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A Parametrization of Supply Rates for Small-Gain Conditions for Global Stability of Interconnected iISS and ISS Systems

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Abstract— **This paper develops a novel tool of establishing global asymptotic stability of nonlinear interconnected systems. The aim is to incorporate useful flexibility in small-gain techniques for integral Input-to-State Stable(iISS) and Input-to-State Stable(ISS) systems. This paper introduces a unique idea of "parametrization of supply rates", which parametrizes a set of many supply rates with which stability of an interconnected system can be derived from a single small-gain condition. The parametrization offers convenient flexibility of free functions to determine proper supply rates that match systems and that lead to stability of the interconnection. The new parametrization covers iISS systems as well as ISS systems.**

I. Introduction

This paper addresses the problem of global asymptotic stability of interconnection of nonlinear dissipative systems. This paper focuses on the question of how to deduce supply rates for establishing the stability of the interconnected systems from simple or natural supply rates and Lyapunov(storage) functions of constituent systems. The unique idea this paper employs to attain this goal is incorporation of new flexibilities into advanced small-gain theorems, and development of new stability criteria based on parametrization of supply rates. This idea originated from the author's paper [5]. In [5], the author has derived a parametrization of supply rates for input-to-state stable(ISS) systems. This paper seeks a similar, but new parametrization of supply rates for integral input-to-state stable(iISS) systems which are more general and allowed to have stronger nonlinearities than ISS systems.

Let us consider a feedback interconnection of two systems Σ_1 and Σ_2 shown in Fig.1. In order to invoke the ISS small-gain theorem[8], [14] for establishing stability of the interconnection, we need to pick and fix supply rates for Σ_1 and Σ_2 beforehand. Suppose that we have chosen a supply rate of Σ_2 naturally. There would be two possible ways to select a supply rate of Σ_1.

- **S1)** calculate an ISS type of supply rate as an upper bound of the time-derivative of a naturally guessed or somehow composed Lyapunov function along the trajectories of Σ_1;
- **S2)** borrow functions from the supply rate of Σ_2, or simply select functions, so that the ISS small-gain condition is fulfilled.

The former supply rate is not guaranteed to meet the ISS small-gain condition(a contraction inequality) although the supply rate is guaranteed to be accepted by Σ_1. The latter one forces the ISS small-gain condition to be satisfied. Such a supply rate is, however, usually incompatible with Σ_1. Indeed, choosing a supply rate leading us to desired stability is not an easy task. To this problem, this paper and [5] pursue a new approach with which we are able to combine advantages of **S1** and **S2**. If we select only one function for the supply rate of Σ_1 in the sense of **S2**, there is usually large gap between the function and supply rates

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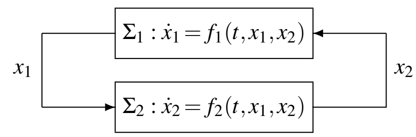

Fig. 1. Feedback interconnected system

accepted by Σ_1 automatically in the sense of **S1**. In order to fill the gap, we generate many candidates from the initial choice obtained in view of **S2**. The primary idea is to parametrize a set of many candidates for supply rate functions that meet the ISS small-gain condition by introducing free functional parameters.

The ISS property has been playing a important role in control of interconnected nonlinear systems[10], [9], [2]. The concept of iISS which is broader than ISS captures a very important characteristic essentially nonlinear systems often have[11]. In contrast to ISS, there are few tools of making use of the iISS in analysis and design of interconnected systems although the property of iISS by itself has been investigated deeply. For instance, stability criteria similar to the ISS small-gain theorem have not been available until a very recent breakthrough accomplished in [6]. The small-gain-like theorem developed in [6] assumes that supply rates of individual subsystems are given and fixed *a priori* as the ISS small-gain theorem does. The issue of how to select successful supply rates remains unaddressed.

The purpose of this paper is to develop a new parametrization of supply rates to establish stability of interconnected systems involving iISS properties. It is demonstrated that contraction inequalities of the ISS small-gain 'condition' type are still applicable to iISS systems although the ISS small-gain 'theorem' limits the applicability to ISS systems. It is shown that the difference between iISS systems and ISS systems appears in the allowable set of free functions in the parametrization. Since iISS is broader or 'weaker' than ISS, the set of admissible free functions for iISS systems becomes smaller than that for ISS systems. Thereby, this paper successfully extends the work in [5] to iISS systems.

