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Stability of Linear Neutral Time-Delay Systems: Exact Conditions via Matrix Pencil Solutions

Peilin Fu, Silviu-Iulian Niculescu and Jie Chen

***Abstract*— In this paper we study the stability properties of linear neutral delay systems. We consider systems described by both neutral differential-difference and state-space equations, and we seek to determine the delay margin of such systems, that is, the largest range of delay values for which a neutral delay system may preserve its stability. In both cases, we show that the delay margin can be found by computing the eigenvalues and generalized eigenvalues of certain constant matrices, which can be executed efficiently and with high precision. The results extend previously known work on retarded systems, and demonstrate that similar stability tests exist for neutral systems; in particular, the tests require essentially the same amount of computation required for retarded systems.**

***Index Terms*— Time delay, neutral delay systems, stability, delay margin, matrix pencil.**

I. INTRODUCTION

In analyzing time-delay systems, it is useful to study how a system's stability may vary with its delay parameters. In such studies, it is customary to assume that the system is stable when free of delay, and where possible, to determine the maximal range of delay so that the system remains stable. This range is furnished by the notion of *delay margin* [5], which is defined by the critical value of delay at which a system becomes unstable. For linear *retarded* systems with *commensurate* delays, it is known that the delay margin can be computed efficiently by solving a matrix pencil problem [3], [5], [11]. When the delay margin is infinity, the system is *stable independent of delay*; otherwise, a delay-dependent, necessary and sufficient stability condition can be readily checked based on this computational solution.

In this paper we consider linear *neutral* delay systems with commensurate delays. While less well-studied, neutral systems find abundant applications as propagation and diffusion models [12], and in approximations of infinite-dimensional, distributed systems [4]; notable examples of such applications are found in e.g., Halanay and Răsvan [7], concerning, e.g., LC electrical lines, lossless steams, water and gas pipes. The stability analysis of a neutral delay system in general poses a harder problem, which is complicated, and hence rendered more difficult by the neutral part of the system. It appears that necessary and sufficient stability conditions for neutral systems are scarce, and are less efficient than their counterparts for retarded systems. The results available to date include the two-variable criteria developed in [10], and the frequency-sweeping tests obtained in [2]; other relevant conditions are found in, e.g., [8]. Useful notwithstanding, these results generally suffer in one way or another in computational efficiency. For example, the two-variable criteria require symbolic computation, while the frequency-sweeping methods may be inadequate in numerical precision.

Our work in the present paper is aimed at developing more efficient stability tests for neutral delay systems. For this purpose, we extend the techniques of [3] and show that similar algorithms can be found for computing the delay margin of neutral systems. Our main results consist of formulas that seek to compute the eigenvalues and generalized eigenvalues of certain constant matrices, which, unlike frequency-sweeping tests, can be executed in finite steps. Hence, analogous to their counterparts for retarded systems, the results obtained herein can also be easily implemented, and they constitute a significant improvement in both efficiency and precision.

Our contribution can be summarized as follows. In Section 2, we introduce the stability notions and present a number of preliminary facts. Section 3 contains our main results, where we first consider neutral delay systems described by differential-difference equations in Section 3.1, and next those modelled by state-space equations in Section 3.2; while these descriptions are mutually exchangeable, from a computational perspective, it will prove advantageous to treat the two cases separately. For both types of systems, we derive a computational procedure consisting of two steps, each of which amounts to computing either the eigenvalues or generalized eigenvalues of constant matrices. These results are then followed by illustrative examples given in Section 4. The paper concludes in Section 5.

