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Newton Observer Design in the Absence of an Exact Discrete-Time Model

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Abstract— **We study the Newton observer design, developed by Moraal and Grizzle [1], when the exact discrete-time model is not known analytically. We eliminate the dependence on this exact model by introducing continuous-time filters that produce the numerical value of the exact model. We then implement the Newton method with *finite-difference* and *secant* approximations for the Jacobian. The proposed method offers flexibility to be implemented with nonuniform, or event-driven, sampling.**

I. Introduction

Progress in nonlinear output feedback control has been hampered by the shortage of constructive tools for observer design. A further difficulty arises when only sampled measurements of the output are available because, then, an exact discrete-time model of the process may be difficult, or impossible, to obtain [2]. In their seminal paper [1], Moraal and Grizzle pose the sampled-data observer problem as a nonlinear equation in which the unmeasured states depend on the past samples of the input and the output through a discrete-time *observability mapping*. These states are then estimated via Newton iterations, with input and output data available in each sampling period. The resulting scheme, referred to as the "Newton observer", is widely applicable and offers ample design flexibility thanks to numerous ramifications of the Newton algorithm. However, in its basic form presented in [1], this observer relies on the availability of an exact discrete-time model, which is seldom available for nonlinear systems.

In the absence of an exact discrete-time model, a common approach is to resort to approximate discretizations, such as Euler's method. This approach, however, results in a residual observer error [2] which, depending on the application and the sampling rate, may be intolerable. An attempt to reduce this error by increasing the sampling rate would compromise the reliability of the Newton observer because, for fast sampling rates, the observability mapping to be inverted would be ill-conditioned (close to singular), and numerical errors would be likely. The alternative approach of refining the approximate models for fixed sampling rates [2] inflates the analytical expressions, rendering them intractable for design.

To circumvent these problems, the approach taken in this paper is to evaluate the exact discrete-time model numerically, rather than analytically. This is achieved by introducing continuous-time filters in the Newton observer, which mimic the solution of the underlying continuous-time plant over one sampling period, thus producing the numerical outcome of the exact discrete-time model for a given initial condition. A separate filter evaluates the observability mapping which is used in Newton iterations. Because the Jacobian of this mapping is also unavailable analytically, we approximate it either with *finite-difference* or *secant* methods within Newton iterations. Broyden's Method, a particular secant technique, has already been applied in [1] to reduce the computational cost of the Newton algorithm. However, this application still relies on analytical expressions for the exact model and the observability mapping.

The modified Newton observer in this paper has a hybrid structure because it combines discrete-time iterations with our continuous-time filters. However, it preserves its sampled-data characteristic because it only employs discrete-time measurements of the output. This design differs from a hybrid variant of the observer presented in [1], in which a continuous-time

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Newton algorithm is employed, and the exact discrete-time model is still required. In a digital implementation, our continuous-time filters would be replaced by powerful numerical integration schemes such as those surveyed in [5]. Compared to the approach of refining approximate models analytically, these numerical integration schemes would offer superior versatility and accuracy.

In Section II we review the Newton observer of [1] and point to the functions that would be unknown in the absence of an exact-discrete time model. In Section III we introduce our continuous-time filters to numerically evaluate these functions and to approximate their Jacobians. Section IV illustrates the resulting hybrid observer on an example. Conclusions are given in Section V.

II. Problem Statement

We consider the system

$$\dot{x} = f(x,u), \quad y = h(x,u) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$. Given a sampling period $T > 0$, we assume that the control u is constant during sampling intervals $[kT,(k+1)T)$ and that the output y is measured at instants kT. The exact discrete-time model of (1) is

$$x_{k+1} = F_T(x_k,u_k), \quad y_k = h(x_k,u_k) \tag{2}$$

where $x_k := x(kT)$, $y_k := y(kT)$, $u_k := u(kT)$ and $F_T(x_k,u_k)$ denotes the solution of (1) with initial condition $x(0) = x_k$. The objective of the Newton observer [1] is to estimate the unmeasured state vector x_k from N consecutive measurements of outputs and inputs, denoted as:

$$Y_k := \begin{bmatrix} y_{k-N+1} \\ y_{k-N+2} \\ \vdots \\ y_k \end{bmatrix}, \qquad U_k := \begin{bmatrix} u_{k-N+1} \\ u_{k-N+2} \\ \vdots \\ u_k \end{bmatrix}. \tag{3}$$