II. Motivating example: idea and usefulness

Consider a nonlinear interconnected system defined by

$$\Sigma_1 : \ \dot{x}_1 = -\mu x_1^3 + \frac{x_1^2 x_2}{x_2+1}, \quad \mu = 1.1, \quad x_1(0) \in \mathbb{R}_+ \quad (1)$$

$$\Sigma_2 : \ \dot{x}_2 = f(x_2, x_1), \qquad x_2(0) \in \mathbb{R}_+ \quad (2)$$

where x_1 and x_2 are scalar. The set $\mathbb{R}_+$ denotes the interval $[0,\infty)$. Assume that the interconnected system has an equilibrium at the origin $x = [x_1, x_2]^T = 0$, and that all trajectories remain in the positive cone $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}_+^2$ for all $t \in \mathbb{R}_+$, i.e.,

$$f_2(0,0) = 0, \qquad f_2(0,x_1) \geq 0, \quad \forall x_1 \in \mathbb{R}_+$$

Supposed that we do not have information about $f_2(x_2, x_1)$ except the existence of a continuously differentiable radially unbounded function $V_2 : x_2 \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\frac{dV_2(x_2)}{dt} \leq -\frac{x_2}{x_2+1} + x_1 \quad (3)$$

along the trajectories of Σ_2. Without any loss of generality, the function V_2 is increasing with respect to $x_2 \in \mathbb{R}_+$. This section addresses the question of whether the $x=0$ of the interconnected system is globally asymptotically stable. The assumption (3) allows systems which are not ISS with respect to input x_1 and state x_2. It only guarantees iISS since V_2 is an iISS Lyapunov function[11]. In contrast, the system Σ_1 is ISS with respect to input x_2 and state x_1, and $V_1 = x_1$ is an ISS Lyapunov function[10], [12]. We cannot resort to the ISS small-gain theorem[8], [14] in order to establish the global asymptotic stability of $x = [x_1, x_2]^T = 0$.

First, let $V_1(x_1)$ be defined as $V_1 = x_1$. It is verified easily from (1) and (3) that for any constant $c > 0$, there are not positive definite functions $\rho_e(x_1, x_2)$ such that

$$\dot{V}_1(x_1) + c\dot{V}_2(x_2) \le -\rho_e(x_1, x_2), \quad \forall x_1, x_2 \in \mathbb{R}_+ \tag{4}$$

is achieved along the trajectories of (1)-(2). . Notice that if (4) was achieved, the function $V_{cl}(x_1, x_2) = V_1(x_1) + cV_2(x_2)$ could be a Lyapunov function establishing the global asymptotic stability. The selection $V_{cl}(x_1, x_2) = V_1(x_1) + cV_2(x_2)$ is, however, not successful.

Time-derivative of $V_1 = x_1$ along the trajectories of (1) satisfies

$$\begin{aligned} \dot{V}_1(x_1) &\le \begin{cases} -\mu x_1^3 + a_1^{-1} x_1^3 & \text{for } x_1 \ge a_1 \frac{x_2}{x_2+1} \\ -\mu x_1^3 + a_1^2 \left(\frac{x_2}{x_2+1}\right)^3 & \text{for } x_1 \le a_1 \frac{x_2}{x_2+1} \end{cases} \\ &\le -\alpha_1(s) + \sigma_1(s) \end{aligned}$$

$$\alpha_1(s) = (\mu - a_1^{-1}) s^3, \quad \sigma_1(s) = a_1^2 \left(\frac{s}{s+1}\right)^3 \tag{5}$$

for any $a_1 > 0$. The nonlinear gain function of Σ_1 with respect to input x_2 and state x_1 is obtained from (5) as

$$\Gamma_1(s) = (1+\varepsilon_1) \left(\frac{a_1^2}{\mu - a_1^{-1}}\right)^{1/3} \left(\frac{s}{s+1}\right)$$

for any $\varepsilon_1 > 0$ [12], [2]. The minimum of the gain function is achieved with $a_1 = 1.5/\mu$ as follows:

$$\Gamma_{1,min}(s) = (1+\varepsilon_1) \frac{(27/4)^{1/3}}{\mu} \left(\frac{s}{s+1}\right) \simeq (1+\varepsilon_1) \frac{1.890}{\mu} \left(\frac{s}{s+1}\right)$$