II. PRELIMINARY RESULTS

We begin with a brief description of our notation. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers. Denote the open right half plane by $\mathbb{C}_+ := \{s : \Re(s) > 0\}$, the closed right half plane by $\overline{\mathbb{C}}_+$, and the imaginary axis by $\partial\mathbb{C}_+$. Similarly, denote the open unit disc by $\mathbb{D}$, the unit circle by $\partial\mathbb{D}$, and the closed exterior of the unit disc by $\mathbb{D}^c$. For any $z \in \mathbb{C}$, we denote its complex conjugate by $\bar{z}$. For a matrix A, denote its spectrum by $\sigma(A)$, and its spectral

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Peilin Fu is with the Department of Electrical Engineering, University of California, Riverside, CA 92521 pfu@ee.ucr.edu

Silviu-Iulian Niculescu is with Université de Technologie de Compiègne, Centre de Recherche de Royallieu, BP 20529, 60205, Compiègne, France niculescu@hds.utc.fr

Jie Chen is with the Department of Electrical Engineering, University of California, Riverside, CA 92521 jchen@ee.ucr.edu

radius by $\rho(A)$. For a matrix pair $(A,\ B)$, denote the set of all its generalized eigenvalues by $\sigma(A,\ B)$, i.e.,

$$\sigma(A,\ B) := \{\lambda \in \mathbb{C} :\ \det(A - \lambda B) = 0\}.$$

Note that the number of finite generalized eigenvalues for $(A,\ B)$ is at most equal to the rank of B. Also, if the rank of B is constant, then the finite generalized eigenvalues of the pair $(A,\ B)$ are continuous with respect to the elements of A and B. Let $A \oplus B$ denote the Kronecker sum, and $A \otimes B$ the Kronecker product, of the matrices A and B. Properties of Kronecker sum and product relevant to our subsequent development are summarized below (see, e.g., [6]):

- For matrices $A,\ B,\ C,\ D$ with compatible dimensions, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$;
- Let A and B be invertible. Then, $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$;
- For square matrices A and B, every eigenvalue of $A \oplus B$ is the sum of the eigenvalues of A and B.

We consider neutral linear, time-invariant systems described by the differential-difference equation

$$y^{(n)}(t) + \sum_{k=1}^{q} b_k y^{(n)}(t - k\tau) + \sum_{i=0}^{n-1} \sum_{k=0}^{q} a_{ki} y^{(i)}(t - k\tau) = 0 \tag{1}$$

where $\tau \geq 0$, the coefficients b_k, $k = 1,\ \cdots,\ q$ and a_{ki}, $k = 0,\ 1,\ \cdots,\ q$, $i = 0,\ 1,\ \cdots,\ n-1$ are known. Similarly, we also consider the state-space representation

$$\dot{x}(t) - \sum_{k=1}^{q} B_k\ \dot{x}(t-k\tau) = A_0\ x(t) + \sum_{k=1}^{q} A_k\ x(t-k\tau), \tag{2}$$

where $\tau \geq 0$, B_k and A_k, $k = 1,\ \cdots,\ q$ are given matrices. Note that to simplify our presentation, in both descriptions we have assumed, with no loss of generality, that the neutral and retarded parts have the same number of delays. This assumption can be easily relaxed, say, by padding zero coefficients or matrices appropriately. It is not difficult to see that these two descriptions are mutually exchangeable. However, transformation from one to another will result in additional computation which is otherwise unnecessary, and hence is less desirable; indeed, it will prove more beneficial to study these descriptions individually.

The characteristic function of the system (1) is given by the quasipolynomial [9], [5]

$$a\left(s,\ e^{-\tau s}\right) = \sum_{k=0}^{q} a_k(s) e^{-k\tau s}, \tag{3}$$

with

$$\begin{aligned} a_0(s) &= s^n + \sum_{i=0}^{n-1} a_{0i} s^i, \\ a_k(s) &= b_k s^n + \sum_{i=0}^{n-1} a_{ki} s^i, \quad k = 1,\ \cdots,\ q. \end{aligned}$$

