U}}$.

Then, for every $\theta\in\Theta$, the *comparison system* $\sum_i^u(\theta)$, is stable with finite gain for all $\delta\in\underline{\mathbb{U}}$.

V. $\mathcal{H}_\infty$ PERFORMANCE

We already defined the *worst case* $\mathcal{H}_\infty$ *performance* γ_d^* of the system Σ_d. The *worst case* $\mathcal{H}_\infty$ *performances* γ_o^* and γ_i^* for *comparison systems* Σ_o and Σ_i are defined in the same way. That is,

$$\begin{aligned} \gamma_o^* &:= \max_{\theta\in\Theta} \left\| T_{zf}^{(o)}(s,\theta) \right\|_\infty \\ \gamma_i^* &:= \max_{\theta\in\Theta} \left\| T_{zf}^{(i)}(s,\theta) \right\|_\infty \end{aligned} \tag{10}$$

where

$$T_{zf}^{(d)}(s,\tau) =$$
$$G_{11}(s)+G_{12}(s)\,h_o(\theta,s)(I_{m_2}-G_{22}(s)\,h_o(\theta,s))^{-1}G_{21}(s)$$
$$\triangleq \mathcal{F}(G, h_o(\theta,s)\, I_{m_2})$$

$T_{zf}^{(i)}(s,\theta)$ is defined analogously. We then have the following Theorem:

Theorem 3: If γ_o^* is a finite number, then

$$\gamma_i^* \le \gamma_d^* \le \gamma_o^* \tag{11}$$

Proof: First, we prove that $\gamma_d^* \le \gamma_o^*$. Consider the comparison system $\Sigma_o^u(\theta)$ (illustrated in Figure 4):

$$\begin{aligned} z &= \frac{1}{\gamma}\mathcal{F}(G, h_o(\theta,s)\, I_{m_2})\, f \\ f &= \delta z + g, \end{aligned} \tag{12}$$

where $g \in \mathbb{L}_{2e}^m[0,\infty)$ and $\gamma = \gamma_o^*$ is a finite number.

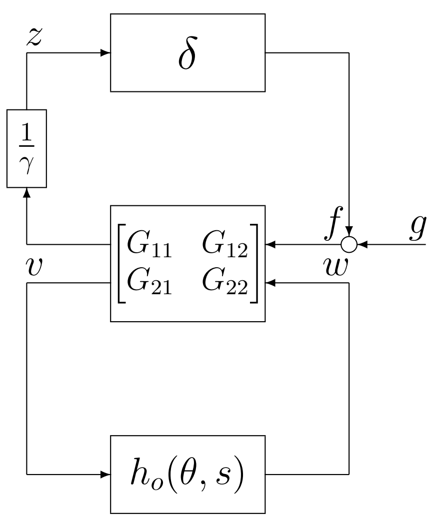

Fig. 4. Comparison system for $\mathcal{H}_\infty$ performance analysis.

$$\begin{aligned} \sup_{\theta\in\Theta} \left\| \mathcal{F}(G, h_o(\theta,s)\, I_{m_2}) \right\|_\infty &= \gamma \\ \left\| \frac{1}{\gamma}\mathcal{F}(G, h_o(\theta,s)\, I_{m_2}) \right\|_\infty &\le 1, \qquad \forall\theta\in\Theta \end{aligned}$$

That is, for every $\theta \in \Theta$, the $\mathcal{L}_2$-gain from g to z is less than or equal to 1. The small gain Theorem then implies that for every $\theta \in \Theta$, $\Sigma_o^u(\theta)$ is robustly stable with finite gain for all admissible $\delta \in \mathbb{U}$. Theorem 1 then implies that the system $\Sigma_d^u(\tau)$,

$$\begin{aligned} z &= \frac{1}{\gamma}\mathcal{F}\left(G, e^{-\tau s} I_{m_2}\right) f \\ f &= \delta z + g \end{aligned}$$

is robustly stable with finite gain on $[\underline{\tau}, \overline{\tau}]$ for all $\delta \in \mathbb{U}$. We conclude from the necessity of the small-gain theorem that

$$\begin{aligned} \left\| \frac{1}{\gamma}\mathcal{F}\left(G, e^{-\tau s} I_q\right) \right\|_\infty &\le 1, \qquad \forall\tau\in[\underline{\tau},\overline{\tau}] \\ \sup_{\tau\in[\underline{\tau},\overline{\tau}]} \left\| \mathcal{F}\left(G, e^{-\tau s} I_q\right) \right\|_\infty &\le \gamma = \gamma_o^* \end{aligned}$$

