Current Observer for Sampled-Data Fuzzy Systems

Ji-Chang Lo and Chien-Hao Su

***Abstract*— We investigate an observer synthesis and stability analysis involving sampled-data fuzzy systems arising from rapid growth of digital observer implementations. The underlying error system is shown to be asymptotically stable when intersampling effects are taken into account. Being a periodically time-varying hybrid (discrete/continuous) system, the Riccati inequality associated with the sampled-data fuzzy system poses difficulties for synthesis and analysis using LMI convex programming. To resolve the difficulties, a convex solution is assumed and the main result is expressed in LMI formulation. Lastly the validity and applicability of the approach are demonstrated by an example.**

Keywords: Current observer; Sampled-data systems; Hybrid systems; TS fuzzy model; Linear matrix inequality (LMI)

I. INTRODUCTION

Sampled-data observer that is capable of observing a general continuous nonlinear system subject to digital (sampled) inputs is of great interest in control community. In this paper the nonlinear system is represented by T-S fuzzy models [1], [2], [3], [4]. In linear system literature, many synthesis methods have been proposed for sampled-data problems. Among them, a widely used approach is based on lifting technique transforming the sampled-data problem into a discrete system with finite-dimensional state vector but with infinite-dimensional input and outputs [5], [6]. A more direct method applying observer theory of time-varying system [7], [8] to the sampled-data system leads to a periodically time-varying hybrid discrete/continuous system and the solution of the problems is obtained in terms of two mixed/coupled discrete/continuous Riccati equations [9], [10]. The lifting method and Riccati-based method have been shown to be connected and lead to identical solutions [10], [11]. Although the two methods yield identical solutions, an advantage of the Riccati-based method is that it is more transparent and it is developed using classic linear quadratic optimal control theory that has been well established in control community. While there are much research on H_2 and H_∞ sampled-data control for the linear systems [12], [13], [14] and nonlinear systems [15] and references therein, only a few results on sampled-data control for the fuzzy systems are reported [16]-[21]. Among them, [16], [20] used locally digital redesign method, matching continuous state to discrete controlled state at every sampling instant whilst reports [17], [18], [19], [21] sought Riccati-based dynamic inequalities without LMI considerations. Inspired by the aforementioned works that stops short of LMI convex solutions, we extend the works into an LMI structure that neither assumed decision variables nor solving a differential dynamic Riccati inequality is required.

Knowing the current development, we adopt Riccati-based notions in our investigation, focusing on sampled-data observer synthesis for fuzzy systems, aiming at synthesizing a stabilizing current observer such that the estimator with intersampling behavior is asymptotically stable.

The organization of this paper is as follows. Section II reviews the fundamentals concerning sampled-data systems and establishes connections between sampled-data systems and jump systems. Section III presents the main results where an LMI-based sufficient condition for the underlying problems is addressed. An illustrative example is considered in Section IV and concluding remarks are made in Section V.

Notations: The symbol $\star$ is used for terms that are induced by symmetry. The notation Y_μ stands for $\sum_{i=1}^{r} \mu_i Y_i$ where the superscript r is the number of fuzzy rules. To ease the presentation, we use shorthand notations k to stand for kT and k^- for kT^- whenever no confusion is made. Here $x(k^-) = x(kT^-) = \lim_{\epsilon\to 0} x(kT-\epsilon)$, meaning the state $\hat{x}(t)$ is continuous from the right and may be left discontinuous. We assume that the pathological sampling is excluded.

II. PRELIMINARIES

Given the following continuous-time system that is represented by T-S fuzzy models whose consequent parts are local submodels characterized by linear systems.

If $z_1(t)$ is M_{i1} and ... and $z_p(t)$ is M_{ip} then

$$\begin{aligned} \dot{x}(t) &= A_i x(t), \quad kT \le t < (k+1)T \\ x(k) &= x(k^-) \\ y(t) &= C_{2i} x(t) \end{aligned}$$

where $z_1, \cdots, z_p$ represent the premise variables depending on the measurable state variables; $M_{i1}, \cdots, M_{ip}$ are fuzzy sets for rule i; A_i and C_{2i} are real constant matrices. Moreover, consider a current observer that samples at every sampling instant kT, $k = 0, 1, 2, \cdots$.

