Design of Unknown Input Observers for Lipschitz Nonlinear Systems

A. M. Pertew* †, H. J. Marquez* † and Q. Zhao *†

* Emails: pertew, marquez, qingzhao @ece.ualberta.ca

† Addresses: Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada, T6G 2V4

Abstract—**The problem of unknown input observer design for Lipschitz nonlinear systems is considered. A new dynamic framework which is a generalization of previously used linear unknown input observers is introduced. The additional degrees of freedom offered by this framework are used to deal with the Lipschitz nonlinearity. The necessary and sufficient condition that ensures asymptotic convergence of the new observer is presented, and the equivalence between this condition and an H_∞ optimal control problem which satisfies the standard regularity assumptions in the H_∞ optimization theory is shown. Based on these results, a design procedure that is solvable using commercially available software is presented.**

I. INTRODUCTION

The observer design problem is a very important problem that has various applications such as output feedback control, system monitoring, process identification and fault detection. The classical solution of this problem for linear time invariant (LTI) systems is the use of the well known *Luenberger* observer structure [1] in which a constant matrix is used to stabilize the observer error dynamics to achieve asymptotic convergence. Modeling errors, plant disturbances and sensor noise, however, corrupt the state reconstruction given by the Luenberger observer and are very difficult to incorporate in this setting. This encouraged more work to be done in the so-called *robust observer design problem* in recent years.

One of the most successful robust observer design techniques is the use of the disturbance decoupling principle, in which the estimation error is designed to be insensitive to unknown disturbances. This problem is also referred to as the *unknown input observer (UIO)* design and it dates back to 1975 where Wang [2] proposed a minimal order UIO structure for linear systems with both known and unknown inputs. After this important work, several approaches for designing reduced order and full order UIOs have been proposed, including the geometric approach by Bhattacharyya [3], the inversion algorithm by Kobayashi and Nakamizo [4], the singular value decomposition technique by Fairman [5] and the algebraic approaches by Hou and Müller in [6] and by Patton, Chen and Zhang in [7]. Achieving less restrictive existence conditions and more direct design procedures has always been a challenge in this area. The UIO application in fault diagnosis has also attracted many researchers. Watanabe and Himmelblau introduced the concept of UIO for robust sensor fault

diagnosis in systems with modeling uncertainty [8]. Their approach was later extended by Wünnenberg and Frank (see [9]) and also by Patton and Chen (see [7], [10], [11]) to the detection of both sensor and actuator faults.

However, most of the previous results are restricted to linear systems and results on nonlinear UIO are scarce. A direct extension of the linear results to the nonlinear case was considered by Wünnenberg in [12]. His approach was referred to as NUIO (Nonlinear UIO) and considered systems with nonlinearities that are functions of inputs and outputs. However, this class of nonlinear systems is rather limited and many physical systems can not be modelled in this way. Another limitation is the difficulty of transforming a general nonlinear system into the required form. An alternative approach referred to as the DDNO (Disturbance Decoupling Nonlinear Observer) was presented in a series of papers by Seliger and Frank [13]-[15]. The class of nonlinear systems considered by the DDNO is more general and the basic idea is the use of nonlinear state transformations to satisfy the decoupling condition. However, the existence conditions for these transformations are derived from the Frobenius theorem and are rather restrictive. Another drawback of the DDNO is that the state transformation leads to another nonlinear system for which an observer design is not a tractable problem.

In this paper, we consider the UIO design problem for the class of nonlinear Lipschitz systems. We extend the result in [7] to this class of nonlinear systems and show that, with the same necessary and sufficient conditions, the Lipschitz UIO (LUIO) design problem is equivalent to an H_∞ optimal control problem that satisfies all of the regularity assumptions. Our formulation employs a new dynamic framework which is a generalization to the one used in previous works. The LUIO synthesis is carried out using H_∞ optimization and can therefore be done using commercially available software packages. The paper is organized as follows: section II introduces some background results and notations. In section III, we introduce our dynamic generalization of the previously used UIO structure and provide an extension of the results in [7] to the new structure. In section IV, we present our main result, where we formulate the Lipschitz unknown input observer (LUIO) design problem as a regular H_∞ problem proving that its solution is necessary and sufficient for the observer stability. Finally, some conclusions are drawn in section V.