The ISS small-gain theorem can tell nothing about stability of the interconnection with the help of this nonlinear gain. However, a stability theorem which is recently developed in [6] and called the iISS-ISS small-gain theorem is applicable to the interconnection although Σ_2 is only iISS. To see this, write (3) as

$$\dot{V}_2(x_2) \le -\alpha_2(x_2) + \sigma_2(x_1), \quad \alpha_2(s) = \frac{s}{s+1}, \ \sigma_2(s) = s \tag{6}$$

and we obtain

$$\sigma_2 \circ \Gamma_{1,min}(s) = (1+\varepsilon_1) \frac{(27/4)^{1/3}}{\mu} \left(\frac{s}{s+1}\right), \quad \frac{[\sigma_2(s)]^k}{\alpha_1(s)} = \frac{3s^{k-3}}{\mu}$$

Thus, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(1+\varepsilon_2) \sigma_2 \circ \Gamma_{1,min}(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{7}$$

holds for $\mu > (27/4)^{1/3} \simeq 1.890$. There exists $k > 0$ such that

$$\frac{[\sigma_2(s)]^k}{\alpha_1(s)} \text{ is non-decreasing.} \tag{8}$$

According to the iISS-ISS small-gain theorem[6], the contraction condition (7) with the help of the non-decreasing property (8) could imply the global asymptotic stability of $x = 0$ if $\mu > (27/4)^{1/3} \simeq 1.890$ held. However, the stability is not established for $\mu = 1.1$ given in (1) using [6].

Next, motivated by

$$\dot{V}_1(x_1) = x_1^2 \left\{ -\mu x_1 + \frac{x_2}{x_2+1} \right\} \tag{9}$$

held for $V_1 = x_1$ along the trajectories of (1), we define

$$\hat{V}_1(x_1) = \int_0^{V_1(x_1)} \frac{1}{s^2} ds \tag{10}$$

For the following choice of α_1 and σ_1

$$\dot{\hat{V}}_1(x_1) = -\mu x_1 + \frac{x_2}{x_2+1}, \quad \alpha_1(s) = \mu s, \ \sigma_1(s) = \frac{s}{s+1} \tag{11}$$

there exist $\varepsilon_1, \varepsilon_2 > 0$ and $k > 0$ such that the contraction condition (7) holds for $\mu > 1$ and the non-decreasing property (8) holds. The function in (10) is, however, not integrable, so that $\hat{V}_1(x_1)$ in (10) is not qualified as a Lyapunov function. It should be stressed that a function in the form of

$$\hat{V}_1(x_1) = \int_0^{V_1(x_1)} \beta(s) ds$$

with a positive-valued function $\beta(s)$ decreasing faster than or as fast as $1/s^2$ toward ∞ is not radially unbounded, so that it cannot be used for proving global properties[13]. Indeed, it is not allowed to use the iISS-ISS small-gain theorem with (10). The selection $V_{cl}(x_1, x_2) = \hat{V}_1(x_1) + cV_2(x_2)$ does not prove the *global* asymptotic stability either.

Finally, we give an affirmative answer to our question of stability by choosing the following candidate for a Lyapunov function of the overall interconnected system.

$$V_{cl}(x_1, x_2) = x_1 + \int_0^{V_2(x_2)} \left(\frac{s}{s+1}\right)^2 ds \tag{12}$$

If we assume that Σ_2 achieves (3) with $V_2(x_2) = x_2$, we obtain

$$\begin{aligned} \dot{V}_{cl}(x_1, x_2) &\le -\mu x_1^3 + \frac{x_1^2 x_2}{x_2+1} - \left(\frac{x_2}{x_2+1}\right)^3 + x_1 \left(\frac{x_2}{x_2+1}\right)^2 \\ &\le -\rho_e(x_1, x_2) \end{aligned} \tag{13}$$

along the trajectories of (1)-(2). It is verified that the inequality (13) holds for some positive definite $\rho_e(x_1, x_2)$ if and only if $\mu > 1$ holds. Since V_{cl} in (12) is positive definite and radially unbounded, the global asymptotic stability of $x = 0$ follows from (13).

The above discussions have suggested the following.