This proves that $\gamma_d^* \le \gamma_o^*$. The proof of $\gamma_i^* \le \gamma_d^*$ is analogous. ■

VI. CANDIDATE FILTERS AND THEIR CONVERGENCE

All the theorems that we presented so far, hold for comparison systems constructed with general parameter-dependent filters satisfying the specified properties. In this section, we provide candidate functions that particularly satisfy the desired properties and can therefore, be employed in construction of the comparison systems. We construct these functions from Padé approximation of e^{-s} in the following way.

A. Outer Parameter-Dependent Filter

Define

$$\tau_m = \frac{\overline{\tau}+\underline{\tau}}{2}, \qquad b = \frac{\overline{\tau}-\underline{\tau}}{2}, \qquad \kappa = \frac{\tau_m}{b}.$$

Let $p_l(s)$ denote the l^{th} order diagonal Padé approximation of e^{-s}. Consider the outer parameter-dependent filter,

$$h_o(\theta,s) := p_l([\tau_m - \alpha_o b]\, s)\, p_l(2\alpha_o\theta s), \qquad \theta\in\Theta \tag{13}$$

where

$$\alpha_o := \min\{\alpha \mid 1<\alpha<\kappa, \qquad \Psi_o(\alpha)=0\} \tag{14}$$

$$\begin{aligned} \Psi_o(\alpha) := {}& \mathrm{Arg}\,(p_l([\kappa-\alpha]\, j\omega_o)\, p_l(2\alpha j\omega_o)) \\ & - \mathrm{Arg}\left(e^{-[\kappa+1]j\omega_o}\right) \end{aligned} \tag{15}$$

$$\omega_o := \min\{\omega > 0 \mid p_l(2j\omega) = 1\} \tag{16}$$

$$\Theta := (0, b] \tag{17}$$

Note that this definition of ω_o implies that $\mathrm{Arg}(p_l(2j\omega_o)) = -2\pi$. At points in the exposition it will be necessary to emphasize the dependence of α_o, Ψ_o, and ω_o upon the Padé order l. To do so, we will write $\alpha_o^{[l]}$, $\Psi_o^{[l]}$ and $\omega_o^{[l]}$. Whenever possible, we will suppress the superscript $[l]$ notation.

It can be shown that for every $l \ge 3$ there exists a frequency ω_o satisfying (16), and for every $\kappa > 1$ there exists an l_κ^o such that for each $l \ge l_\kappa^o \ge 3$ there exists an $\alpha_o^{[l]} < 1 + \frac{\pi}{\omega_o^{[l]}}$ satisfying (14). Suppose that $h_o(\theta,s)$ is as specified in (13) with Padé order $l \ge l_\kappa^o$. Then $h_o(\theta,s)$ satisfies properties P_O-1 to P_O-3. (See [28]).

B. Inner Parameter-Dependent Filter

Consider the inner parameter-dependent filter,

$$h_i(\theta, s) := p_l\left(\left[\tau_m - \frac{1}{\alpha_i}b\right]s\right)p_l\left(\frac{2}{\alpha_i}\theta s\right), \qquad \theta \in \Theta \tag{18}$$

where

$$\alpha_i := \min\{\alpha \mid 1 < \alpha,\ \Psi_i(\alpha) = 0\} \tag{19}$$

$$\Psi_i(\alpha) := \mathrm{Arg}\left(p_l\left(\left[\kappa - \frac{1}{\alpha}\right]j\pi\right)\right) - \mathrm{Arg}\left(e^{-[\kappa-1]j\pi}\right) \tag{20}$$

and Θ is defined as before.