To express Y_k as a function of x_{k-N+1} and U_k, denote $F_T^{u_k}(x_k) := F_T(x_k,u_k)$ and $h^{u_k}(x_k) := h(x_k,u_k)$ as in [1], and note from (2) that

$$\begin{aligned} Y_k &= H_T(x_{k-N+1},U_k) \\ &:= \begin{bmatrix} h^{u_{k-N+1}}(x_{k-N+1}) \\ h^{u_{k-N+2}} \circ F_T^{u_{k-N+1}}(x_{k-N+1}) \\ \vdots \\ h^{u_k} \circ F_T^{u_{k-1}} \circ \ldots \circ F_T^{u_{k-N+1}}(x_{k-N+1}) \end{bmatrix} \end{aligned} \tag{4}$$

where "$\circ$" denotes composition and $H_T(\cdot,U_k) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the "observability mapping" of the discrete-time model (2). When this mapping is invertible, the observer problem consists in solving the N^{th} order nonlinear equation

$$Y_k - H_T(x_{k-N+1},U_k) = 0 \tag{5}$$

in x_{k-N+1}. This is achieved in [1] with the Newton iterations:

$$\begin{aligned} w_k^{i+1} &= w_k^i + \left[\ \frac{\partial H_T}{\partial w}(w_k^i,U_k) \ \right]^{-1} (Y_k - H_T(w_k^i,U_k)) \\ i &= 0,...,d-1, \end{aligned} \tag{6}$$

where the number of iterates, d, is a design parameter. In this paper, we assume that H_T is square, i.e $N = n$, otherwise inverses should be replaced by pseudo-inverses. The final estimate w_k^d of x_{k-N+1} is propagated in time by $N-1$ steps to obtain

$$\hat{x}_k = F_T^{u_{k-1}} \circ F_T^{u_{k-2}} \circ \ldots \circ F_T^{u_{k-N+1}}(w_k^d), \tag{7}$$

and the initial condition for the next sampling period is assigned to be

$$w_{k+1}^0 = F_T(w_k^d, u_{k-N+1}). \tag{8}$$

A shortcoming of this algorithm is that it relies on an analytical expression for the exact discrete-time model F_T in (2). Indeed (7)-(8) directly require the knowledge of F_T, while (6) relies on the knowledge of H_T and its Jacobian, which also depend on F_T as in (4). To calculate the exact model F_T analytically, however, we need a closed form solution to the initial value problem

$$\dot{x} = f(x,u_k), \quad x(0) = x_k \tag{9}$$

over one sampling interval $[kT,(k+1)T)$, which is impossible to obtain in general. In the next section, we solve this problem by numerically integrating (9) to compute H_T and F_T within each sampling period, and approximate $\frac{\partial H_T}{\partial w}$ via *finite-difference* or *secant* methods.

III. Numerical Evaluation of the Discrete-Time Model

To numerically evaluate $F_T(w_k^d, u_{k-N+1})$ in (8) at the k^{th} sampling period we employ the continuous-time filter:

$$\begin{aligned} \tau_1 \dot{\xi} &= f(\xi,u) \\ \xi(t_0) &= w_k^d, \quad u = u_{k-N+1}, \quad t_0 = kT. \end{aligned} \tag{10}$$

The solution of (10) at $t = t_0 + \tau_1 T$ coincides with the solution of the original system (1) at $t = t_0 + T$ and, thus,

$$\xi(t_0 + \tau_1 T) = F_T(w_k^d, u_{k-N+1}). \tag{11}$$

This means that $F_T(w_k^d, u_{k-N+1})$ is evaluated numerically with the filter (10), without the need for an analytical expression. The scaling factor $\tau_1 < 1$ ensures that F_T is obtained quickly enough that other calculations discussed below can be carried out within the same sampling period.