II. Preliminaries and notation

The linear UIO design problem considers a class of systems in which the system uncertainty can be expressed as an additive unknown disturbance term $d(t)$ as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{1}$$
$$y(t) = Cx(t) \tag{2}$$

where $A \in \Re^{n \times n}, B \in \Re^{n \times m}, E \in \Re^{n \times r}$ and $C \in \Re^{p \times n}$. The matrix E is referred to as the *unknown input distribution matrix* and is assumed to be a known full column rank matrix (with $r \leq p$). The term $Ed(t)$ can actually be used to describe additive disturbances as well as many kinds of modeling uncertainties such as noise, nonlinear or time-varying terms, linearization and model reduction errors, parameter variations, etc. It can also represent system inputs which are inaccessible (or unmeasurable) [10]. It is important to note that much work has been done on estimating the distribution matrix E when it is not fully known (see for example [11], [16], [17]). In all the literature available for this problem, the observer proposed fall in the class of Luenberger-like observers, namely:

$$\dot{z}(t) = Fz(t) + Ly(t) + TBu(t) \tag{3}$$
$$\hat{x}(t) = z(t) + Hy(t) \tag{4}$$

where $F \in \Re^{n \times n}, L \in \Re^{n \times p}, T \in \Re^{n \times n}$ and $H \in \Re^{n \times p}$. This observer can be represented by the structure in Fig. 1.
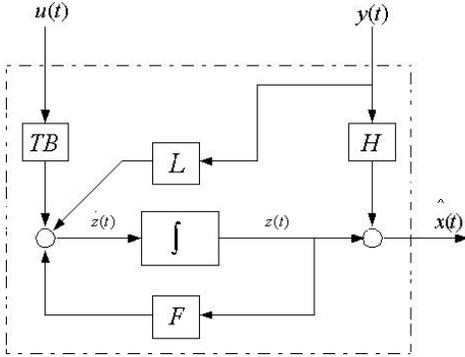

Fig. 1. The structure of UIO.

As mentioned in section I, several approaches for designing a UIO (particularly, for designing F, L, T and H to guarantee observer stability) have been developed. In this section, we focus on the technique developed by Chen, Patton and Zhang in [7]. In their work, the necessary and sufficient existence conditions of UIO were presented. Compared with other UIO design methods, these conditions are easy to verify and the design procedure is relatively simple.

Their results can be summarized as follows. By noting

that if the following matrix equations are satisfied:

$$HCE = E \tag{5}$$
$$T = I - HC \tag{6}$$
$$F = A - HCA - L_1C; \quad \text{with } F \text{ stable} \tag{7}$$
$$L_2 = FH \tag{8}$$
$$L = L_1 + L_2 \tag{9}$$

then the observer in (3)-(4) has the stable undisturbed error dynamics $\dot{e} = Fe$, they defined (3)-(4) as a UIO for the system (1)-(2) if it satisfies (5)-(9). Based on that definition, their main result was given by the following theorem:

Theorem 1: [7] Necessary and sufficient conditions for (3)-(4) to be a UIO for the system (1)-(2) are:
(i) rank (CE) = rank(E). (ii) (A_1, C) is a detectable pair, where $A_1 = A - E\left[(CE)^T CE\right]^{-1} (CE)^T CA$.

They also presented a systematic design procedure to compute the matrices F, L, T and H that satisfy (5)-(9).

In this paper, we study the nonlinear UIO design problem for the class of nonlinear Lipschitz systems of the form:

$$\dot{x}(t) = Ax(t) + \Gamma(y, u, t) + \Phi(x, u, t) + Ed(t) \tag{10}$$
$$y(t) = Cx(t) \tag{11}$$

and where the function $\Phi(x, u, t)$ satisfies a uniform Lipschitz continuity globally in x, i.e,

$$\| \Phi(x_1, u, t) - \Phi(x_2, u, t) \| \leq \alpha \| x_1 - x_2 \| \tag{12}$$

for all $u \in \Re^m$ and $t \in \Re$ and for all x_1 and $x_2 \in \Re^n$. Here $\alpha \in \Re$ is referred to as the *Lipschitz constant* and is independent of x, u and t. Lipschitz systems constitute a very important class. Notice that any system of the form $\dot{x} = f(x, u)$ can be expressed in the form of (10) as long as $f(x, u)$ is continuously differentiable with respect to x. Many nonlinearities satisfy (12) at least locally. Examples include trigonometric nonlinearities occurring in robotic applications, the nonlinearities which are square or cubic in nature, etc. Therefore, much research has been done on the observer design problem for Lipschitz systems without additive disturbances (see for example [18], [19]). However, in this paper we consider the case of nonzero additive disturbances, i.e, the Lipschitz unknown input observer (LUIO) design problem. Our approach is based on the extension of the UIO structure in (3)-(4) to a more general dynamic framework. We further tackle the LUIO design problem by showing that, using the new dynamic observer, this problem is equivalent to a standard H_∞ control problem. The extra design freedom offered by the dynamic formulation is used to deal with the nonlinearities. The following definitions and notations will be used throughout the paper:

Definition 1: ($\mathcal{L}_2$ Space) The space $\mathcal{L}_2$ consists of all $\mathcal{L}$ebesque measurable functions $u : \Re^+ \to \Re^q$ having a finite $\mathcal{L}_2$ norm $\|u\|_{\mathcal{L}_2}$, where $\|u\|_{\mathcal{L}_2} \triangleq \sqrt{\int_0^\infty \| u(t) \|^2 \ dt}$, with $\|u(t)\|$ as the *Euclidean* norm of the vector $u(t)$.
For a system $H : \mathcal{L}_2 \to \mathcal{L}_2$, $\gamma(H)$ will represent its $\mathcal{L}_2$ gain defined as: $\gamma(H) = \sup_u \frac{\|Hu\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$. It is well known that,

for a linear system $H : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ with a transfer matrix $\hat{H}(s)$, $\gamma(H)$ is equivalent to the H_∞ norm of $\hat{H}(s)$ defined as: $\gamma(H) \triangleq \parallel \hat{H}(s) \parallel_\infty = \sup_{\omega \in \Re} \sigma_{\max}(\hat{H}(j\omega))$, where $\sigma_{\max}$ represents the maximum singular value of $\hat{H}(j\omega)$. The matrices I_n, 0_n and 0_{nm} are used to represent the identity matrix of order n, the zero square matrix of order n and the zero n by m matrix respectively. The symbol $\hat{T}_{yu}$ is used to represent the transfer matrix from input u to output y. The partitioned matrix $K = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ (when used as an operator from y to u, i.e, $u = Ky$) represents the state space representation ($\dot{\xi} = A\,\xi + B\,y,\ u = C\,\xi + D\,y$).

III. A NEW UIO DYNAMIC DESIGN

In this section, we first present a dynamic generalization of the UIO structure in (3)-(4) developing conditions that guarantee the observer stability in this case. We then extend the results in [7] to this new framework by proving that the same conditions of Theorem 1 are necessary and sufficient for the new UIO dynamic design.

A. Dynamic generalization of the classical UIO structure

Throughout this paper, we will make use of dynamical observers with the following structure:

$$\dot{z}(t) = w_1(t) + w_2(t) + TB\,u(t) \qquad (13)$$
$$\hat{x}(t) = z(t) + H\,y(t) \qquad (14)$$

where $w_1(t)$ and $w_2(t)$ are obtained by applying dynamical compensators on the vectors z and y respectively. In other words, w_1 and w_2 are given from:

$$\dot{\xi}_1 = A_F\,\xi_1 + B_F\,z,\ w_1 = C_F\,\xi_1 + D_F\,z \qquad (15)$$
$$\dot{\xi}_2 = A_L\,\xi_2 + B_L\,y,\ w_2 = C_L\,\xi_2 + D_L\,y \qquad (16)$$

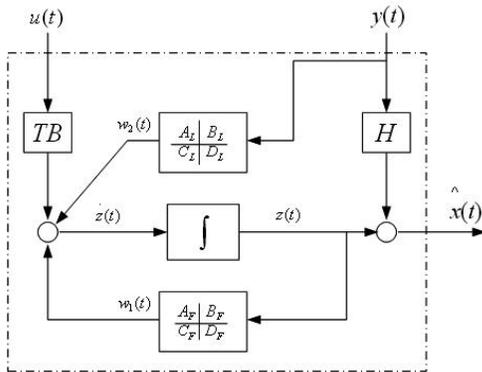This can be represented by the structure in Fig. 2.

Fig. 2.   Dynamic structure of UIO.

Compared with the one in Fig. 1, this new structure offers more dynamics which will be used in section IV to tackle the LUIO design problem. Towards that goal, we consider first its use as a UIO for the system (1)-(2). The following lemma develops conditions for observer stability in this case (see proof in Appendix I).

Lemma 1: The error dynamics of (13)-(16) as an observer for (1)-(2) is asymptotically stable and decoupled from the disturbance term $d(t)$ if the following conditions are satisfied:

$$HCE = E \qquad (17)$$
$$T = I_n - HC \qquad (18)$$

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \text{stable; with} \begin{cases} A_F = & A_L \\ B_F = & B_{L1}C \\ C_F = & -C_L \\ D_F = & A - HCA - D_{L1}C \end{cases}$$
$$\qquad (19)$$

$$D_{L2} = D_F H \qquad (20)$$
$$B_{L2} = -B_F H \qquad (21)$$
$$B_L = B_{L1} + B_{L2} \qquad (22)$$
$$D_L = D_{L1} + D_{L2} \qquad (23)$$

In the next subsection, we show that same conditions of theorem 1 (which were necessary and sufficient to satisfy (5)-(9)) are still necessary and sufficient for (17)-(23).

B. Dynamic UIO: definition and existence conditions

We start by introducing the following definition:

Definition 2: (Dynamic UIO) An observer of the form (13)-(16) is referred to as a dynamic UIO for the system (1)-(2) if it satisfies conditions (17)-(23).