For every $\kappa > 1$ there exists an integer l_κ^i such that for each integer $l \geq l_\kappa^i$ there exists an α_i satisfying (19). Suppose that $h_i(\theta, s)$ is as specified in (18) with Padé order $l \geq l_\kappa^i$. Then $h_i(\theta, s)$ satisfies properties P_I-1 and P_I-2. (See [28]).

Throughout the rest of this paper, $h_o(\theta, s)$ and $h_i(\theta, s)$ denote the specifically designed Padé-based filters. Also, $\Sigma_o(\theta, s)$ and $\Sigma_i(\theta, s)$ denote the necessary and sufficient comparison systems formed by replacing the delay elements with outer and inner Padé-based parameter-dependent filters of order l. We suppress showing the dependence of these system on Padé order l, to avoid complications in notation.

Definition 4: The *delay margin* ξ^* for the system Σ_d^u about a mean delay value of τ_m, is defined by

$$\xi^* := \sup\left\{\begin{array}{c}\xi < \tau_m \mid \Sigma_d^u(\tau) \text{ is stable with finite-gain on} \\ [\tau_m - \xi, \tau_m + \xi],\ \text{for all } \delta \in \mathbb{U}.\end{array}\right\}$$

Theorem 4: Let ξ^* be the *delay margin* about a mean delay value of τ_m for the (finite-gain stable) system Σ_d^u. Then, for any positive $b < \xi^*$, there exists a comparison system $\Sigma_o^u(\theta, s)$ (developed with high enough Padé order) that proves finite-gain stability of the system Σ_d^u on $[\tau_m - b, \tau_m + b]$ for all $\delta \in \mathbb{U}$.

Proof: Proof is given in [28] and is omitted here for brevity. ■

VII. Analysis

For convenience in notation, throughout the rest of this exposition we denote $\rho := (\tau_m - \alpha_o b)^{-1}$ and $\eta := \frac{1}{2}\alpha_o^{-1}$. Let $\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right]$ be a minimal realization of $p_l(s)I_{m_2}$. Also $n_p = lm_2$ denotes the order of A_p

Theorem 5: Define

$$A_L(\theta) = \begin{bmatrix} A_{11} & A_{12} \\ \theta^{-1}A_{21} & \theta^{-1}A_{22} \end{bmatrix} \tag{21}$$

where:

$$A_{11} = \begin{bmatrix} A + H_d D_p^2 F_d & H_d C_p \\ \rho B_p D_p F_d & \rho A_p \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} H_d D_p C_p \\ \rho B_p C_p \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} \eta B_p F_d & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} \eta A_p \end{bmatrix}$$

The system Σ_d is asymptotically stable for any constant time-delay $\tau \in [\tau_m - b, \tau_m + b]$ if for every $\theta \in (0, b]$ there exists a symmetric and positive definite matrix $X(\theta) \in \mathbb{R}^{(n+2n_p)\times(n+2n_p)}$ satisfying

$$A_L(\theta)^T X(\theta) + X(\theta)A_L(\theta) < 0$$

Proof: See [28]. ■

Theorem 6: The system $\Sigma_d(\tau, s)$ is stable for all $\tau \in [\underline{\tau}, \overline{\tau}]$, and satisfies the worst-case $\mathcal{H}_\infty$ performance bound

$$\gamma_d^* \leq \gamma,$$

if there exist symmetric matrices $X_2 \in \mathbb{R}^{(n+n_p)\times(n+n_p)}$, $X_3 \in \mathbb{R}^{n_p\times n_p}$, a positive definite matrix $X_1 \in \mathbb{R}^{(n+n_p)\times(n+n_p)}$, a negative definite matrix $X_4 \in \mathbb{R}^{n_p\times n_p}$ and a matrix $Z \in \mathbb{R}^{(n+n_p)\times n_p}$ such that:

$$\Lambda(0) < 0,\ \Lambda(b) < 0. \tag{22}$$

and

$$X(b) > 0 \tag{23}$$

where

$$X(\theta) = \begin{bmatrix} X_1 + \theta X_2 & \theta Z \\ \theta Z^T & \theta X_3 + \theta^2 X_4 \end{bmatrix}$$

and

$$\Lambda(\theta) := \begin{bmatrix} \begin{matrix} \Lambda_{11}(\theta) & \Lambda_{12}(\theta) \\ * & \Lambda_{22}(\theta) \end{matrix} & X(\theta)\begin{bmatrix} B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} C_1^T \\ 0 \end{bmatrix} \\ * & -\gamma I_q & D_{11}^T \\ * & * & -\gamma I_q \end{bmatrix}$$

