2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Low-Order Unknown Input Observers

Stefen Hui * Stanislaw H. Żak †

Abstract

A projection operator approach to the state estimation of dynamical systems is proposed resulting in new types of low-order observers for systems with unknown inputs. The state of the system, whose state is to be estimated, is decomposed into known and unknown components. The unknown component is a projection, not necessarily orthogonal, of the whole state along the subspace in which the available state component resides. Then a dynamical system to estimate the unknown component is constructed. Combining the output of the dynamical system, which estimates the unknown state component, with the available state information results in the observer that estimates the whole state. It is shown that some previously proposed unknown input observer (UIO) architectures can be obtained using the projection operator approach developed in this paper.

1 Introduction

Observers use the plant input and output signals to generate an estimate of the plant's state, which can then be employed to close the control loop. Observers can also be utilized to augment or replace sensors in a control system. The observer was first proposed and developed by Luenberger in the early sixties of the last century [1, 2]. In this paper, we utilize deterministic methods to estimate the state of an uncertain dynamical system employing only the known system input and output signals. Since the early developments, which concentrated on observers for systems without uncertainties, observers for plants with both known and unknown inputs have been developed resulting in the so-called unknown input observer (UIO) architectures. More recently, different observer architectures for uncertain systems, utilizing the concept of sliding modes were proposed, see, for example [3].

We propose full- and reduced-order observers for a class of dynamical systems where some or all of the inputs are unknown. The unknown input can be a combination of unmeasurable or unmeasured disturbances, unknown control action, and unmodeled system dynamics. We employ a projection operator approach to the state estimation where the state of the system, whose state is to be estimated, is decomposed into known and unknown components. The unknown component is, in general, a skew projection, that is, not necessarily orthogonal, of the whole state along the subspace in which the available state component resides. We then construct a dynamical system to estimate the unknown component. Finally, we combine the output of the dynamical system, which estimates the unknown state component, with the available state information to obtain the observer that estimates the whole state. We add that the order of the proposed reduced-order observer is $(n - m_2)$ rather than $(n - p)$, where n is the dimension of the plant state vector, m_2 is the number of the unknown inputs and p is the number of the plant's outputs.

2 Plant Model

The class of uncertain dynamical systems that we consider is modeled by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \quad (1)$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \quad (2)$$

where the input matrix $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ and the output matrix $\boldsymbol{C} \in \mathbb{R}^{p\times n}$. We assume that the model parameters $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ are known. We further assume that some or all of the inputs are unknown and that the last m_2 inputs are unkown and the remaining $m_1 = m - m_2$ inputs are known. We partition the input matrix $\boldsymbol{B}$ corresponding to the known and unknown inputs as

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{bmatrix},$$

*Department of Mathematical Sciences, San Diego State University, San Diego, CA 92182

†School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907-2035

where $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$ and $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$. Let

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1^T & \boldsymbol{u}_2^T \end{bmatrix}^T .$$

Then, the system model (1) can be represented as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2, \tag{3}$$

where $\boldsymbol{u}_1$ is known and $\boldsymbol{u}_2$ is unknown. Without loss of generality, we assume that $\boldsymbol{B}_2$ is of full rank.

3 A Projection Operator Approach to State Estimation

Since the system output $\boldsymbol{y}$ is known, it would seem reasonable to decompose the state $\boldsymbol{x}$ as

$$\boldsymbol{x} = (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{x}+\boldsymbol{M}\boldsymbol{C}\boldsymbol{x} = (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{x}+\boldsymbol{M}\boldsymbol{y}, \tag{4}$$

where $\boldsymbol{M}$ is an $n \times p$ real matrix, and the unknown part of the decomposition is $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}$. Let $\boldsymbol{q} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}$. Then $\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}$ and we have

$$\begin{aligned} \dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\dot{\boldsymbol{x}} \\ &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}_1\boldsymbol{u}_1+\boldsymbol{B}_2\boldsymbol{u}_2) \\ &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1) \\ &\quad +(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2. \end{aligned}$$

If $\boldsymbol{M}$ is chosen so that $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$, then the dynamics of $\boldsymbol{q}$ depend only on the known quantities $\boldsymbol{u}_1$ and $\boldsymbol{y}$:

$$\dot{\boldsymbol{q}} = (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1). \tag{5}$$