This definition accommodates the observer defined in [7] as a special case and we here prove that the same conditions of theorem 1 hold for this more general definition as follows:

Theorem 2: There exists a dynamic UIO for the system (1)-(2) (according to Definition 2) if and only if :

(i) rank $(CE) = $ rank (E) $\qquad (24)$

(ii) (A_1, C) is a detectable pair, where

$$A_1 = A - E\left[(CE)^T CE\right]^{-1} (CE)^T CA. \qquad (25)$$

Proof: The proof is constructive (shows the steps needed to design a dynamic UIO). It is a direct result of the proof of theorem 1 and of the interpretation of conditions (19)-(23) as follows. It was proved in [10] that (17) is solvable iff (24) is satisfied and that the general solution is:

$$H = E(CE)^+ + H_0\left[I_p - CE(CE)^+\right] \qquad (26)$$

where H_0 is an arbitrary n by p matrix and $(CE)^+$ is the left inverse of (CE) which is:

$$(CE)^+ = \left[(CE)^T CE\right]^{-1} (CE)^T$$

The rest of the proof follows by noting that satisfying (19) is equivalent to finding A_L, C_L, B_{L1} and D_{L1} such that $\begin{bmatrix} A_L & B_{L1}C \\ -C_L & A - HCA - D_{L1}C \end{bmatrix}$ is stable. This is equivalent to the stabilization problem in Fig. 3 which is solvable iff $(A - HCA, C)$ is detectable.

Finally, notice that for any H that satisfies (26), $(A - HCA, C)$ is detectable iff (A_1, C) is detectable (as proven in [10]) where A_1 is given by (25). Therefore,
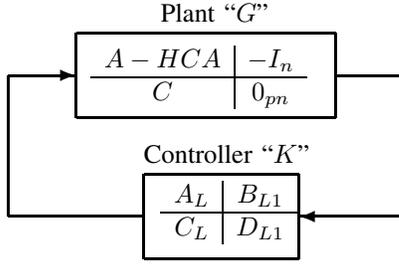
Fig. 3. Stabilization problem.

conditions (i) and (ii) are necessary and sufficient to satisfy (17)-(23) and the proof is complete. △

As a conclusion, with the same existence conditions of the UIO in [7], the dynamic UIO design is reduced to the stabilization problem in Fig. 3 which has a *set* of controllers [20] as its solution. We will make use of this extra design freedom in section IV when dealing with the Lipschitz case.

IV. DESIGN OF LIPSCHITZ UNKNOWN INPUT OBSERVERS

For the Lipschitz unknown input observer (LUIO) design problem, we first prove that solving an H_∞ problem is necessary and sufficient for observer stability. We then show that this problem is equivalent to a standard H_∞ problem. Finally, we present a systematic design procedure for the LUIO within the H_∞ framework.

A. An H_∞ formulation of the LUIO design problem

For a Lipschitz system of the form (10)-(11) and where $\Phi(x, u, t)$ satisfies (12) with a Lipschitz constant α, we consider the use of the following observer:

$$\dot{z}(t) = w_1(t) + w_2(t) + T\Gamma(y, u, t) + T\Phi(\hat{x}, u, t) \quad (27)$$

$$\hat{x}(t) = z(t) + Hy(t) \quad (28)$$

where $w_1(t)$ and $w_2(t)$ are obtained from (15) and (16) respectively and where the parameters of the observer satisfy the conditions (17)-(23). Similar to the linear case (see proof of lemma 1 in Appendix I), it can be seen that the error dynamics of this observer is given from:

$$\dot{\xi} = A_F \, \xi + B_F \, e \quad (29)$$

$$\dot{e} = C_F \, \xi + D_F \, e + T \left(\Phi(x, u, t) - \Phi(\hat{x}, u, t) \right) \quad (30)$$

which (by using (19)) can also be represented by the transfer function $\hat{T}_{\zeta\omega}$ in the following so-called standard form:

$$\dot{\psi} = [A - HCA] \, \psi + \begin{bmatrix} T & -I_n \end{bmatrix} \begin{bmatrix} \omega \\ \nu \end{bmatrix} \quad (31)$$

$$\begin{bmatrix} \zeta \\ \varphi \end{bmatrix} = \begin{bmatrix} I_n \\ C \end{bmatrix} \psi + \begin{bmatrix} 0_n & 0_n \\ 0_{pn} & 0_{pn} \end{bmatrix} \begin{bmatrix} \omega \\ \nu \end{bmatrix} \quad (32)$$

where

$$\begin{aligned} \omega &= \tilde{\phi} = \Phi(x, u, t) - \Phi(\hat{x}, u, t) \\ \nu &= K \, (y - \hat{y}) \\ \zeta &= e = x - \hat{x} \\ \varphi &= y - \hat{y} \end{aligned} \quad (33)$$

and where K is the dynamic controller

$$K = \left[\begin{array}{c|c} A_L & B_{L1} \\ \hline C_L & D_{L1} \end{array} \right] \quad (34)$$

This can also be represented by Fig. 4 where the plant G has the state space representation in (35) with the matrices in (31)-(32) and where the controller K is given from (34).

$$\hat{g}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (35)$$
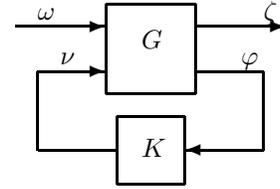

Fig. 4. Standard setup.