$\Lambda_{11}(\theta) = \mathrm{He}\left(ZA_{21} + (X_1 + \theta X_2)A_{11}\right)$,
$\Lambda_{12}(\theta) = ZA_{22} + \theta A_{11}^T Z + A_{21}^T(X_3 + \theta X_4) + (X_1 + \theta X_2)A_{12}$,
$\Lambda_{22}(\theta) = \mathrm{He}\left(\theta Z^T A_{12} + (X_3 + \theta X_4)A_{22}\right)$.

Proof: First, notice that $X(\theta)$ is concave in θ, and $\Lambda(\theta)$ is affine in θ. Condition (23) along with $X_1 > 0$, and $X_4 < 0$ implies that $X(\theta) > 0$ for all $\theta \in (0, b]$. Similarly, condition (22) implies that $\Lambda(\theta) < 0$ for all $\theta \in (0, b]$. It follows that

$$\begin{bmatrix} \Lambda_{11}(\theta) & \Lambda_{12}(\theta) \\ * & \Lambda_{22}(\theta) \end{bmatrix} < 0, \forall\theta \in (0, b]$$

Equivalently,

$$A_L(\theta)^T X(\theta) + X(\theta)A_L(\theta) < 0, \forall\theta \in (0, b]$$

By Theorem 5, the system $\Sigma_d(\tau, s)$ is stable for all $\tau \in [\underline{\tau}, \overline{\tau}]$. In addition, by Bounded Real Lemma, condition $\Lambda(\theta) < 0$ is equivalent to

$$\left\|\begin{bmatrix} C_1 & 0 \end{bmatrix}[sI - A_L(\theta)]^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + D_{11}\right\|_\infty < \gamma,\ \forall\theta$$

Which immediately implies that

$$\gamma_o^* < \gamma$$

The conclusion then follows from Theorem 3. ■

VIII. Numerical Example

Consider the system

$$A = \begin{bmatrix} -3.09 & 2.67 \\ -9.80 & 2.83 \end{bmatrix}, \; A_d = \begin{bmatrix} 0.57 & 0.02 \\ 1.26 & 0.80 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}, \; D_{11} = 0.5$$

which is unstable for $\tau < 0.2319$ and stable for $\tau \in [0.2319, 0.8609]$. The worst case $\mathcal{H}_\infty$ performance was determined for three intervals within the stable range using exhaustive sweeping of τ and ω; in each case, an upper bound on this was determined via Theorem 6. The results, presented in the table below, demonstrate the effectiveness of the LMI condition.

$[\underline{\tau}, \overline{\tau}]$	γ_d^*	γ (Theorem 6)
$[0.28, 0.60]$	4.962	5.200
$[0.35, 0.75]$	1.873	1.917
$[0.45, 0.75]$	1.051	1.055

IX. Conclusions

The results in this paper extend the results of [14] and [15] to time-delay systems with dynamical uncertainties. Sufficient and necessary comparison systems were formed by replacing the delay elements with parameter-dependent filters, in a similar manner to that previously employed in [14], [15], [28]. It was shown that robust stability of the (finite-dimensional) parameter-dependent comparison system guarantees robust stability of the time-delay system. Moreover, it was shown that the worst case $\mathcal{H}_\infty$ performance of the nominal time-delay system (without dynamical uncertainty) is bounded from above by that of the sufficient *comparison system* and from below by that of the necessary *comparison system*. Finally, it should be pointed out that the results can be readily extended to *robust* performance analysis by introducing a fictitious uncertainty block (see [33]).

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