Note that if we start the above dynamical system with the initial condition $\boldsymbol{q}(0) = (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(0)$, then $\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}$ for all $t \geq 0$. But since $\boldsymbol{x}(0)$ is assumed to be unknown, then $\tilde{\boldsymbol{x}} = \boldsymbol{q}+\boldsymbol{M}\boldsymbol{y}$ is only an estimate of $\boldsymbol{x}$. To improve convergence rate or to ensure convergence, we add an extra term to the right-hand side of (5) to obtain

$$\begin{aligned} \dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1 \\ &\quad +\boldsymbol{L}(\boldsymbol{y}-\boldsymbol{C}\boldsymbol{q}-\boldsymbol{C}\boldsymbol{M}\boldsymbol{y})) \\ &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{y}+\boldsymbol{B}_1\boldsymbol{u}_1 \\ &\quad +\boldsymbol{L}\boldsymbol{C}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{M}\boldsymbol{y})). \end{aligned} \tag{6}$$

Let $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$. We will show that

$$\dot{\boldsymbol{e}} = (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\boldsymbol{e}$$

and $\boldsymbol{e}(t) \to \boldsymbol{0}$ as $t \to \infty$ under mild conditions, which we state in Section 4.

Since

$$\text{rank }(\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}_2) \leq \text{rank }(\boldsymbol{C}\boldsymbol{B}_2) \leq \text{rank }(\boldsymbol{B}_2),$$

the equality $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ makes it necessary that rank $(\boldsymbol{C}\boldsymbol{B}_2) = \text{rank}(\boldsymbol{B}_2)$, which we assume throughout the paper. This rank condition also implies that there must be at least as many independent outputs as unknown inputs for the method to work.

4 Background Results

In this section, we first analyze convergence properties of the proposed full-order unknown input observer (UIO) and then we use the results of our analysis to propose a new type of reduced-order UIO. Consider (6) and let

$$\tilde{\boldsymbol{x}} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}. \tag{7}$$

We will now show that $\tilde{\boldsymbol{x}} \to \boldsymbol{x}$ as $t \to \infty$. To this end let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ denote the estimation error. Then, using $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$, we have

$$\begin{aligned} \frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x}-\tilde{\boldsymbol{x}}) \\ &= \frac{d}{dt}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\ &= \frac{d}{dt}((\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{x}-\boldsymbol{q}) \\ &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}_1\boldsymbol{u}_1) \\ &\quad +(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2 \\ &\quad -(\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q}+\boldsymbol{A}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x}+\boldsymbol{B}_1\boldsymbol{u}_1 \\ &\quad +\boldsymbol{L}(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{C}\boldsymbol{q}-\boldsymbol{C}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x})) \\ &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\ &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\,\boldsymbol{e}. \end{aligned} \tag{8}$$

Our objective is to specify $\boldsymbol{M}$ and $\boldsymbol{L}$ and a set of initial conditions so that $\boldsymbol{e}(t) \to \boldsymbol{0}$ as $t \to \infty$. A particular class of solutions to $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ is given by

$$\boldsymbol{M} = \boldsymbol{B}_2\left((\boldsymbol{C}\boldsymbol{B}_2)^{\dagger} + \boldsymbol{H}_0\left(\boldsymbol{I}_p - (\boldsymbol{C}\boldsymbol{B}_2)(\boldsymbol{C}\boldsymbol{B}_2)^{\dagger}\right)\right)$$

where the superscript † denotes the Moore-Penrose pseudo-inverse operation and $\boldsymbol{H}_0 \in \mathbb{R}^{m_2\times p}$ is a design parameter matrix. Since, by assumption, rank $(\boldsymbol{C}\boldsymbol{B}_2)$ = rank $\boldsymbol{B}_2$ and $\boldsymbol{B}_2$ has full rank, we have $(\boldsymbol{CB}_2)^\dagger(\boldsymbol{CB}_2) = \boldsymbol{I}_{m_2}$. If $\boldsymbol{CB}_2$ is a square matrix, then $\boldsymbol{CB}_2$ is invertible by assumption and the above $\boldsymbol{M}$ reduces to $\boldsymbol{B}_2(\boldsymbol{CB}_2)^{-1}$. Furthermore, it is easy to check that for the above class of $\boldsymbol{M}$, the product $\boldsymbol{MC}$ is a projection (not necessarily orthogonal), that is, $(\boldsymbol{MC})^2 = \boldsymbol{MC}$. It follows that $\tilde{\boldsymbol{P}} = \boldsymbol{I} - \boldsymbol{MC}$ is also a projection.