To compute $H_T(w_k^i, U_k)$ in (6) at the k^{th} sampling period and $(i+1)^{st}$ Newton step, we let $\bar{u}$ denote the zero-order-hold output of u_k, that is $\bar{u}(t) := u_{k-N+m}$, $(m-1)T \le t < mT$, $m = 1, ..., N$ and employ the filter

$$\begin{aligned} \tau_2 \dot{\zeta}_0 &= f(\zeta_0, \bar{u}(t - t_0)) \\ y_0 &= h(\zeta_0, \bar{u}(t - t_0)) \\ \zeta_0(t_0) &= w_k^i, \; t_0{=}kT + \tau_1 T + i(N-1)\tau_2 T, \end{aligned} \tag{12}$$

where, again, $\tau_2 < 1$. The output y_0 of this filter at $t = t_0 + \tau_2 T$ coincides with the output y of (1) at $t = t_0 + T$ and, hence, H_T in (4) can be constructed via consecutive measurements of y_0:

$$H_T(w_k^i, U_k) := \begin{bmatrix} y_0(t_0) \\ y_0(t_0 + \tau_2 T) \\ \vdots \\ y_0(t_0 + \tau_2(N-1)T) \end{bmatrix}. \tag{13}$$

Next, to find a finite difference approximation of the Jacobian $\frac{\partial H_T}{\partial w}$ as in [3], at each Newton step we dither w_i by ρe_j where e_j is the j^{th} unit vector and ρ is a small perturbation. Thus, with n parallel filters with initial conditions $w_k^i + \rho e_j$, $j = 1, ..., n$ we obtain an approximation to $\frac{\partial H_T}{\partial w}$ as follows:

$$\begin{aligned} \tau_2 \dot{\zeta}_j &= f(\zeta_j, \bar{u}) \\ y_j &= h(\zeta_j, \bar{u}) \\ \zeta_j(t_0) &= w_k^i + \rho e_j \\ t_0 &= kT + \tau_1 T + i(N-1)\tau_2 T \\ i &= 0, 1, ..., d-1 \end{aligned}$$

$$H_T(w_k^i + \rho e_j, U_k) = \begin{bmatrix} y_j(t_0) \\ y_j(t_0 + \tau_2 T) \\ \vdots \\ y_j(t_0 + \tau_2(n-1)T) \end{bmatrix}$$

$$\frac{\partial H_T}{\partial w}(w_k^i, U_k) \approx \begin{bmatrix} \left(\frac{H(w_k^i + \rho e_1, U_k) - H(w_k^i, U_k)}{\rho}\right)^T \\ \vdots \\ \left(\frac{H(w_k^i + \rho e_n, U_k) - H(w_k^i, U_k)}{\rho}\right)^T \end{bmatrix}^T \tag{14}$$

Finally, after d iterations of (12) and (14), we mimic (7) with the filter

$$\begin{aligned} \tau_3 \dot{\psi} &= f(\psi, \bar{u}) \\ \psi(t_0) &= w_k^d, \;\; t_0{=}kT{+}\tau_1 T{+}d(N{-}1)\tau_2 T \\ \hat{x}_k &= \psi(t_0 + \tau_3 T), \quad \tau_3 < 1. \end{aligned} \tag{15}$$

Thus, each sampling period of our algorithm starts with filter (10) which is run for $\tau_1 T$ seconds, followed by the $n+1$ filters (12)-(14) run in parallel d times, each for $\tau_2(N-1)T$ seconds, and concludes with filter (15) run for $\tau_3 T$ seconds. This means that the scaling factors τ_i, $i = 1, 2, 3$, are to be selected such that:

$$\tau_1 + d(N-1)\tau_2 + \tau_3 \le 1. \tag{16}$$

It is important to note that, despite the introduction of continuous-time updates, the observer only makes use of discrete time measurements of the output y. With a simple modification of the algorithm, we can replace the one-sided approximation (14) of the Jacobian with

$$\frac{\partial H_T}{\partial w}(w_k^i, U_k) \approx \begin{bmatrix} \left(\frac{H(w_k^i + \rho e_1, U_k) - H(w_k^i - \rho e_1, U_k)}{2\rho}\right)^T \\ \vdots \\ \left(\frac{H(w_k^i + \rho e_n, U_k) - H(w_k^i - \rho e_n, U_k)}{2\rho}\right)^T \end{bmatrix}^T \tag{17}$$

which, as shown in [4, pp. 168-169], reduces the approximation error from $\mathcal{O}(\rho)$ to $\mathcal{O}(\rho^2)$. However, for this implementation, the number of filters in (14) must be raised from n to $2n$, which increases the complexity of the observer.