It is well-known that the system is stable for a given $\tau \geq 0$ if and only if for some $\epsilon < 0$,

$$a\left(s,\ e^{-\tau s}\right) \neq 0, \quad \forall s \in \overline{\mathbb{C}}_{\epsilon+}, \tag{4}$$

where $\mathbb{C}_{\epsilon+} := \{s :\ \Re(s) > \epsilon\}$. The system is said to be *stable independent of delay* if the condition (4) holds for all $\tau \geq 0$. Analogously, the characteristic quasipolynomial of the system (2) is

$$p\left(s,\ e^{-\tau s}\right) = \det\left(s\left(I - \sum_{k=1}^{q} B_k e^{-k\tau s}\right) - \sum_{k=0}^{q} A_k e^{-k\tau s}\right) \tag{5}$$

and the system is stable for a given $\tau \geq 0$ if and only if for some $\epsilon < 0$,

$$p\left(s,\ e^{-\tau s}\right) \neq 0, \quad \forall s \in \overline{\mathbb{C}}_{\epsilon+}, \tag{6}$$

and stable independent of delay if (6) holds for all $\tau \geq 0$.

It is known that for a neutral delay system to be stable, it is necessary that its neutral part must be stable. For the systems (1) and (2), this requirement concerns the stability of the difference equations

$$y(t) + \sum_{k=1}^{q} b_k y(t - k\tau) = 0, \tag{7}$$

and

$$x(t) - \sum_{k=1}^{q} B_k\ x(t - k\tau) = 0. \tag{8}$$

An important fact (see, e.g., [9], [5]) states that the stability of these difference equations construes a global property with respect to the delay parameter $\tau \geq 0$; that is, whenever the equations admit stable solutions for some $\tau \geq 0$, they will be stable for all $\tau \geq 0$, or equivalently, stable independent of delay. The following facts provide necessary and sufficient conditions for the stability of (7) and (8), respectively.

Fact 1 *The difference equation (7) is stable for all $\tau \geq 0$ if and only if*

$$\rho(N_d) < 1, \tag{9}$$

where

$$N_d := \begin{bmatrix} -b_1 & \cdots & -b_{q-1} & -b_q \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Equivalently, define the polynomial

$$b(z) = z^q + b_1 z^{q-1} + \cdots + b_q. \tag{10}$$

Then (7) is stable for all $\tau \geq 0$ if and only if $b(z)$ is Schur-stable; that is, $b(z)$ has all its zeros in $\mathbb{D}$.

Fact 2 *The difference equation (8) is stable for all $\tau \geq 0$ if and only if*

$$\rho(N_s) < 1, \tag{11}$$

where

$$N_s := \begin{bmatrix} B_1 & \cdots & B_{q-1} & B_q \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}.$$

Throughout this paper, we shall assume that the conditions (9) and (11) hold. It is important to point out that under these assumptions, the conditions (4) and (6) need to hold only for $\mathbb{C}_+$; in other words, $\mathbb{C}_{\epsilon+}$ can be replaced by $\mathbb{C}_+$. We shall also assume that both the system (1) and (2) are stable at $\tau = 0$; that is, in the delay-free case, both systems are stable. Our task is to find the delay margin for these systems, defined as

$$\tau_d := \inf\left\{\tau :\ a\left(s,\ e^{-\tau s}\right) = 0 \text{ for some } s \in \overline{\mathbb{C}}_+\right\}$$

and

$$\tau_s := \inf\left\{\tau :\ p\left(s,\ e^{-\tau s}\right) = 0 \text{ for some } s \in \overline{\mathbb{C}}_+\right\},$$

respectively. In other words, we want to determine the smallest deviation of the delay value so that the system becomes unstable.

III. Main Results

We now present our main results. We shall first consider the system (1) in Section 3.1, and next the system (2) in Section 3.2. In both cases, we derive readily computable expressions for the delay margin, which require only the computation of eigenvalues and generalized eigenvalues.