To proceed further, we need the following lemma.

Lemma 1 *Let $\tilde{\boldsymbol{P}} : \mathbb{R}^n \to \mathbb{R}^n$ be a projection, that is, $\tilde{\boldsymbol{P}}^2 = \tilde{\boldsymbol{P}}$, and let rank $\tilde{\boldsymbol{P}} = n - m_2$. Then $\tilde{\boldsymbol{P}}$ has $(n - m_2)$ eigenvalues equal to 1 while the remaining m_2 eigenvalues are located at 0 and there is a basis of $\mathbb{R}^n$ in which the matrix $\tilde{\boldsymbol{P}}$ relative to this basis has the form*

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix},$$

that is, there is an invertible matrix $\boldsymbol{Q}$ whose columns are eigenvectors of $\tilde{\boldsymbol{P}}$ such that

$$\boldsymbol{Q}^{-1}\tilde{\boldsymbol{P}}\boldsymbol{Q} = \boldsymbol{P} = \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}.$$

Proof See Smith [4, pp. 156–158 and pp. 194–195]. □

5 Full-Order Unknown Input Observer

We begin this section by introducing the following coordinate transformation,

$$\tilde{\boldsymbol{e}} = \boldsymbol{Q}^{-1}\boldsymbol{e}, \tag{9}$$

where the transformation matrix $\boldsymbol{Q}$ is obtained, using Lemma 1, from the representation of the projection operator $\tilde{\boldsymbol{P}}$ in the form

$$\tilde{\boldsymbol{P}} = \boldsymbol{QPQ}^{-1}. \tag{10}$$

Applying the coordinate transformation (9) to the error equation (8) gives

$$\begin{aligned} \dot{\tilde{\boldsymbol{e}}} &= \boldsymbol{PQ}^{-1}(\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{Q}\tilde{\boldsymbol{e}} \\ &= \boldsymbol{P}\left(\boldsymbol{Q}^{-1}\boldsymbol{AQ} - \left(\boldsymbol{Q}^{-1}\boldsymbol{L}\right)(\boldsymbol{CQ})\right)\tilde{\boldsymbol{e}}. \end{aligned} \tag{11}$$

Let

$$\left.\begin{aligned} \tilde{\boldsymbol{A}} &= \boldsymbol{Q}^{-1}\boldsymbol{AQ} = \begin{bmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} \\ \tilde{\boldsymbol{A}}_{21} & \tilde{\boldsymbol{A}}_{21} \end{bmatrix} \\ \tilde{\boldsymbol{L}} &= \boldsymbol{Q}^{-1}\boldsymbol{L} = \begin{bmatrix} \tilde{\boldsymbol{L}}_1 \\ \tilde{\boldsymbol{L}}_2 \end{bmatrix} \\ \tilde{\boldsymbol{C}} &= \boldsymbol{CQ} = \begin{bmatrix} \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 \end{bmatrix}, \end{aligned}\right\} \tag{12}$$

where $\tilde{\boldsymbol{A}}_{11} \in \mathbb{R}^{(n-m_2)\times(n-m_2)}$, $\tilde{\boldsymbol{L}}_1 \in \mathbb{R}^{(n-m_2)\times p}$, $\tilde{\boldsymbol{C}}_1 \in \mathbb{R}^{p\times(n-m_2)}$, and the remaining block submatrices are of appropriate dimensions. Using the above notation, we represent (11) in the form,

$$\begin{aligned} \dot{\tilde{\boldsymbol{e}}} &= \boldsymbol{P}\left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}\right)\tilde{\boldsymbol{e}} \\ &= \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}\left(\begin{bmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} \\ \tilde{\boldsymbol{A}}_{21} & \tilde{\boldsymbol{A}}_{21} \end{bmatrix} - \begin{bmatrix} \tilde{\boldsymbol{L}}_1 \\ \tilde{\boldsymbol{L}}_2 \end{bmatrix}\begin{bmatrix} \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 \end{bmatrix}\right)\tilde{\boldsymbol{e}} \\ &= \begin{bmatrix} \tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{A}}_{12} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_2 \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}\tilde{\boldsymbol{e}}. \end{aligned} \tag{13}$$