An alternative Jacobian approximation, which eliminates the need for the "dither filters" (14), is the *secant* method [3, Chapter 8.4] :

$$w_k^{i+1} = w_k^i + (J_k^i)^{-1}[Y_k - H_T(w_k^i, U_k)] \qquad (18)$$

which employs the "surrogate" J_k^i for the Jacobian, and updates it in each Newton step such that

$$J_k^{i+1} s_k^i = v_k^i \qquad (19)$$

where $s_k^i := w_k^{i+1} - w_k^i$ and $v_k^i := H_T(w_k^i, U_k) - H_T(w_k^{i+1}, U_k)$. Among numerous update laws that comply with (19), an advantageous one is Broyden's method, given by

$$J_k^{i+1} = J_k^i + \frac{(v_k^i - J_k^i s_k^i)(s_k^i)^T}{(s_k^i)^T s_k^i} \qquad (20)$$

and incorporated in Newton Observer design in [1, Section 6]. Although our implementation differs from [1, Section 6], in that H_T and F_T are evaluated numerically with filters (10), (12) and (15) rather than analytically, its convergence properties are the same as those proved in [1, Lemma 6.1].

For the *finite-difference* approximation (14), it follows from a combination of the proof of [6, Theorem 3.2] and [3,Theorem 5.4.1] that the conclusion of [1, Theorem 3.2] holds, with the size of the region of attraction (specified with number δ in [1, Theorem 3.2]) reduced by the size of dither, ρ. Although these convergence proofs are local, it would not be difficult to obtain global results by employing global Newton schemes, such as those discussed in [3, Chapter 6].

IV. Design Example

We now pursue Newton observer designs for the Rössler equation [7, p. 376]:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b + x_1 x_3 - c x_3 \\ y &= x_2 \end{aligned} \qquad (21)$$

which, for the parameter values $a = b = 0.2$ and $c = 4$, exhibits periodic solutions. For comparison to the numerical evaluation method of Section III, we first apply the analytical design (6)-(8) with the Euler approximation of the exact discrete-time model for (21), given by:

$$\begin{aligned} x_{k+1} &= F_T^a(x_k) := \begin{bmatrix} x_{1,k} + T(-x_{2,k} - x_{3,k}) \\ x_{2,k} + T(x_{1,k} + a x_{2,k}) \\ x_{3,k} + T(b + x_{1,k} x_{3,k} - c x_{3,k}) \end{bmatrix} \\ y_k &= x_{2,k}, \end{aligned} \qquad (22)$$

where x_k denotes the state vector at the k^{th} sampling instant and $x_{1,k}$, $x_{2,k}$ and $x_{3,k}$ denote its components. The Newton observer obtained by letting $N = 3$, $d = 5$ and using

$$H_T^a(x_{k-2}, U_k) := \begin{bmatrix} x_{2,k-2} \\ x_{2,k-2} + T(x_{1,k-2} + a x_{2,k-2}) \\ \begin{pmatrix} x_{2,k-2} + T(x_{1,k-2} + a x_{2,k-2}) \\ +T(x_{1,k-2} + T(-x_{2,k-2} - x_{3,k-2})) \\ +aT(x_{2,k-2} + T(x_{1,k-2} + a x_{2,k-2})) \end{pmatrix} \end{bmatrix} \qquad (23)$$

$$\frac{\partial H_T^a}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ T & 1 + Ta & 0 \\ 2T + T^2 a & 1 + 2Ta - T^2 + T^2 a^2 & -T^2 \end{bmatrix} \qquad (24)$$

from (22) results in a residual observer error as in Figures 1-2. An improvement is achieved in Figures 3-4 by incorporating $\mathcal{O}(T^2)$ terms in the approximation (22); that is, by replacing the Euler Formula $x_{k+1} = x_k + T f(x_k)$ with $x_{k+1} = x_k + T f(x_k) + \frac{T^2}{2} \frac{\partial f(x_k)}{\partial x} f(x_k)$.