If $z_1(t)$ is M_{i1} and ... and $z_p(t)$ is M_{ip} then

$$\begin{aligned} \dot{\hat{\xi}}(t) &= A_i \hat{x}(t), \quad kT \le t < (k+1)T && (1) \\ \hat{x}(k) &= \hat{\xi}(k^-) + L_i(k)[y(k) - \hat{y}(k)] && (2) \\ \hat{y}(k) &= C_{2i} \hat{x}(k) \end{aligned}$$

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where $L_i(k)$ is the observer gain matrix to be determined. Here $\hat{\xi}(t)$ is a first estimate of the state at time t, based on the dynamic of the system (1). This first estimate is then corrected in (2) when the measurement at time k arrives. By using a standard fuzzy inference method, the inferred fuzzy model of interest, after normalization, is described by the following defuzzified jump fuzzy system

$$\begin{cases} \dot{x}(t) = A_\mu x(t), \quad kT \le t < (k+1)T \\ x(k) = x(k^-) \\ y(k) = C_{2\mu} x(k) \end{cases} \tag{3}$$

$$\begin{cases} \dot{\hat{\xi}}(t) = A_\mu \hat{x}(t), \quad kT \le t < (k+1)T \\ \hat{x}(k) = \hat{\xi}(k^-) + L_\mu(k)[y(k) - \hat{y}(k)] \\ \hat{y}(k) = C_{2\mu}\hat{\xi}(k) \end{cases} \tag{4}$$

where $x \in R^n$ and $\mu_i \ge 0$ are the state, control input, and grade of membership, respectively. Here, we assume (A_i, C_i) pairs are completely observable. To continue, define an error state variable

$$e(\cdot) = x(\cdot) - \hat{x}(\cdot)$$

such that the system (3)-(4) is recast into an error system formulation.

$$\dot{e}(t) = A_\mu e(t), \quad kT \le t < (k+1)T \tag{5}$$

$$e(k) = \bar{A}_{d\mu\mu} e(k^-), \quad k = 0, 1, 2, \cdots \tag{6}$$

where $\bar{A}_{d\mu\mu} = (I - L_\mu(k)C_{2\mu})$ and the state is assumed to be right continuous with possible discontinuity on the left at $t = (k+1)T$. Simply note that setting $\bar{A}_{d\mu\mu} = I$ results in a continuous-time system, while setting $A_\mu = 0$ leads to a discrete-time system. A focal point to make is that the jump fuzzy system (5)-(6) of continuous/discrete-time expressions, respectively, whose stability problem can be analyzed by Lyapunov theory. Thus, continuous- and discrete-time Riccati equations are considered simultaneously under the framework of jump systems [10].

III. MAIN RESULTS

The stability analysis and observer synthesis problem for the sampled-data nonlinear fuzzy systems are stated formally as follows.

Theorem 1: The error dynamic fuzzy system (5)-(6) is globally asymptotically stable over a finite horizon $0 \le t < NT$ if for each interval $kT \le t < (k+1)T$, $k = 0, 1, 2, \cdots, N-1$, there exist two positive definite matrices $Q^{-1}(k^-) > 0$, $Q(k) > 0$, slope $Y(k)$ and observer gain $L_i(k)$ such that the following LMI conditions are satisfied:

$$Q(k)A_i' + A_i Q(k) - Y(k) < 0 \tag{7}$$

$$Q(k)A_i' + A_i Q(k) - Y(k) + T(Y(k)A_i' + A_i Y(k)) < 0 \tag{8}$$

and

$$\begin{bmatrix} -Q(k) & I - G_{ij} - G_{ji} \\ I - G_{ij}' - G_{ji}' & -Q^{-1}(k^-) \end{bmatrix} < 0 \tag{9}$$

where $G_{ij} = L_i(k)C_{2j}/2$.

Proof: The proof is proceeded in two stages because the energy is considered separately, as given below.

$$V(t) = e'(t)P(t)e(t), \quad kT \le t < (k+1)T$$
$$V(k) = e'(k)P(k)e(k), \quad kT^- < k \le kT.$$

where $k = 0, 1, 2, \cdots, N-1$.

It is worth emphasizing that the aforementioned Lyapunov function is continuous over the specified ranges and is continuously differentiable except at the finite sampling instants where the discrete Lyapunov comes to establish decreases of energy at each jump state. This two-stage proof is distinct from most Lyapunov methods for continuous-time systems where finite jumps do not occur. Employing both continuous and discrete Lyapunov functions, we guarantee stable jumps at sampling instants and stable trajectory within intervals as well.