Let

$$\tilde{\boldsymbol{e}} = \begin{bmatrix} \tilde{\boldsymbol{e}}_1^T & \tilde{\boldsymbol{e}}_2^T \end{bmatrix}^T, \tag{14}$$

where $\tilde{\boldsymbol{e}}_1 \in \mathbb{R}^{n-m_2}$. Note that $\dot{\tilde{\boldsymbol{e}}}_2 = \boldsymbol{0}$. Hence if $\tilde{\boldsymbol{e}}_2(0) = \boldsymbol{0}$, then $\tilde{\boldsymbol{e}}_2 = \boldsymbol{0}$ for all $t \geq 0$. Thus if $\tilde{\boldsymbol{e}}_2 = \boldsymbol{0}$ then $\dot{\tilde{\boldsymbol{e}}}_1 = \left(\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1\right)\tilde{\boldsymbol{e}}_1$, and so if $\tilde{\boldsymbol{e}}_2 = \boldsymbol{0}$ and $\tilde{\boldsymbol{e}}_1 \to \boldsymbol{0}$ then $\tilde{\boldsymbol{e}} \to \boldsymbol{0}$. Obviously $\tilde{\boldsymbol{e}}_1 \to \boldsymbol{0}$ for arbitrary $\tilde{\boldsymbol{e}}_1(0)$ if and only if the matrix $\left(\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1\right)$ is asymptotically stable.

We now give a condition on $\boldsymbol{q}(0)$ that guarantees that $\tilde{\boldsymbol{e}}_2 = \boldsymbol{0}$. We have

$$\boldsymbol{MC} = \boldsymbol{I} - \tilde{\boldsymbol{P}} = \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{Q}^{-1}.$$

Hence

$$\begin{aligned} \begin{bmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{e}}_2 \end{bmatrix} &= (\boldsymbol{I}_n - \boldsymbol{P})\boldsymbol{Q}^{-1}\boldsymbol{e} \\ &= \boldsymbol{Q}^{-1}\boldsymbol{MCe} \\ &= \boldsymbol{Q}^{-1}(\boldsymbol{MCx} - \boldsymbol{MCq} - (\boldsymbol{MC})^2\boldsymbol{x}) \\ &= \boldsymbol{Q}^{-1}\boldsymbol{MCq}. \end{aligned} \tag{15}$$

Therefore $\tilde{\boldsymbol{e}}_2(0) = \boldsymbol{0}$ if and only if $\boldsymbol{MCq}(0) = \boldsymbol{0}$, which is equivalent to $\boldsymbol{q}(0) = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{v}$ for arbitrary $\boldsymbol{v} \in \mathbb{R}^n$. In particular $\boldsymbol{q}(0) = \boldsymbol{0}$ satisfies the above condition.

In summary, we proved the following theorem,

Theorem 1 *If the following conditions are satisfied:*

1. *rank $(\boldsymbol{CB}_2) =$ rank $\boldsymbol{B}_2$;*
2. *the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{C}}_1)$ defined in (12) is detectable;*
3. *$\boldsymbol{q}(0) = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{v}$ for arbitrary $\boldsymbol{v} \in \mathbb{R}^n$,*

then there exists a gain matrix $\boldsymbol{L}$ such that the estimation error, $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$, of the full-order observer given by

$$\begin{aligned}\dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1 \\ &\quad + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\ \tilde{\boldsymbol{x}} &= \boldsymbol{q} + \boldsymbol{My}\end{aligned}$$

converges to $\mathbf{0}$ *as* $t \to \infty$.

Theorem 2 *The second condition of Theorem 1 that states that the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{C}}_1)$ defined in (12) is detectable, is equivalent to*

$$\mathit{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{O}\end{bmatrix} = n + m_2$$

for all s such that $Re(s) \geq 0$.