The following theorem is the main result of this section:

Theorem 3: Given the Lipschitz system of equations (10)-(11), the state $\hat{x}$ of the observer (27)-(28) (satisfying conditions (17)-(23)) globally asymptotically converges to the system state x for all Φ satisfying (12) with Lipschitz constant α if and only if K in (34) satisfies

$$sup_{\omega \in \Re} \, \sigma_{max} \left[\hat{T}_{\zeta\omega}(j\omega) \right] < \frac{1}{\alpha} \quad (36)$$

Proof: (Sufficiency) Using the variable definitions in (33) and the matrices in (31), (32) and (34), $\hat{T}_{\zeta\omega}$ can be represented as follows:

$$\hat{T}_{\zeta\omega} = \hat{T}_{e\tilde{\phi}} = \left[\begin{array}{cc|c} A - HCA - D_{L1}C & -C_L & T \\ B_{L1}C & A_L & 0_n \\ \hline I_n & 0_n & 0_n \end{array} \right] \quad (37)$$

and is such that $\gamma(\hat{T}_{e\tilde{\phi}}) = \| \hat{T}_{e\tilde{\phi}} \|_\infty < \frac{1}{\alpha}$ according to (36). The proof of sufficiency follows by noting that the estimation error e is given from the feedback interconnection of $\hat{T}_{e\tilde{\phi}}$ and Δ as shown in Fig. 5 where Δ is the static nonlinear time-varying operator defined as follows:

$$\begin{aligned} \Delta(t) : e \rightarrow \tilde{\phi} &= \Phi(x, u, t) - \Phi(\hat{x}, u, t) \\ &= \Phi(e + \hat{x}(t), u(t), t) - \Phi(\hat{x}(t), u(t), t) \end{aligned}$$
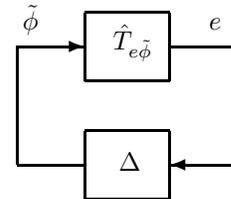

Fig. 5. Feedback interconnection.

In this loop, $\gamma(\hat{T}_{e\tilde{\phi}}) < \frac{1}{\alpha}$ as mentioned earlier and, although an exact expression for Δ is not available, we

4201

have $\gamma(\Delta) \leq \alpha$ because from (12) it follows that

$$\gamma(\Delta) \leq \frac{\sqrt{\int_0^\infty \alpha^2 \parallel x - \hat{x} \parallel^2 \, \mathrm{d}t}}{\sqrt{\int_0^\infty \parallel x - \hat{x} \parallel^2 \, \mathrm{d}t}} \leq \alpha$$

Using the bounds on the $\mathcal{L}_2$ gains of the $\hat{T}_{e\tilde{\phi}}$ and Δ, we will make use of a dissipativity argument by noting that the following properties are satisfied for the feedback interconnection of Fig. 5:
(i) Δ is a static nonlinearity (no internal states) and $\hat{T}_{e\tilde{\phi}}$ is given from (37). (ii) The mappings $\hat{T}_{e\tilde{\phi}} : \tilde{\phi} \rightarrow e$ and $\Delta : e \rightarrow \tilde{\phi}$ have finite $\mathcal{L}_2$ gains $\gamma(\hat{T}_{e\tilde{\phi}})$ and $\gamma(\Delta)$, and moreover they satisfy $\gamma(\hat{T}_{e\tilde{\phi}}).\gamma(\Delta) < 1$. (iii) $\hat{T}_{e\tilde{\phi}}$ and Δ are dissipative with respect to the supply rates $\omega_1 = -e^T e + \gamma(\hat{T}_{e\tilde{\phi}})^2 \tilde{\phi}^T \tilde{\phi}$ and $\omega_2 = -\tilde{\phi}^T \tilde{\phi} + \alpha^2 e^T e$ respectively. We will denote by S_1 and S_2 the storage functions associated with these supply rates.