(A) For the continuous system (5) over the interval $kT \le t < (k+1)T$, we define a Lyapunov function $V = e'(t)P(t)e(t)$ whose time derivative is expressed as

$$\begin{aligned} \dot{V}(t) &= \dot{e}'(t)P(t)e(t) + e'(t)\dot{P}(t)e(t) + e'(t)P(t)\dot{e}(t) \\ &= e'(t)[A_\mu' P(t) + P(t)A_\mu + \dot{P}(t)]e(t). \end{aligned}$$

By Lyapunov theorem, a sufficient condition to the continuous part is if there exists a time-varying symmetric matrix $P(t) > 0$ satisfying the following *dynamic* Riccati inequality

$$A_\mu' P(t) + P(t)A_\mu + \dot{P}(t) < 0, \quad kT \le t < (k+1)T$$

which is equivalent to

$$Q(t)A_\mu' + A_\mu Q(t) - \dot{Q}(t) < 0, \quad kT \le t < (k+1)T \tag{10}$$

where $P^{-1}(t) = Q(t)$, $P^{-1}(t)\dot{P}(t)P^{-1}(t) = -\dot{Q}(t)$.

We note that the differential inequality (10) is moving forwards in time with possible discontinuity at sampling instants. It is also noted that the time-varying $Q(t)$ makes the application of LMI solver difficult. To tackle the difficulties, we define $Q(t)$ over the interval $kT \le t < (k+1)T$ to be a piecewise linear continuous function as in

$$Q(t) = Q(k) + (t - kT)Y(k) \tag{11}$$

which is a convex combination of two vertices $Q(kT)$ and $Q((k+1)T^-)$.

Substituting the linear combination (11) into (10), a simple manipulation yields

$$\begin{aligned} &Q(k)A_\mu' + A_\mu Q(k) - Y(k) \\ &+(t - kT)(Y(k)A_\mu' + A_\mu Y(k)) < 0 \end{aligned}$$

where $kT \le t < (k+1)T$. The argument shows that the negativeness of (10) over the interval $kT \le t < (k+1)T$ is assured by checking the negativeness of two vertices at $t = kT$ and $t = (k+1)T^-$, respectively, which can be found to be

$$Q(k)A_\mu' + A_\mu Q(k) - Y(k) < 0$$

and

$$Q(k)A'_\mu + A_\mu Q(k) - Y(k) + T(Y(k)A'_\mu + A_\mu Y(k)) < 0.$$

Thus, satisfaction of

$$Q(k)A'_i + A_i Q(k) - Y(k) < 0 \quad (12)$$

and

$$\begin{aligned} &Q(k)A'_i + A_i Q(k) - Y(k) \\ &+T(Y(k)A'_i + A_i Y(k)) < 0 \end{aligned} \quad (13)$$

suffices to establish (10). Therefore, we have the desired LMI results (7)-(8) immediately.

(B) For the discrete-time system (6) at the jump points kT, $k = 0, 1, 2, \cdots$, define a Lyapunov function $V(k) = e'(k)P(k)e(k)$ and the time difference of $V(k) - V(k^-)$ is calculated as

$$\begin{aligned} \triangle V &= e'(k)P(k)e(k) - e'(k^-)P(k^-)e(k^-) \\ &= e'(k^-)[\bar{A}'_{d\mu\mu}P(k)\bar{A}_{d\mu\mu} - P(k^-)]e(k^-) < 0. \end{aligned}$$

¿From which a sufficient condition, after executing Schur complement twice, is obtained as

$$\begin{bmatrix} -Q(k) & I - L_\mu(k)C_{2\mu} \\ (I - L_\mu(k)C_{2\mu})' & -Q^{-1}(k^-) \end{bmatrix} < 0$$

where $Q(k) = P^{-1}(k)$ and $Q^{-1}(k^-) = P(k^-)$. The sufficient condition is written as

$$\sum_{i=1}^{r} \mu_i^2 M_{ii} + 2\sum_{i=1}^{r}\sum_{i<j} \mu_i\mu_j \frac{(M_{ij} + M_{ji})}{2} < 0$$

where the first term M_{ii} is given by LHS (9) when $i = j$ and the second term $(M_{ij} + M_{ji})/2$ is given by LHS (9) when $i \neq j$. We note in passing that $Q(k)$ and $Q^{-1}(k^-)$ are different decision variables in time scale and, moreover, $Q^{-1}(k^-)$ serves as a start point for each interval. This completes the proof. ■