Proof See Hui and Żak [5]. □

6 Reduced-Order Unknown Input Observer

The error dynamics of the full-order observer that we analyzed above are given by equation (13):

$$\dot{\tilde{\boldsymbol{e}}} = \begin{bmatrix} \tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{A}}_{12} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_2 \\ \boldsymbol{O} & \boldsymbol{O}\end{bmatrix}\tilde{\boldsymbol{e}}.$$

We note that because we choose the initial condition for $\boldsymbol{q}$ to force $\tilde{\boldsymbol{e}}_2(t) = \mathbf{0}$ for $t \geq 0$, the dynamics of the error are completely determined by the dynamics of $\tilde{\boldsymbol{e}}_1$, which are given by

$$\dot{\tilde{\boldsymbol{e}}}_1 = \left(\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1\right)\tilde{\boldsymbol{e}}_1, \qquad (16)$$

an $(n - m_2)$-dimensional system. This motivates us to apply the transformation from $\boldsymbol{e}$ to $\tilde{\boldsymbol{e}}$ to $\boldsymbol{q}$ of the form, $\tilde{\boldsymbol{q}} = \boldsymbol{Q}^{-1}\boldsymbol{q}$. Next, from equation (6) and noting that $\boldsymbol{I} - \boldsymbol{MC} = \boldsymbol{QPQ}^{-1}$, we obtain

$$\begin{aligned}\dot{\tilde{\boldsymbol{q}}} &= \boldsymbol{P}(\boldsymbol{Q}^{-1}\boldsymbol{AQ}\tilde{\boldsymbol{q}} + \boldsymbol{Q}^{-1}\boldsymbol{AMy} \\ &\quad + \boldsymbol{Q}^{-1}\boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{Q}^{-1}\boldsymbol{L}(\boldsymbol{y} - \boldsymbol{CQ}\tilde{\boldsymbol{q}} \\ &\quad - \boldsymbol{CMy})) \\ &= \boldsymbol{P}\left(\boldsymbol{Q}^{-1}\boldsymbol{AQ} - \boldsymbol{Q}^{-1}\boldsymbol{LCQ}\right)\tilde{\boldsymbol{q}} \\ &\quad + \boldsymbol{P}\left(\boldsymbol{Q}^{-1}\boldsymbol{AM} + \boldsymbol{Q}^{-1}\boldsymbol{L}\right. \\ &\quad \left. - \boldsymbol{Q}^{-1}\boldsymbol{LCM}\right)\boldsymbol{y} + \boldsymbol{PQ}^{-1}\boldsymbol{B}_1\boldsymbol{u}_1.\end{aligned}$$

Using the notation defined in (12), we have

$$\begin{aligned}\dot{\tilde{\boldsymbol{q}}} &= \boldsymbol{P}\left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}\right)\tilde{\boldsymbol{q}} + \boldsymbol{PQ}^{-1}[(\boldsymbol{AM} \\ &\quad + \boldsymbol{Q}\tilde{\boldsymbol{L}}(\boldsymbol{I}_p - \boldsymbol{CM}))\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1].\end{aligned}$$

Let

$$\tilde{\boldsymbol{q}} = \begin{bmatrix} \tilde{\boldsymbol{q}}_1^T & \tilde{\boldsymbol{q}}_2^T \end{bmatrix}^T,$$

where $\tilde{\boldsymbol{q}}_1 \in \mathbb{R}^{n-m_2}$ and $\tilde{\boldsymbol{q}}_2 \in \mathbb{R}^{m_2}$. Since

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O}\end{bmatrix},$$

we have $\dot{\tilde{\boldsymbol{q}}}_2(t) = \mathbf{0}$. Therefore, setting $\tilde{\boldsymbol{q}}_2(0) = \mathbf{0}$ ensures that $\tilde{\boldsymbol{q}}_2(t) = \mathbf{0}$ for $t \geq 0$. We thus can remove m_2 observer states from the observer dynamics. Let $\tilde{\boldsymbol{G}} = \boldsymbol{AM} + \boldsymbol{Q}\tilde{\boldsymbol{L}}(\boldsymbol{I}_p - \boldsymbol{CM})$. Then the resulting reduced-order observer takes the form,