A. The existence of a positive definite function $V_1(x_1)$ which is radially unbounded and satisfies

$$\dot{V}_1 \le x_1^2 \left\{ -\mu x_1 + \frac{x_2}{x_2+1} \right\} \tag{14}$$

$$1/\mu < 1 \tag{15}$$

implies the global asymptotic stability of the interconnected system under the assumption (3).

The claim **A** is trivial due to the iISS-ISS small-gain theorem if (14) is replaced by immediate copies of functions in (3) as follows:

$$\dot{V}_1 \le -\mu x_1 + \frac{x_2}{x_2+1} \tag{16}$$

Indeed, the stability is also verified with $V_{cl}(x_1, x_2) = V_1(x_1) + cV_2(x_2)$. A significant point suggested by the claim **A** is that the properties of global asymptotic stability may be often established even if the property (16) is relaxed in the form of

$$\dot{V}_1 \le \hat{\lambda}(x_1) \left\{ -\mu x_1 + \frac{x_2}{x_2+1} \right\} \tag{17}$$

It is desirable if we can predict when and what kind of functions $\hat{\lambda}(x_1)$ are allowed without discovering an ad hoc Lyapunov function $V_{cl}(x_1,x_2)$. In the above example, the usage of a particular $\hat{\lambda}(x_1)$ is justified in a heuristic manner only when $V_2(x_2)=x_2$. It is greatly useful if the appropriateness of incorporating $\hat{\lambda}(x_1)$ is answered without knowing V_2. The rest of this paper addresses

- the claim **A** is justified independently of the choice of V_2;
- the idea of (17) is formulated precisely so that the problem of stability can be answered without constructing a heuristic Lyapunov function V_{cl} of the interconnected system;
- the degree of freedom to choose $\hat{\lambda}(x_1)$ is investigated.

III. Formulation of objective

Consider the interconnected system in Fig.1 consisting of

$$\Sigma_1 : \ \dot{x}_1 = f_1(t,x_1,u_1), \quad u_1 = x_2 \in \mathbb{R}^{n_2} \tag{18}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(t,x_2,u_2), \quad u_2 = x_1 \in \mathbb{R}^{n_1} \tag{19}$$

For each $i=1,2$, we assume that $f_i(t,0,0)=0$ holds for all $t\in[t_0,\infty)$, $t_0\in\mathbb{R}_+$, and $f_i(t,x_i,u_i)$ is piecewise continuous in t, and locally Lipschitz in the other arguments. The state vector of the interconnected system is given by $x=[x_1^T,x_2^T]^T\in\mathbb{R}^n$.

Stability criteria for interconnected systems based on the dissipative (or Lyapunov) systems theory, such as the passivity theorems, the $\mathscr{L}_2$ small-gain theorem and the circle and Popov criteria, first assume that supply rates of individual subsystems are given a priori[15], [1], [2]. To put it shortly, a continuous function $\rho_i(x_i.u_i)$ is said to be a supply rate of Σ_i if

$$\frac{\partial V_i}{\partial t}+\frac{\partial V_i}{\partial x_i}f_i(t,x_i,u_i)\le\rho_i(x_i,u_i),\ \forall x_i\in\mathbb{R}^{n_i},u_i\in\mathbb{R}^{n_{ui}},t\in\mathbb{R}_+$$

holds for a positive semi-definite function $V(t,x_i)$. In order to obtain specific properties of stability, we often impose additional constraints on V. The ISS small-gain theorem can be also explained in terms of dissipation and Lyapunov functions[7], [3]. Recently, a framework of state-dependent scaling has been proposed in [4], [6] and it unifies and extends these existing stability criteria. However, it also assumes supply rates fixed *a priori*. In general, supply rates of systems are not given automatically. Although we are occasionally able to obtain supply rates intuitively from a physical point of view, there are many practical cases where natural supply rates of individual subsystems prevent us from establishing desired stability of interconnected systems. It would be nice if there were some guidelines for selecting judicious supply rate successfully.