A. Delay Margin for Differential-Difference Equations

Our development seeks to generalize that of [3]. Consider the quasipolynomial (3). Under the assumption that (9) holds, it is known that the zeros of $a\left(s,\ e^{-\tau s}\right)$ vary continuously with $\tau \geq 0$. As such, the delay margin τ_d reduces to

$$\tau_d = \inf\left\{\tau :\ a\left(j\omega,\ e^{-j\tau\omega}\right) = 0 \text{ for some } \omega \in \mathbb{R}_+\right\}, \tag{12}$$

where it suffices to consider $\omega \in \mathbb{R}_+$ since the complex zeros of $a\left(s,\ e^{-\tau s}\right)$ are conjugate symmetric. We call such $\omega \in \mathbb{R}_+$ that $a\left(j\omega,\ e^{-\tau\omega}\right) = 0$ the crossing frequencies of $a(s, e^{-\tau s})$. The expression (12) suggests that τ_d can be determined once the crossing frequencies are determined, or alternatively, when all the zeros $(s,\ z)$ of the bivariate polynomial $a(s,\ z)$ are located, such that $s \in \partial\mathbb{C}_+$ and $z \in \partial\mathbb{D}$. Our following result sets out precisely to accomplish this goal.

Theorem 1 *Suppose that the system (1) is stable at $\tau = 0$, and that the condition (9) holds. Define*

$$\begin{aligned}
T_n &:= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ b_{q-1} & b_{q-2} & \cdots & 1 \end{bmatrix}, \\
T_i &:= \begin{bmatrix} a_{0i} & 0 & \cdots & 0 \\ a_{1i} & a_{0i} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{q-1,i} & a_{q-2,i} & \cdots & a_{0i} \end{bmatrix},\ i = 0, \cdots, n-1, \\
H_n &:= \begin{bmatrix} b_q & b_{q-1} & \cdots & b_1 \\ 0 & b_q & \cdots & b_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_q \end{bmatrix}, \\
H_i &:= \begin{bmatrix} a_{qi} & a_{q-1,i} & \cdots & a_{1i} \\ 0 & a_{qi} & \cdots & a_{2i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{qi} \end{bmatrix},\ i = 0, 1, \cdots, n-1, \\
P_i &:= \begin{bmatrix} (j)^i T_i & (j)^i H_i \\ (-j)^i H_i^T & (-j)^i T_i^T \end{bmatrix},\ i = 0, 1, \cdots, n.
\end{aligned}$$

Then P_n is invertible. Define further

$$P := \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -P_n^{-1}P_0 & -P_n^{-1}P_1 & \cdots & -P_n^{-1}P_{n-1} \end{bmatrix}.$$

Then, $\tau_d = \infty$ if $\sigma(P) \cap \mathbb{R}_+ = \emptyset$, or $\sigma(P) \cap \mathbb{R}_+ = \{0\}$. Otherwise, let $\sigma(P) \cap \mathbb{R}_+ = \{\omega_k :\ \omega_k \neq 0,\ k = 1, \cdots, l,\ l \leq 2nq\}$, and define

$$\begin{aligned}
F(s) &:= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0(s) & -a_1(s) & \cdots & -a_{q-1}(s) \end{bmatrix}, \\
G(s) &:= \mathit{diag}\left(1 \ \cdots \ 1 \ \ a_q(s)\right).
\end{aligned}$$

If $\sigma(F(j\omega_k),\ G(j\omega_k)) \cap \partial\mathbb{D} = \emptyset$ for all $k = 1, \cdots, l$, then $\tau_d = \infty$; otherwise,

$$\tau_d = \min_k \min_i \frac{\alpha_k^{(i)}}{\omega_k},$$

with $\sigma(F(j\omega_k),\ G(j\omega_k)) \cap \partial\mathbb{D} = \{e^{-j\alpha_k^{(i)}} :\ \alpha_k^{(i)} \in [0,\ 2\pi],\ i = 1, \cdots, m,\ m \leq q\}$.