$$\begin{aligned}\dot{\tilde{\boldsymbol{q}}}_1 &= (\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1)\tilde{\boldsymbol{q}}_1 \\ &\quad + \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O}_{m_2}\end{bmatrix}\boldsymbol{Q}^{-1}(\tilde{\boldsymbol{G}}\boldsymbol{y} \\ &\quad + \boldsymbol{B}_1\boldsymbol{u}_1), \quad \tilde{\boldsymbol{q}}_1(0) = \mathbf{0} \\ \tilde{\boldsymbol{x}} &= \boldsymbol{Q}\begin{bmatrix} \boldsymbol{I}_{n-m_2} \\ \boldsymbol{O}_{m\times(n-m_2)}\end{bmatrix}\tilde{\boldsymbol{q}}_1 + \boldsymbol{My},\end{aligned}$$

where the vector $\tilde{\boldsymbol{x}}$ is the estimate of the plant state $\boldsymbol{x}$. We now summarize the above considerations in the form of the following design algorithm.

Reduced-Order Unknown Input Observer Design Algorithm

For a given a quadruple of matrices $(\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C})$ modeling the plant,

1. Check that rank $(\boldsymbol{CB}_2) = \operatorname{rank}\boldsymbol{B}_2$
   If rank $(\boldsymbol{CB}_2) < \operatorname{rank}\boldsymbol{B}_2$, STOP. The observer does not exist.
2. Compute
   $$\boldsymbol{M} = \boldsymbol{B}_2\left((\boldsymbol{CB}_2)^\dagger + \boldsymbol{H}_0\left(\boldsymbol{I}_p - (\boldsymbol{CB}_2)(\boldsymbol{CB}_2)^\dagger\right)\right)$$
3. Compute the projector, $\tilde{\boldsymbol{P}} = \boldsymbol{I}_n - \boldsymbol{MC}$.
4. Represent $\tilde{\boldsymbol{P}}$ as $\tilde{\boldsymbol{P}} = \boldsymbol{QPQ}^{-1}$, where
   $$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O}\end{bmatrix}.$$
5. Compute
   $$\begin{aligned}\tilde{\boldsymbol{A}} &= \boldsymbol{Q}^{-1}\boldsymbol{AQ} = \begin{bmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} \\ \tilde{\boldsymbol{A}}_{21} & \tilde{\boldsymbol{A}}_{22}\end{bmatrix}, \\ \tilde{\boldsymbol{C}} &= \boldsymbol{CQ} = \begin{bmatrix} \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 \end{bmatrix},\end{aligned}$$

where $\tilde{\boldsymbol{A}}_{11} \in \mathbb{R}^{(n-m_2)\times(n-m_2)}$ and $\tilde{\boldsymbol{C}}_1 \in \mathbb{R}^{p\times(n-m_2)}$

6. Check detectability of the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{C}}_1)$
If the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{C}}_1)$ is not detectable, STOP. The observer does not exist.
Note that if the matrix $\tilde{\boldsymbol{A}}_{11}$ is asymptotically stable, then the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{C}}_1)$ is detectable for an arbitrary matrix $\tilde{\boldsymbol{C}}_1$.

7. If there are eigenvalues of $\tilde{\boldsymbol{A}}_{11}$ that are not asymptotically stable, construct $\tilde{\boldsymbol{L}}_1$ so that the matrix $(\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1)$ has its eigenvalues in locations as close to the desired eigenvalues as possible.

8. Form
$$\tilde{\boldsymbol{L}} = \begin{bmatrix} \tilde{\boldsymbol{L}}_1 \\ \boldsymbol{O}_{m_2\times p} \end{bmatrix},$$
where $\boldsymbol{O}_{m_2\times p}$ is an $m_2 \times p$ zero matrix.