Toward this end, this paper treats Σ_1 and Σ_2 differently. It is supposed that a supply rate of one system Σ_2 is given and fixed *a priori*, while we do not know a proper supply rate of the other system Σ_1. The underlying aim in the process of selecting a supply rate of Σ_1 is the establishment of global asymptotic stability of the equilibrium at the origin $x=0$. The supply rate of Σ_1 is not determined *a priori*. We need to search for a supply rate which establishes the stability of the interconnection of Σ_1 and Σ_2 using the information of Σ_2 given *a priori*. We assume that there exists and we know a pair of continuous functions α_2 and σ_2 such that

$$\underline{\alpha}_2(|x_2|)\le V_2(t,x_2)\le\bar{\alpha}_2(|x_2|) \tag{20}$$

$$\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial x_2}f_2(t,x_1,x_2)\le-\alpha_2(|x_2|)+\sigma_2(|x_1|) \tag{21}$$

are satisfied with some $\mathbf{C}^1$ function $V_2:\mathbb{R}_+\times\mathbb{R}^{n_2}\to\mathbb{R}_+$ and some continuous functions $\underline{\alpha}_2,\bar{\alpha}_2\in\mathscr{K}_\infty$ for all $x\in\mathbb{R}^n$ and $t\in\mathbb{R}_+$. The pair of α_2 and σ_2 determines degrees of stability of the system Σ_2. The functions α_2 and σ_2 are supposed to be defined on $\mathbb{R}_+$ and satisfy $\alpha_2(0)=\sigma_2(0)=0$. We will assume that each of α_2 and σ_2 belongs either the class $\mathscr{K}$ or the class $\mathscr{K}_\infty$. Remember that a class $\mathscr{K}$ function $\gamma(\cdot)$ is a continuous, increasing function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ such that $\gamma(0)=0$. A class $\mathscr{K}_\infty$ function $\gamma(\cdot)$ is a class $\mathscr{K}$ function which satisfies $\lim_{s\to\infty}\gamma(s)=\infty$. By definition, for continuous functions of a single non-negative variable,

$$\mathscr{K}_\infty\subset\mathscr{K}\subset\text{positive definite functions} \tag{22}$$

holds. The system Σ_2 is said to be integral input-to-state stable (iISS) if α_2 is a positive definite function and σ_2 is a class $\mathscr{K}$ function. In this paper, such a pair is called the iISS supply rate. The function $V_2(t,x_2)$ satisfying (20) and (21) for an iISS supply rate is called the $\mathbf{C}^1$ iISS Lyapunov function[11]. The system Σ_i is said to be input-to-state stable (ISS) if α_i is a class $\mathscr{K}_\infty$ function and σ_2 is a class $\mathscr{K}$ function. Such a pair is called the ISS supply rate. The function $V_2(t,x_2)$ satisfying (20) and (21) for an ISS supply rate is called the $\mathbf{C}^1$ ISS Lyapunov function[12]. By definition, ISS implies iISS. The converse is not true.

The purpose of this paper is to extend the following theorem proposed basically in [5] to more general classes of supply rates.

Theorem 1: The system Σ_2 is supposed to accept an ISS supply rate (21) defined with

$$\alpha_2\in\mathscr{K}_\infty,\quad \sigma_2\in\mathscr{K}_\infty \tag{23}$$

Suppose that real numbers $\nu_i>0$, $c_i>1$, $i=1,2$ satisfy

$$\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\sigma_2^{-1}\circ c_1\frac{\nu_2}{\nu_1}\alpha_2\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ c_2\sigma_2(s)\le s \quad \forall s\in\mathbb{R}_+ \tag{24}$$

If there exist a continuous function $\hat{\lambda}:\mathbb{R}_+\to\mathbb{R}_+$ and a $\mathbf{C}^1$ function $V_1:\mathbb{R}_+\times\mathbb{R}^{n_1}\to\mathbb{R}_+$ such that

$$\hat{\lambda}(s)>0,\quad \forall s\in(0,\infty) \tag{25}$$

$$\underline{\alpha}_1(|x_1|)\le V_1(t,x_1)\le\bar{\alpha}_1(|x_1|) \tag{26}$$

$$\frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f_1(t,x_1,x_2)\le\hat{\lambda}(V_1(t,x_1))\left[-\nu_1\sigma_2(|x_1|)+\nu_2\alpha_2(|x_2|)\right] \tag{27}$$

hold for all $x\in\mathbb{R}^n$ and $t\in\mathbb{R}_+$ with some $\underline{\alpha}_1,\bar{\alpha}_1\in\mathscr{K}_\infty$, the equilibrium $x=0$ of the interconnected system (18)-(19) is globally uniformly asymptotically stable.