Consider for a fixed $s \in \mathbb{C}$ the bivariate polynomial $a(s, z)$. Before establishing Theorem 1, we first construct the Schur-Cohn matrix [1] associated with the complex polynomial $a(s, z)$, defined as

$$\Delta(s) := \begin{bmatrix} \Delta_1(s) & \Delta_2(s) \\ \Delta_2^H(s) & \Delta_1^H(s) \end{bmatrix},$$

where

$$\Delta_1(s) := \begin{bmatrix} a_0(s) & 0 & \cdots & 0 \\ a_1(s) & a_0(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{q-1}(s) & a_{q-2}(s) & \cdots & a_0(s) \end{bmatrix},$$

$$\Delta_2(s) := \begin{bmatrix} a_q(s) & a_{q-1}(s) & \cdots & a_1(s) \\ 0 & a_q(s) & \cdots & a_2(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_q(s) \end{bmatrix}.$$

It follows from the *Orlando formula* [1] that

$$\det(\Delta(s)) = (-1)^q |a_q(s)|^{2q} \prod_{i,j=1}^{q} (1 - z_i \bar{z}_j), \tag{13}$$

where for a given s, z_i, $i = 1, \cdots, q$, are the zeros of the polynomial $a(s, z)$. From this formula, it is clear that for any $s \in \mathbb{C}$, $\det(\Delta(s)) = 0$ whenever $a(s, z)$ have zeros in $\mathbb{D}^c$. Consequently, by finding the solutions to

$$\det(\Delta(j\omega)) = 0, \tag{14}$$

we can determine all $\omega \in \mathbb{R}_+$ such that $a(j\omega, z) = 0$ for some $z \in \partial\mathbb{D}$. This underlines our proof for Theorem 1.

Like its counterpart for retarded systems [3], Theorem 1 shows that the delay margin for the neutral differential-difference equation (1) (or equivalently, the quasipolynomial (7)) can be determined by first computing the eigenvalues of a constant matrix, and next the generalized the eigenvalues of a constant matrix pair, both of which can be executed efficiently and reliably. It is especially reassuring that the computation herein is the same as that required for computing the delay margin of a retarded differential-difference equation (cf. [3]). In other words, the complication incurred by the neutral dynamics in fact does not lead to any additional computational load.

B. Delay Margin for State-Space Models

We now consider neutral systems described by the state-space form (2). In a similar manner, we attempt to determine the critical frequencies $\omega \in \mathbb{R}_+$ at which the characteristic quasipolynomial $p(j\omega, e^{-j\tau\omega}) = 0$, that is, ω are crossing frequencies of $p(s, e^{-\tau s})$. Denote

$$\mathcal{A}(z) := \left(I - \sum_{k=1}^{q} B_k z^k \right)^{-1} \sum_{k=0}^{q} A_k z^k.$$

Then our task amounts to finding all such $z_k \in \partial\mathbb{D}$ that

$$\sigma(\mathcal{A}(z_k)) \cap \partial\mathbb{C}_+ \neq \emptyset.$$

Whenever this is the case, there will exist $j\omega_k \in \partial\mathbb{C}_+$ and $z_k \in \partial\mathbb{D}$ such that $j\omega_k \in \sigma(\mathcal{A}(z_k))$, or more explicitly,

$$\det(j\omega_k I - \mathcal{A}(z_k)) = 0. \tag{15}$$

By finding all such $\omega_k \in \mathbb{R}_+$ and $z_k \in \partial\mathbb{D}$, we may compute the delay margin τ_s analogously as in Theorem 1.

Noting the property of the Kronecker sum alluded to at the end of Section 2, it is useful to observe that the condition (15) is equivalent to

$$\det\left[\mathcal{A}(z_k) \oplus \mathcal{A}^H(z_k)\right] = 0. \tag{16}$$

Let

$$A(z) = \sum_{k=0}^{q} A_k z^k, \qquad B(z) = I - \sum_{k=1}^{q} B_k z^k.$$

Invoking the properties of the Kronecker product, it follows that for any $z \in \partial\mathbb{D}$,