9. Compute matrix
$$\tilde{\boldsymbol{G}} = \boldsymbol{AM} + \boldsymbol{Q}\tilde{\boldsymbol{L}}(\boldsymbol{I}_p - \boldsymbol{CM}).$$

10. Construct the observer,
$$\begin{aligned} \dot{\tilde{\boldsymbol{q}}}_1 &= (\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1)\tilde{\boldsymbol{q}}_1 \\ &\quad + \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O}_{m_2} \end{bmatrix} \boldsymbol{Q}^{-1}(\tilde{\boldsymbol{G}}\boldsymbol{y} \\ &\quad + \boldsymbol{B}_1\boldsymbol{u}_1), \quad \tilde{\boldsymbol{q}}_1(0) = \boldsymbol{0} \\ \tilde{\boldsymbol{x}} &= \boldsymbol{Q}\begin{bmatrix} \boldsymbol{I}_{n-m_2} \\ \boldsymbol{O}_{m_2\times(n-m_2)} \end{bmatrix}\tilde{\boldsymbol{q}}_1 + \boldsymbol{My}. \end{aligned}$$
The vector $\tilde{\boldsymbol{x}}$ is the estimate of the state $\boldsymbol{x}$.

7 Summary and Relation With Other Observer Architectures

In this paper, we concentrated on the analysis and the design of full-order observers that can be used to construct reduced-order observers. Our analysis can be extended to cover the case

$$\begin{aligned} \dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) \\ &\quad + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy}), \end{aligned} \tag{17}$$

where the term $\boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})$ is not multiplied by $(\boldsymbol{I} - \boldsymbol{MC})$. However, this case leads to the observer analyzed in [6, 7] even though the approach there is quite different. Indeed, we can equivalently represent the dynamics of the proposed in this paper full-order observer as follows:

$$\begin{aligned} \dot{\boldsymbol{q}} &= ((\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{q} \\ &\quad + ([(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{A} - \boldsymbol{LC}]\boldsymbol{M} + \boldsymbol{L})\boldsymbol{y} \\ &\quad + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_1\boldsymbol{u}_1 \\ &= (\boldsymbol{TA} - \boldsymbol{LC})\boldsymbol{q} + \boldsymbol{Ky} + \boldsymbol{TB}_1\boldsymbol{u}_1, \\ \tilde{\boldsymbol{x}} &= \boldsymbol{q} + \boldsymbol{My}, \end{aligned}$$

where, using the notation similar to that in [6, 7],

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{I} - \boldsymbol{MC}, \quad \boldsymbol{K}_1 = \boldsymbol{L}, \\ \boldsymbol{K}_2 &= [\boldsymbol{TA} - \boldsymbol{LC}]\boldsymbol{M}, \quad \boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2. \end{aligned}$$

In addition, the conditions for the existence of the full-order observer presented in [6, 7] and our observers are equivalent.

The observer given by (17) is also the same as the one proposed by Yang and Wilde [8] and further analyzed by Darouach, Zasadzinski, and Xu [9]. The connections are as follows: (i) $\boldsymbol{M}$ is called $-\boldsymbol{E}$ in [8, 9], (ii) $(\boldsymbol{I} - \boldsymbol{MC})$ corresponds to $\boldsymbol{P}$ there, (iii) $\boldsymbol{B}_1$ is $\boldsymbol{B}$ and $\boldsymbol{B}_2$ is $\boldsymbol{D}$ in [8, 9], (iv) $(\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A} - \boldsymbol{LC})$ corresponds to $\boldsymbol{N}$.

We now compare the reduced-order UIO proposed by Hou and Müller [10] with our reduced-order UIO. Somewhat similair approach is proposed by Kudva et al. [11]. Hou and Müller first transform the system (3) into the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{u}_1 + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{m_2} \end{bmatrix}\boldsymbol{u}_2 \\ &= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{12} \end{bmatrix}\boldsymbol{u}_1 \\ &\quad + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{m_2} \end{bmatrix}\boldsymbol{u}_2. \end{aligned}$$

Note that $\boldsymbol{x}_1$ in the new coordinates is independent of $\boldsymbol{u}_2$ and we have

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 + \boldsymbol{B}_{11}\boldsymbol{u}_1. \tag{18}$$

Let
$$\boldsymbol{y} = \boldsymbol{Cx} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix},$$
where $\boldsymbol{C}_2 \in \mathbb{R}^{p\times m_2}$. Because by assumption, $\text{rank}(\boldsymbol{CB}_2) = \text{rank}\boldsymbol{B}_2 = m$, the submatrix $\boldsymbol{C}_2$ has a left inverse, $\boldsymbol{C}_2^\dagger$. Hence we can compute

$$\boldsymbol{x}_2 = -\boldsymbol{C}_2^\dagger\boldsymbol{C}_1\boldsymbol{x}_1 + \boldsymbol{C}_2^\dagger\boldsymbol{y}. \tag{19}$$