Remarks:

1) Unlike [17] where bilinear matrix inequalities (BMI) were obtained, the result presented in this paper is an LMI itself.
2) Solving a set of dynamic Riccati inequality in forward direction requires a start point condition $Q^{-1}(0^-)$ to activate the dynamic Riccati inequality. Moreover, the convex solution expression (11) is defined differently when compared to [17], leading to an LMI expression.
3) It is noted that (7)-(9) are convex in decision variables $(Q(k), Y(k), Q^{-1}(k^-), L_i(k))$ and can be solved by LMI convex analysis. However, to solve a finite horizon case, we need to check existence of interval-wise solutions $(Q(k), Y(k), L_i(k))$, given a start point $Q^{-1}(k^-)$ for each interval.

An algorithm in pseudocode is stated below:

Step 0: Given a final time index N and a uniform sampling period T, set $k = 0$.

Step 1: Given a start point $Q^{-1}(k^-)$, solve for $(Q(k)$,$Y(k)$,$L_i(k))$ via LMIs (7)-(9).

Step 2: Use (11) to determine $Q(k+1^-)(= Q(k)+TY(k))$ which will be inverted to obtain the start point of the next interval in a forward direction.

Step 3: Set $k = k + 1$, go to Step 1 until $k = N$.

IV. AN EXAMPLE

In this section, we develop a current observer for a nonlinear system adapted from [15]. The nonlinear system is repeated here for convenience. (Note that the uncertainty and disturbance are removed in this example.)

$$\begin{aligned} \dot{x}_1(t) &= -10x_1(t) - 20x_2(t) - \sin(x_1(t)) \\ \dot{x}_2(t) &= x_1(t) \\ y(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(k). \end{aligned}$$

Since the nonlinear term $\sin(x_1(t))$ can be exactly modeled by two fuzzy rules [22], we have the following two-rule fuzzy model representing the underlying nonlinear system.

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i A_i x.$$

where μ_i, $i = 1, 2$, are firing strength/fuzzy sets for $x_1(t)$ respectively,

$$\mu_1 = \begin{cases} \frac{\sin(x_1(t)) - (2/\pi)x_1(t)}{(1-2/\pi)x_1(t)}, & x_1(t) \neq 0 \\ 1, \quad x_1(t) = 0 \end{cases}$$

$$\mu_2 = \begin{cases} \frac{x_1(t) - \sin(x_1(t))}{(1-2/\pi)x_1(t)}, & x_1(t) \neq 0 \\ 0, \quad x_1(t) = 0 \end{cases}$$

and the system matrices are

$$A_i = \begin{bmatrix} (-10 - b_i) & -20 \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

where $b_1 = 1$ and $b_2 = 1/\pi$.

A current observer is designed as

$$\begin{aligned} \dot{\hat{\xi}}(t) &= A_i\hat{x}(t), \quad kT \leq t < (k+1)T \\ \hat{x}(k) &= \hat{\xi}(k^-) + L_i(k)[y(k) - \hat{y}(k)] \\ \hat{y}(k) &= C_{2i}\hat{\xi}(k). \end{aligned}$$

Assume $T = 0.5$ second and let the finite horizon be $k = 0, 1, 2, \cdots, 7$ $(N = 8)$. Let

$$Q(0^-) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and solving the feasibility problem (7)-(9) via the LMI solver in MATLAB, we find decision variables

$$Y(0) = \begin{bmatrix} 1.1119 & -0.5909 \\ -0.5909 & 0.3885 \end{bmatrix}$$

$$Q(0) = \begin{bmatrix} 1.7128 & -0.9029 \\ -0.9029 & 0.6094 \end{bmatrix}$$

$$L_1(0) = L_2(0) = \begin{bmatrix} 0.3512 \\ 0.6488 \end{bmatrix}.$$

Use (11) to find

$$Q(1^-) = \begin{bmatrix} 2.2687 & -1.1984 \\ -1.1984 & 0.8036 \end{bmatrix}$$

then solve (7)-(9) for

$$Y(1) = \begin{bmatrix} 4.1959 & -2.2409 \\ -2.2409 & 1.5020 \end{bmatrix}$$

$$Q(1) = \begin{bmatrix} 3.8399 & -2.0635 \\ -2.0635 & 1.4604 \end{bmatrix}$$

$$L_1(1) = L_2(1) = \begin{bmatrix} 0.3975 \\ 0.6393 \end{bmatrix}.$$

Continue the procedure for each interval forwardly, we find the following decision variables at $k = 7$.