When we pick $\hat{\lambda}=1$, Theorem 1 reduces to the ISS small-gain 'theorem'[8], [14], and the contraction inequality (24) is the same as the ISS small-gain 'condition'. In contrast to the ISS small-gain 'theorem', the supply rate employed in Theorem 1 provides a free parameter $\hat{\lambda}$. Theorem 1 does not require Σ_1 to take a supply rate given directly by functions appearing in the ISS small-gain 'condition'. The function $\hat{\lambda}$ in (27) provides us with flexibility of 'state-dependently scaling' an initial choice $-\nu_1\sigma_2(|x_1|)+\nu_2\alpha_2(|x_2|)$ for a supply rate of Σ_1 borrowed from functions of Σ_2 in (21). The advantages of using Theorem 1 is that the freedom of $\hat{\lambda}$ can be utilized to render (27) acceptable for Σ_1. Theorem 1 is considered as a parametrization of supply rates for Σ_1 to establish the stability of the interconnected system.

The assumption of (23) determines the broadness of systems to which Theorem 1 is applicable. The requirement (23) imposed on Σ_2 is equivalent to ISS although the original definition of ISS is given with $\sigma_2\in\mathscr{K}$ instead of $\sigma_2\in\mathscr{K}_\infty$[2]. Nevertheless, the difference between $\sigma_2\in\mathscr{K}$ and $\sigma_2\in\mathscr{K}_\infty$ is crucial in terms of the property imposed on Σ_1. The supply rate given by $-\nu_1\sigma_2(|x_1|)+\nu_2\alpha_2(|x_2|)$ to be applied to Σ_1 is ISS if $\sigma_2\in\mathscr{K}_\infty$. In the case of $\sigma_2\in\mathscr{K}$, the supply rate $-\nu_1\sigma_2(|x_1|)+\nu_2\alpha_2(|x_2|)$ only requires

Σ_1 to be iISS. Thus, the assignment of $\sigma_2 \in \mathcal{K}$ would allow us to deal with a more general class of Σ_1 than $\sigma_2 \in \mathcal{K}_\infty$. On the other hand, the example in Section II is not covered by Theorem 1. To answer the example, we need to handle a class $\mathcal{K}$ function α_2 which is not class $\mathcal{K}_\infty$. The objective of this paper is to extend Theorem 1 to such broader classes of systems.

Remark 1: Theorem 1 is better than the result previously presented in [5]. The previous version of Theorem 1 uses an assumption $(c_1-1)(c_2-1) > 1$ in addition to (24). A further study has yielded that the technical assumption can be removed.

IV. Main results

The pair of α_2 and σ_2 this section focuses on is broader than (23) considered in Theorem 1. It is expected naturally that the set of admissible free functions $\hat{\lambda}$ is required to be smaller for less restricted α_2 and σ_2. This section answers the question of how the set of $\hat{\lambda}$ needs to be narrowed for less restrictive α_2 and σ_2.

Theorem 2: The system Σ_2 is supposed to accept an ISS supply rate (21) defined with

$$\alpha_2 \in \mathcal{K}_\infty, \quad \sigma_2 \in \mathcal{K} \tag{28}$$

Suppose that real numbers $\nu_i > 0$, $c_i > 1$, $i = 1,2$ satisfy

$$c_1 \frac{\nu_2}{\nu_1} \alpha_2 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2 \sigma_2(s) \leq \sigma_2 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s) \quad , \forall s \in \mathbb{R}_+ \tag{29}$$

If there exists a $\mathbf{C}^1$ function $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+$ and a continuous function $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\underline{\alpha}_1(|x_1|) \leq V_1(t,x_1) \leq \bar{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}, \forall t \in \mathbb{R}_+ \tag{30}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1,x_2) \leq \hat{\lambda}(V_1(t,x_1))\left[-\nu_1 \sigma_2(|x_1|) + \nu_2 \alpha_2(|x_2|)\right] \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, \forall t \in \mathbb{R}_+ \tag{31}$$

$$\hat{\lambda}(s) > 0, \quad \forall s \in (0,\infty) \tag{32}$$

$$\lim_{s\to 0^+} \frac{[\sigma_2 \circ \bar{\alpha}_1^{-1}(s)]^m}{\hat{\lambda}(s)} < \infty \tag{33}$$

$$\liminf_{s\to\infty} \frac{s[\sigma_2 \circ \bar{\alpha}_1^{-1}(s)]^m}{\hat{\lambda}(s)} > 0 \tag{34}$$

hold with some $\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty$ and some $m \geq 0$, the equilibrium $x = 0$ of the interconnected system (18)-(19) is globally uniformly asymptotically stable.