It is an straightforward application of Corollary 1 in reference [21] that $S_1 + aS_2$, $a > 0$, is a Lyapunov function for the feedback system of Fig. 5, and that, since $\gamma(\hat{T}_{e\tilde{\phi}})\gamma(\Delta) < 1$, the composite system is asymptotically stable. This implies that $e \rightarrow 0$ as $t \rightarrow \infty$.
(Necessity) This is a direct result of the small gain theorem for LTI systems (see the proof of Theorem 4 in [19] for more details) which implies that if $\gamma(\hat{T}_{e\tilde{\phi}}) \geq \frac{1}{\alpha}$ in Fig. 5, then there exists $\Phi(x, u, t) = M(t)x$ with $\parallel \hat{M}(s) \parallel_\infty \leq \alpha$ such that the closed loop system in Fig. 5 is unstable. $\triangle$

B. Problem regularization

Despite the result in theorem 3, observer synthesis can not be carried out directly using the standard H_∞ solution since the standard form in (31)-(32) does not satisfy all of the regularity assumptions in the H_∞ framework [20]. However, by considering a "weighted" disturbance term in the output, i.e using (38) instead of (11):

$$y(t) = Cx(t) + \epsilon \, \eta(t), \qquad \epsilon > 0 \qquad (38)$$

then the standard form in (31)-(32) has now the form:

$$\dot{\psi} = \begin{bmatrix} A - HCA \end{bmatrix} \psi + \begin{bmatrix} \begin{bmatrix} T & 0_{np} \end{bmatrix} & -I_n \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ \eta(t) \end{bmatrix} \\ \nu \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} \zeta \\ \varphi \end{bmatrix} = \begin{bmatrix} I_n \\ C \end{bmatrix} \psi + \begin{bmatrix} 0_n & 0_{np} & 0_n \\ 0_{pn} & \epsilon I_p & 0_{pn} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ \eta(t) \end{bmatrix} \\ \nu \end{bmatrix} \tag{40}$$

This can also be represented by the setup in Fig. 4, except for redefining the matrices of $\hat{g}(s)$ in (35) and replacing ω by $\bar{\omega}$ which is defined as: $\bar{\omega} \triangleq \begin{bmatrix} \omega & \eta(t) \end{bmatrix}^T$. This standard form, however, still does not satisfy the regularity assumptions in the H_∞ problem since $D_{12}^T D_{12}$ is singular. Fortunately, regularization can be done by simply extending the external output ζ to include the "weighted" vector $\beta\nu$, $\beta > 0$. This adds another change in Fig. 4 consisting of

replacing ζ by $\bar{\zeta}$ defined as: $\bar{\zeta} \triangleq \begin{bmatrix} \zeta & \beta\nu \end{bmatrix}^T$.
The entries of $\hat{g}(s)$ in (35) are then given by the following state space realization (where $\bar{A} = A - HCA$)

$$\dot{\psi} = \begin{bmatrix} \bar{A} \end{bmatrix} \psi + \begin{bmatrix} \begin{bmatrix} T & 0_{np} \end{bmatrix} & -I_n \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ \eta(t) \end{bmatrix} \\ \nu \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} \begin{bmatrix} \zeta \\ \beta\nu \end{bmatrix} \\ \varphi \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_n \\ 0_n \end{bmatrix} \\ C \end{bmatrix} \psi + \begin{bmatrix} \begin{bmatrix} 0_n & 0_{np} \\ 0_n & 0_{np} \end{bmatrix} & \begin{bmatrix} 0_n \\ \beta I_n \end{bmatrix} \\ \begin{bmatrix} 0_{pn} & \epsilon I_p \end{bmatrix} & 0_{pn} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ \eta(t) \end{bmatrix} \\ \nu \end{bmatrix} \tag{42}$$

It is now straightforward to see that all of the regularity assumptions summarized below [20] are now satisfied:
  1) (A,B_2) stabilizable: satisfied for any matrix $\bar{A}$.
      (C_2,A) detectable: satisfied iff $(\bar{A},C)$ is detectable.
  2) $D_{21}D_{21}^T = \epsilon^2 I_{p \times p}$ which is nonsingular.
      $D_{12}^T D_{12} = \beta^2 I_{n \times n}$ which is nonsingular.
  3) rank $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = 2n = $ full column rank.
      rank $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p = $ full row rank.
  4) $D_{22} = 0$.

It follows that all the regularity assumptions are satisfied iff $(A - HCA, C)$ is detectable (with no new design restrictions over the design conditions of theorem 2). We also prove that this regular H_∞ problem is actually equivalent to the original one introduced by theorem 3 in the following sense: Let T_1 be the setup in Fig. 4 associated with (31)-(32), T_2 the one associated with (39)-(40) and T_3 the one associated with (41)-(42). And let $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ be their corresponding transfer matrices. The following two lemmas demonstrate a certain equivalence between these setups (the proof is omitted and is included in [22]).

Lemma 2: Consider a stabilizing controller K for T_1 and T_2, then $\parallel \hat{T}_1 \parallel_\infty < \gamma$ iff $\exists \, \epsilon > 0$ such that $\parallel \hat{T}_2 \parallel_\infty < \gamma$.