$$\begin{aligned} & \det\left[\mathcal{A}(z) \oplus \mathcal{A}^H(z)\right] \\ = \; & \det\left[\left(B^{-1}(z)A(z)\right) \otimes I + I \otimes \left(A^H(z)B^{-H}(z)\right)\right] \\ = \; & \det\left[\left(B^{-1}(z) \otimes I\right)\left(A(z) \otimes I\right) + \left(I \otimes A^H(z)\right) \right. \\ & \left. \left(I \otimes B^{-H}(z)\right)\right] \\ = \; & \det\left[\left(B(z) \otimes I\right)^{-1}\left(A(z) \otimes I\right) + \left(I \otimes A^H(z)\right) \right. \\ & \left. \left(I \otimes B^H(z)\right)^{-1}\right] \\ = \; & \det\left[\left(B(z) \otimes I\right)^{-1}\right] \det\left[A(z) \otimes B^H(z) \right. \\ & \left. + B(z) \otimes A^H(z)\right] \det\left[\left(I \otimes B^H(z)\right)^{-1}\right]. \end{aligned}$$

We are thus led to the following theorem.

Theorem 2 *Suppose that the system (2) is stable at $\tau = 0$, and that the condition (11) holds. Let*

$$H_k = \sum_{i=\max\{0,k-q\}}^{\min\{k,q\}} \left[A_{k-i} \otimes B_{q-i}^T + B_{k-i} \otimes A_{q-i}^T\right], \quad k = 0, 1, \cdots, 2q,$$

$$Q_k = \begin{cases} I \otimes A_{q-k}^T - H_k & k = 0, 1, \cdots, q-1, \\ A_0 \oplus A_0^T - H_q & k = q \\ A_{k-q} \otimes I - H_k & k = q+1, \cdots, 2q \end{cases},$$

with $B_0 = 0$. Define further

$$U := \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & Q_{2q} \end{bmatrix},$$

$$V := \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -Q_0 & -Q_1 & \cdots & -Q_{2q-1} \end{bmatrix}.$$

Then, $\tau_s = \infty$ if $\sigma(V, U) \cap \partial\mathbb{D} = \emptyset$. If, however, $\sigma(V, U) \cap \partial\mathbb{D} \neq \emptyset$ and $\sigma(\mathcal{A}(z_k)) = \{0\}$ for all $z_k \in \sigma(V, U) \cap \partial\mathbb{D}$, then $\tau_s = \infty$ as well. Otherwise, let $\sigma(V, U) \cap \partial\mathbb{D} = \{e^{j\alpha_k} : \alpha_k \in [0, 2\pi],\ k = 1, \cdots, m,\ m \leq 2n^2 q\}$. If $\sigma(\mathcal{A}(e^{-j\alpha_k})) \cap \partial\mathbb{C}_+ = \emptyset$ for all $k = 1, \cdots, m$, then $\tau_s = \infty$; otherwise

$$\tau_s = \min_k \min_i \frac{\alpha_k}{\omega_k^{(i)}},$$

with $\omega_k^{(i)} \in \mathbb{R}_+$, $\omega_k^{(i)} \neq 0$ *and* $j\omega_k^{(i)} \in \sigma(\mathcal{A}(e^{-j\alpha_k})) \cap \partial\mathbb{C}_+$ *for* $i = 1, \cdots, l, \; l \leq m$.

To see Theorem 2 in a more transparent light, it is instructive to consider the special case $q = 1$, that is, the neutral system given by

$$\dot{x}(t) - B_1\, \dot{x}(t-\tau) = A_0\, x(t) + A_1\, x(t-\tau), \qquad \tau \geq 0. \quad (17)$$

In this case, $U, \; V$ are $2n^2 \times 2n^2$ real constant matrices, and

$$\mathcal{A}(z) = (I - B_1 z)^{-1}\, (A_0 + A_1 z)$$

is an $n \times n$ complex matrix. In general, Theorem 2 states that to determine the delay margin, it suffices to compute first the generalized eigenvalues of the $2n^2q \times 2n^2q$ matrix pair $(V, \; U)$, and subsequently the eigenvalues of the $n \times n$ matrix $\mathcal{A}(z_k)$, where $z_k \in \partial\mathbb{D}$ is a generalized eigenvalue of the pair $(V, \; U)$. Moreover, it is evident that when $B_k = 0$ for $k = 1, \; \cdots, \; q$, we have $H_k = 0$ for $k = 0, \; 1, \; \cdots, \; 2q$, and so Theorem 2 reduces to its counterpart for retarded systems, given in [3].