Substituting the above into (18) gives

$$\dot{\boldsymbol{x}}_1 = \left(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{C}_2^{\dagger}\boldsymbol{C}_1\right)\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{C}_2^{\dagger}\boldsymbol{y} + \boldsymbol{B}_{11}\boldsymbol{u}_1.$$

Hou and Müller [10] propose now to construct an observer for $\boldsymbol{x}_1$ using only known signals and then substitute the estimate of $\boldsymbol{x}_1$ into (19) to obtain an estimate of $\boldsymbol{x}_2$. Thus the resulting architecture of the reduced-order UIO proposed by Hou and Müller [10], as well as their approach differs from our design. Yet another approach to constructing reduced-order UIOs can be found in [12].

8 Future Work

Effectiveness of the unknown input observers (UIOs) in real-life applications need to be investigated. A successful application of the UIOs to a DC servo motor system was reported by Chang et al. [13]. On the the other hand, Millerioux and Daafouz [14] proposed UIO architectures for switched linear discrete systems. Röbenack and Lynch [15] presented a method to the observer design for a class of nonlinear plants which yields almost linear observation error dynamics. This method looks like a promising tool to be used to extend our approach to a class of nonlinear plants. Another promising application of the proposed in this paper UIOs is in the area of fault detection and isolation—see, for example, [16] for an application of sliding mode observers for fault detection and isolation.

References

[1] D. G. Luenberger, "Observers for multivariable systems," *IEEE Transactions on Automatic Control*, vol. AC-11, pp. 190–197, April 1966.

[2] D. G. Luenberger, "An introduction to observers," *IEEE Transactions on Automatic Control*, vol. AC-16, pp. 596–602, December 1971.

[3] C. Edwards and S. K. Spurgeon, *Sliding Mode Control: Theory and Applications*. Systems and Control Book Series, London, England: Taylor & Francis Ltd, 1998.

[4] L. Smith, *Linear Algebra*. New York: Springer-Verlag, second ed., 1984.

[5] S. Hui and S. H. Żak, "Observer design for uncertain systems." Submitted for publication, 2005.

[6] J. Chen, R. J. Patton, and H.-Y. Zhang, "Design of unknown input observers and robust fault detection filters," *Int. J. Control*, vol. 63, no. 1, pp. 85–105, 1996.

[7] J. Chen and R. J. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*. Boston: Kluwer Academic Publishers, 1999.

[8] F. Yang and R. W. Wilde, "Observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 33, pp. 677–681, July 1988.

[9] M. Darouach, M. Zasadzinski, and S. J. Xu, "Full-order observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 39, pp. 606–609, March 1994.

[10] M. Hou and P. C. Müller, "Design of observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 37, pp. 871–875, June 1992.

[11] P. Kudva, N. Viswanadham, and A. Ramakrishna, "Observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. AC-25, pp. 113–115, February 1980.

[12] S. Hui and S. H. Żak, "Low-order state estimators and compensators for dynamical systems with unknown inputs," *Systems & Control Letters*, vol. 21, no. 6, pp. 493–502, 1993.

[13] S.-K. Chang, W.-T. You, and P.-L. Hsu, "Design of general structured observers for linear systems with unknown inputs," *Journal of the Franklin Institute*, vol. 334B, pp. 213–232, March 1997.

[14] G. Millerioux and J. Daafouz, "Unknown input observers for switched linear discrete time systems," in *Proceedings of the 2004 American Control Conference, Boston, Massachusetts*, pp. 5802–5805, June 30–July 2, 2004.

[15] K. Röbenack and A. F. Lynch, "An efficient method for observer design with approximately linear error dynamics," *Int. J. Control*, vol. 77, pp. 607–612, 10 May 2004.

[16] C. Edwards, S. K. Spurgeon, and R. J. Patton, "Sliding mode observers for fault detection and isolation," *Automatica*, vol. 36, pp. 541–553, April 2000.