Lemma 3: Given $\epsilon > 0$ and a stabilizing controller K for T_2 and T_3, then $\parallel \hat{T}_2 \parallel_\infty < \gamma$ iff $\exists \, \beta > 0$ s.t $\parallel \hat{T}_3 \parallel_\infty < \gamma$.
The previous results can be used to extend the results in theorem 3 showing that the observer gain K needed to stabilize the estimation error must solve a regular H_∞ optimal control problem. To this end, we define the regular continuous H_∞ control problem "Problem 1" as follows:
Problem 1 *Given $\epsilon > 0$ and $\beta > 0$, find S, the set of admissible controllers K satisfying $\parallel \hat{T}_{\bar{\zeta}\bar{\omega}} \parallel_\infty < \gamma$ for the standard setup in Fig. 4 with G having the state space representation (35) along with the matrices in (41)-(42).*

Now we extend the result of theorem 3 as follows:

Theorem 4: Given the Lipschitz system of equations (10)-(11), the state $\hat{x}$ of the observer (27)-(28) (satisfying the conditions (17)-(23)) globally asymptotically converges to the system state x for all Φ satisfying (12) with a Lipschitz constant α if and only if $\exists \, \epsilon$, $\beta > 0$ such that K (the controller in (34)) $\in S^*$ (the set of controllers solving "Problem 1" defined above with $\gamma = \frac{1}{\alpha}$).
Proof: A direct result of Theorem 3, Lemmas 2 and 3. $\triangle$

C. A LUIO design procedure

The following iterative "binary search" procedure is then proposed to design the LUIO (27)-(28) for the system (10)-(11) satisfying the conditions (12), (24) and (25).

Design procedure

Step 1 Calculate $H = E(CE)^+ + H_0 [I_p - CE(CE)^+]$; where H_0 is an arbitrary n by p matrix and where $(CE)^+$ is the left inverse of CE which is given from $(CE)^+ = [(CE)^T CE]^{-1} (CE)^T$.

Step 2 Set $T = I_n - HC$. **Step 3** Set $\epsilon, \ \beta > 0$ and $\gamma \leftarrow \frac{1}{\alpha}$.

Step 4 Test solvability of "Problem 1" (the H_∞ problem defined in sec IV-B). If test fails, go to Step 6; otherwise solve the problem (using available software packages) and any $K \in S$ is candidate for (34).

Step 5 Calculate all the remaining gains of the LUIO using (19)-(23) and go to Step 7 . **Step 6** Set $\epsilon = \frac{\epsilon}{2}$ and $\beta = \frac{\beta}{2}$. If ϵ or $\beta <$ a threshold, then *go to step 7 (a LUIO does not exist)*; otherwise go to Step 4. **Step 7** Stop.

Comments:

- When the H_∞ problem is not solvable, one can increase γ by decreasing the Lipschitz constant α. The statement in *italic* in step 6 can then be replaced by: *decrease α and go to Step 3*. With this change, the algorithm is guaranteed to work as $\alpha \to 0$.

- Same design can be used when the output is disturbed as in (38). The small gain theorem guarantees that the estimation error $e(t) \in \mathcal{L}_2$ if $\eta(t) \in \mathcal{L}_2$.

V. CONCLUSION

In this paper we considered the design of unknown input observers (UIO) for nonlinear Lipschitz systems. A new H_∞ approach is proposed and is shown to be feasible with the same existence conditions of the linear UIO design. Our approach makes use of dynamical observers, an approach that shows promise given the additional degrees of freedom available to the designer. The introduced design procedure addresses the nonlinearity and the disturbance decoupling problems in the same framework and can be carried out using commercially available software, such as MATLAB.

APPENDIX I
Proof of Lemma 1

Using (13)-(16) as an observer for the system (1)-(2) and defining the error $e_z = Tx - z$ and the state $\xi = \xi_2 - \xi_1$ we have (with the help of (17)-(19)):

$$
\begin{aligned}
\dot{e_z} &= T(Ax + Bu + Ed) - w_1 - w_2 - TBu \\
&= -C_F\xi_1 - C_L\xi_2 - [D_F(Tx - e_z) - TAx + D_LCx] \\
&= C_F\xi + D_Fe_z - (D_FT - TA + D_LC)x
\end{aligned}
$$

But from (18)-(20) and (23): $TA - D_FT = D_LC$. Then, $\dot{e_z} = C_F\xi + D_Fe_z$. Also, for the state ξ, we have:

$$
\begin{aligned}
\dot{\xi} &= A_L\xi_2 + B_Ly - A_F\xi_1 - B_Fz \\
&= A_L\xi_2 + B_LCx - A_F\xi_1 - B_F(Tx - e_z) \\
&= A_F\xi + B_Fe_z - (B_FT - B_LC)x
\end{aligned}
$$

But from (18), (19), (21) and (22): $B_FT = B_LC$. Then, $\dot{\xi} = A_F\xi + B_Fe_z$. Therefore, the observer error dynamics is ($\dot{\xi} = A_F\xi + B_Fe_z, \ \dot{e_z} = C_F\xi + D_Fe_z$). From the stability of the matrix in (19), it follows that the error dynamics is asymptotically stable and decoupled from $d(t)$, and the proof is completed by noting that $e = x - \hat{x} = x - (z + Hy) = (I - HC)x - z = e_z$. △

REFERENCES

[1] D. G. Luenberger, "An introduction to observers," *IEEE transactions on Automatic Control* , vol. 16, pp. 596-602, 1971.