IV. Illustrative Examples

In this section we use a number of examples to illustrate the preceding results. The first example below serves to demonstrate the numerical effectiveness of our method, especially for high order systems with multiple delays.

Example 1 Consider the neutral system described by the quasipolynomial

$$\begin{aligned} a(s, e^{-\tau s}) \;\; &= \;\; (s^4 + 2s^3 + 5s^2 + 3s + 2) + (0.6s^4 + s^2 \\ &\quad +2)e^{-\tau s} + (0.11s^4 + s^2 + s + 2)e^{-2\tau s} \\ &\quad +(0.006s^4 + 2s^3 + 5s)e^{-3\tau s}. \qquad (18) \end{aligned}$$

which is a 4th-order quasipolynomial with three commensurate delays. The polynomial $b(z)$ resulting from the neutral part is given by

$$b(z) = z^3 + 0.6z^2 + 0.11z + 0.006.$$

A direct computation shows that $\rho(N_d) = 0.3000$. As such, the polynomial $b(z)$ is stable; in fact, its zeros are located at $z = -0.3000, \; -0.2000, \; -0.1000$. It is also easy to verify that the quasipolynomial (18) is stable at $\tau = 0$. The resultant polynomial at $\tau = 0$ is

$$a(s) = 1.716s^4 + 4s^3 + 7s^2 + 9s + 6,$$

which has zeros at $s = -0.0594 \pm 1.5117i, \; -1.1061 \pm 0.5515i$. We proceed to compute the delay margin τ_d based on Theorem 1. Toward this end, we first find the imaginary eigenvalues $j\omega_k$ of the 24×24 matrix P, such that $\omega_k > 0$. The corresponding unitary generalized eigenvalues of the 3×3 matrix pencil $(F(j\omega_k), \; G(j\omega_k))$, along with their phase angles, are then computed. Table I gives the computation results. From these computations, we found immediately $\tau_d = 0.1356$.

TABLE I

Crossing Frequencies, Generalized Eigenvalues and Phase Angles in Example 1

Crossing Frequency ω_k	Generalized Eigenvalue z_k	Phase Angle α_k
3.0171	0.0554 + 0.9985i	4.7678
1.8723	0.6123 + 0.7906i	5.3714
1.4432	0.9809 - 0.1945i	0.1957
0.8692	-0.9717 - 0.2362i	2.9032
0.2740	-0.8990 + 0.4380i	3.5950

Alternatively, we may also compute the delay margin based on the formula in Theorem 2. For this purpose, we represent the system in the state-space form (2), with

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.11 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.006 \end{bmatrix}.$$

It follows readily by computation that $\rho(N_s) = \rho(N_d) = 0.3000$. We then compute the generalized eigenvalues of the 96×96 matrix pencil $(V, \; U)$, and the eigenvalues of the 4×4 matrices $\mathcal{A}(z_k)$, where z_k are the unitary generalized eigenvalues of $(V, \; U)$. The computation generates the same set of crossing frequencies and unitary complex numbers as those given in Table 1, which consequently lead to $\tau_s = \tau_d = 0.1356$. ■

It is of interest to compare the above system to its retarded counterpart

$$\begin{aligned} a(s, \; e^{-\tau s}) \;\; &= \;\; (s^4 + 2s^3 + 5s^2 + 3s + 2) + (s^2 + 2)e^{-\tau s} \\ &+ \;\; (s^2 + s + 2)e^{-2\tau s} + (2s^3 + 5s)e^{-3\tau s}. \end{aligned}$$

This corresponds to replacing the coefficients $b_k, \; k = 1, \; 2, \; 3$ by zeros. It is known from [3], [5] that in this case the delay margin is 0.3786. Example 1 thus shows that the presence of neutral dynamics renders the delay margin smaller. This is plausible; indeed, in the extreme, it is possible that the neutral part may even destabilize an otherwise stable retarded system at $\tau = 0$, lest that any delay margin may exist. Nevertheless, it is also possible that the delay margin may increase due to the neutral effect, as shown by the following example.