Rather than pursue higher order approximations, which lead to cumbersome analytical expressions, we now apply the numerical evaluation method of Section III. We run filter (10) to compute F_T, and filters (12)-(14) with $\rho = 0.01$ to obtain H_T and $\frac{\partial H_T}{\partial x}$. The output of (12) after $d = 5$ iterations is propagated in time by filter (15) to obtain $\hat{x}_k$. This hybrid design eliminates the residual error as shown in Figures 5-6.

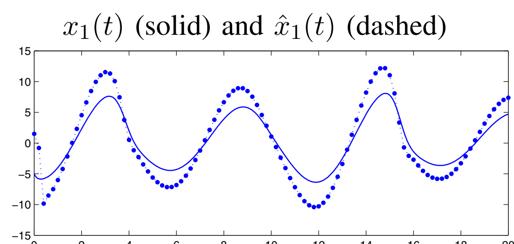

Fig. 1. Newton observer with Euler approximation (22)-(24), T=0.2 sec.

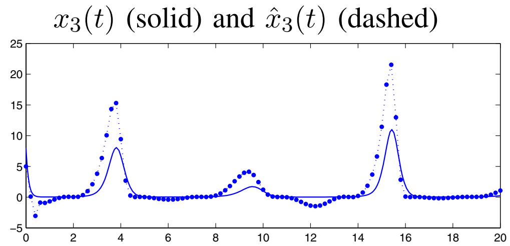

Fig. 2. Newton observer with Euler approximation (22)-(24), T=0.2 sec.

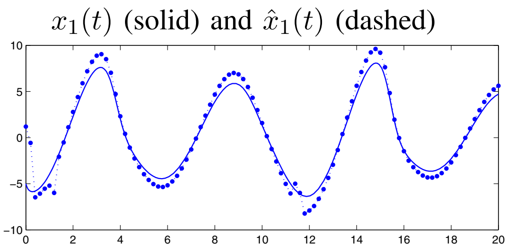

Fig. 3. Newton observer with the addition of $\mathcal{O}(T^2)$ terms in the approximation, T=0.2 sec.

V. Conclusions

We have eliminated the dependence of the Newton observer [1] on the knowledge of the exact discrete-time plant model. This has been achieved by carrying out the model-based calculations at each step numerically, with the help of continuous-time filters. An important feature of this design is that it can be extended, upon suitable modifications, to systems with nonuniform and event-driven sampling. Indeed, the duration for which the filters (10), (12), (15) are run, and their time-scale factors τ_i, $i = 1, 2, 3$, can be adjusted on-line according to the varying sampling period. Another important research direction is to explore other classes of sampled-data observers and controllers to which our numerical evaluation method can be applied.

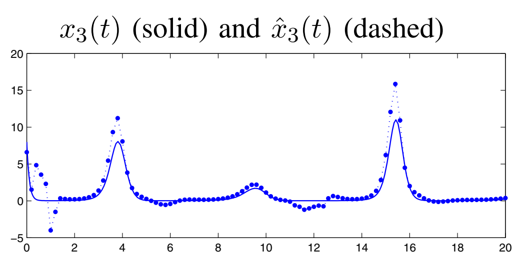

Fig. 4. Newton observer with the addition of $\mathcal{O}(T^2)$ terms in the approximation, T=0.2 sec.

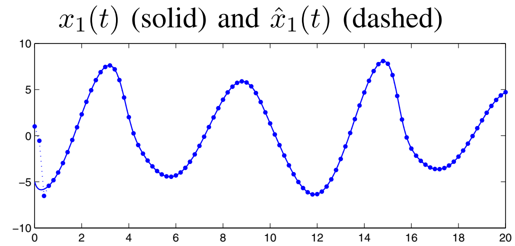

Fig. 5. Newton observer with numerical evaluation, T=0.2 sec.

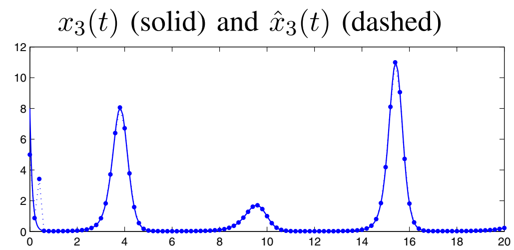

Fig. 6. Newton observer with numerical evaluation, T=0.2 sec.

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