$$Q(7^-) = 10^3 * \begin{bmatrix} 5.2428 & -2.7851 \\ -2.7851 & 2.0548 \end{bmatrix}$$

$$Y(7) = 10^4 * \begin{bmatrix} 1.4173 & -0.7428 \\ -0.7428 & 0.5245 \end{bmatrix}$$

$$Q(7) = 10^3 * \begin{bmatrix} 9.6946 & -5.2208 \\ -5.2208 & 3.9385 \end{bmatrix}$$

$$L_1(7) = L_2(7) = \begin{bmatrix} 0.3705 \\ 0.6295 \end{bmatrix}.$$

The simulation with an initial condition $x(0^-)$=[0.9 0.7]$'$ and $\hat{x}(0)$=[0 0]$'$ is shown in Fig. 1 for system state and estimated state, respectively. Fig. 2 clearly shows that the error $(x(t) - \hat{x}(t))$ converges to zero, indicating the digital fuzzy state observer can capture the system state successfully. While the finite horizon problem is addressed in this paper, another problem concerning feasibility of the infinite time horizon case is currently under investigation.

V. CONCLUSION

A complete solution to an observation problem involving sampled-data fuzzy systems is presented. The sampled-data fuzzy system is first formulated in an equivalent jump system, and then a set of hybrid continuous/discrete LMI test conditions for fuzzy current observer is derived based on Lyapunov approach. The focus of this paper addresses solving a hybrid (continuous/discrete) Riccati inequality associated with sampled-data fuzzy systems via LMI techniques. An advantage of such approach is that the Riccati-based method is transparent and well known in control community.

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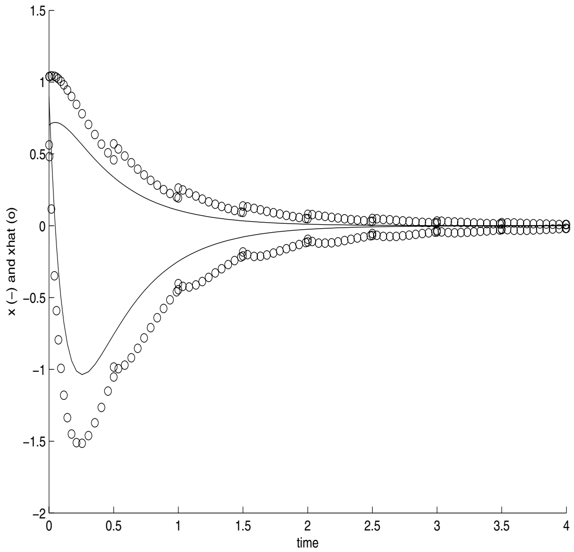

Fig. 1. Finite horizon: Converging trajectories of system state (solid line) and estimated state (circles).

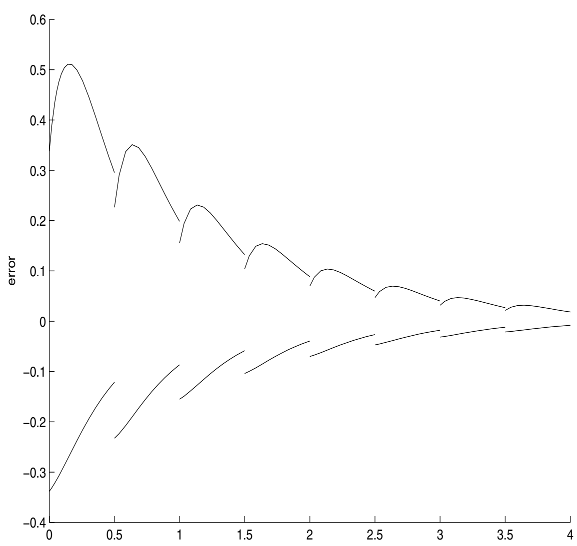

Fig. 2. Finite horizon: Converging trajectories of error state.