[2] S. H. Wang, E. J. Davison, P. Dorato, "Observing the states of systems with unmeasurable disturbances," *IEEE transactions on Automatic Control* , vol. 20, pp. 716-717, 1975.

[3] S. P. Bhattacharyya, "Observer design for linear systems with unknown inputs," *IEEE transactions on Automatic Control* , vol. 23, pp. 483-484, 1978.

[4] N. Kbayashi, R. Nakamizo, "An observer design for linear systems with unknown inputs," *International Journal of Control* , vol. 35, pp. 605-619, 1982.

[5] F. W. Fairman, S. S. Mahil, L. Luk, "Disturbance decoupled observer design via singular value decomposition," *IEEE transactions on Automatic Control* , vol. 29, No.1, pp. 84-86, 1984.

[6] M. Hou, P. C. Müller, "Design of observers for linear systems with unknown inputs," *IEEE transactions on Automatic Control* , vol. 37, No.6, pp. 871-875, 1992.

[7] J. Chen, R. J. Patton, H. Zhang, "Design of unknown input observers and robust fault detection filters," *International Journal of Control* , vol. 63, No.1, pp. 85-105, 1996.

[8] K. Watanabe, D. M. Himmelblau, "Instrument Fault Detection in Systems with Uncertainty", *Int. J. Sys. Sci*, Vol 13, pp. 137-158, 1982.

[9] J. Wünnenberg, P. M. Frank, "Sensor Fault Using Detection via Robust Observer", *System Fault Diagnosis, Reliability& Related Knowledge-Based Approaches*, S.G. Tzafestas, M. G. Singh and G. Schmidt (eds), Reidel Press, Dordrecht, pp. 147-160, 1987.

[10] J. Chen, R. J. Patton, "Robust model-based fault diagnosis for dynamic systems", Kluwer Academic Publishers, 1999.

[11] R. J. Patton, J. Chen, "Optimal selection of unknown input distribution matrix in the design of robust observers for fault diagnosis", *Automatica*, vol. 29, No. 4, pp 837-841, 1991.

[12] J. Wünnenberg, "Observer-based fault detection in dynamic systems", PhD thesis, University of Duisburg, Germany, 1990.

[13] P. M. Frank, R. Seliger, "Fault detection and isolation in automatic processes", *in C. Leondes (ed.), Control and dynamic systems*, vol. 49, Academic press, pp 241-287, 1991.

[14] R. Seliger, P. M. Frank, "Fault diagnosis by disturbance decoupled nonlinear observers", *Proceedings of the 30th IEEE conference on decision and control*, Brighton, UK, pp 2248-2255, 1991.

[15] R. Seliger, P. M. Frank, "Robust residual evaluation by thersohld selction and a performance index for nonlinear observer-based fault diagnosis", *Proceedings of the International conference on fault diagnosis: TOOLDIAG'93*, Toulouse, pp 496-504, 1993.

[16] J. Gertler, M. K. Kunwer, "Optimal residual decoupling for robust fault diagnosis", *International Journal of Control*, vol. 61, No. 2, pp 395-421, 1995.

[17] A. Edelmayer, J. Bokor, F. Szigeti, L. Keviczky, "Robust detection filter design in the presence of time-varying system perturbations", *Automatica*, vol. 33, No. 3, pp 471-475, 1997.

[18] S. Raghavan, J. K. Hedrick, "Observer design for a class of nonlinear systems," *International Journal of Control*, Vol. 59, No. 2, pp. 515-528, 1994.

[19] R. Rajamani, "Observers for Lipschitz nonlinear systems," *IEEE transactions on Automatic Control*, vol. 43, No. 3, pp. 397-401, March 1998.

[20] K. Zhou, J. C. Doyle, "Essentials of robust control", Prentice-Hall, NY, 1998.

[21] D. J. Hill and P. J. Moylan, "Stability Results for Nonlinear Feedback Systems," *Automatica*, Vol. 13, pp. 377–382, July 1977.

[22] A. M. Pertew, H. J. Marquez and Q. Zhao, "H_∞ synthesis of unknown input observers for nonlinear Lipschitz systems," *submitted to the International Journal of Control*, 2004.