Example 2 We examine the first-order neutral system

$$\dot{x}(t) + \beta\dot{x}(t-\tau) = -ax(t) - bx(t-\tau),$$

where we assume that $b > 0$, $a < 0$, and $|\beta| < 1$. It is clear that under these assumptions, the neutral part is stable, and that the system is stable at $\tau = 0$ if and only if $a + b > 0$. These conditions together imply that $b > |a|$. In the case $\beta = 0$, i.e, when the system is only retarded, its delay margin is given by (pp. 40, [5])

$$\tau_r = \frac{\cos^{-1}(\frac{|a|}{b})}{\sqrt{b^2 - a^2}}.$$

More generally, for any β such that $|\beta| < 1$, we may calculate the delay margin using Theorem 1. We begin with the matrix

$$P = -\frac{j}{1-\beta^2}\begin{bmatrix} -(a+\beta b) & -(b+\beta a) \\ b+\beta a & a+\beta b \end{bmatrix},$$

which has a single positive eigenvalue

$$\omega^* = \sqrt{\frac{b^2 - a^2}{1-\beta^2}}\ .$$

The matrix pair $(F(s),\ G(s))$ is given as

$$F(s) = -(s + a), \qquad G(s) = \beta s + b,$$

whose generalized eigenvalue for $s = j\omega^*$ is found to be

$$\lambda^* = -\frac{j\omega^* + a}{j\beta\omega^* + b}.$$

It is trivial to verify that $\lambda^* \in \partial\mathbb{D}$. Let $\lambda^* = e^{-j\alpha^*}$. It is evident that

$$\alpha^* = \pi - \angle\left(\frac{j\omega^* + a}{j\beta\omega^* + b}\right).$$

In view of Theorem 1, the delay margin can then be determined as $\tau_d = \alpha^*/\omega^*$. Set in particular $a = -(\sqrt{2}/2)$, $b = 1$, and plot τ_d vs β. Fig. 1 shows how τ_d may vary with the value of β.

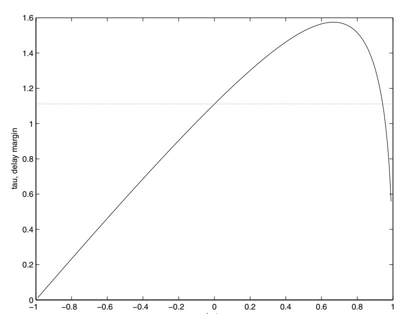

Fig. 1. The delay margin τ_d vs parameter β in Example 2.

Note that in this figure, the dotted line indicates the delay margin ($\tau_d = 1.1107,\ \beta = 0$) for the system's retarded part. The example exhibits that even in rather simple instances, the neutral dynamics may either increase or reduce the delay margin.

V. Conclusion

In this paper we have studied the stability of linear neutral delay systems, modelled both as differential-difference equations or in state-space form. Our purpose is to compute the exact delay margin of such systems, which defines a critical value of the time delay at which the system loses its stability. We extended and further developed earlier matrix pencil techniques to tackle this problem. Our main results consist of readily implementable, computation-oriented formulas, requiring only the solutions of eigenvalue and generalized eigenvalue problems associated with certain constant matrices. As a consequence, our results insure that the delay margin can be computed efficiently and with high numerical precision.

Future extension of this work can be pursued in a more general setting, e.g., for singular, neutral delay systems. Our techniques can also be extended to 2-D systems, and to analyze stability properties in the entire range of delay values. The latter analysis seeks to characterize the stability and instability of a time-delay system over all possible delay values partitioned into successive intervals. These extensions are currently ongoing and will be reported elsewhere.

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