Proof: Consider an inequality given by

$$\lambda_1(V_1(t,x_1))\{-\nu_1 \sigma_2(|x_1|) + \nu_2 \alpha_2(|x_2|)\} + \lambda_2(V_2(t,x_2))\{-\alpha_2(|x_2|) + \sigma_2(|x_1|)\} \leq \rho_e(x) \quad , \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, t \in \mathbb{R}_+ \tag{35}$$

where the continuous function $\rho_e(x)$ is

$$\rho_e(x) = -(1-\delta)\left[\lambda_1(\underline{\alpha}_1(|x_1|))\nu_1\sigma_2(|x_1|) + \frac{\tau-1}{\tau}\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right], \quad 0<\delta<1, \ 1<\tau \tag{36}$$

Suppose that λ_1, $\lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing continuous functions which have yet to be determined. Define $\theta_2 \in \mathcal{K}$ by

$$\theta_2(s) = \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \tau\sigma_2(s),, \quad \tau > 1 \tag{37}$$

Using Young's inequality, we obtain the following inequality.

$$\lambda_1(V_1(t,x_1))\{-\nu_1\sigma_2(|x_1|) + \nu_2\alpha_2(|x_2|)\} \leq -\lambda_1(V_1(t,x_1))\nu_1\sigma_2(|x_1|) + \frac{\lambda_1(V_1(t,x_1))^p}{p\mu^p} + \frac{\mu^q \nu_2 \alpha_2(|x_2|)^q}{q}$$

for arbitrary $\mu > 0$ and $q > 1$ satisfying $(1/p) + (1/q) = 1$. Combining calculations for two cases separated by $\alpha_2(|x_2|) \geq \tau\sigma_2(|x_1|)$ and $\alpha_2(|x_2|) < \tau\sigma_2(|x_1|)$, we obtain

$$\lambda_2(V_2(t,x_2))\{-\alpha_2(|x_2|) + \sigma_2(|x_1|)\} \leq -\frac{\tau-1}{\tau}\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + \lambda_2(\theta_1(|x_1|))\sigma_2(|x_1|)$$

Thus, the inequality (35) holds if

$$\frac{1}{p\mu^p}\lambda_1(s)^p - \delta\lambda_1(s)\nu_1\sigma_2(\bar{\alpha}_1^{-1}(s)) + \lambda_2(\theta_2(\underline{\alpha}_1^{-1}(s)))\sigma_2(\underline{\alpha}_1^{-1}(s)) \leq 0, \ \forall s \in \mathbb{R}_+ \tag{38}$$

$$-\delta\frac{\tau-1}{\tau}\lambda_2(s) + \frac{\mu^q}{q}[\nu_2\alpha_2(\underline{\alpha}_2^{-1}(s))]^{q-1} \leq 0, \quad \forall s \in \mathbb{R}_+ \tag{39}$$

are satisfied. The inequality (39) holds if and only if

$$\lambda_2(s) \geq \frac{\mu^q\tau}{\delta q(\tau-1)}[\nu_2\alpha_2(\underline{\alpha}_2^{-1}(s))]^{q-1}, \quad \forall s \in \mathbb{R}_+ \tag{40}$$

is achieved by λ_2. Define an increasing function of s as

$$\lambda_1(s) = \mu^{p/(p-1)}[\delta\nu_1\sigma_2(\bar{\alpha}_1^{-1}(s))]^{1/(p-1)} \tag{41}$$

The inequality (38) holds if and only if $\lambda_2(\cdot)$ satisfies

$$\lambda_2(\theta_2((s)) \leq \frac{\mu^q}{q}\frac{[\delta\nu_1\sigma_2(\bar{\alpha}_1^{-1}(\underline{\alpha}_1(s)))]^q}{\sigma_2(s)}, \quad \forall s \in \mathbb{R}_+ \tag{42}$$

Let $d = \lim_{s\to\infty}\theta_2(s) \in (0,\infty]$. Let $\theta_2^{-1}(\cdot)$ denote a continuous function such that $\theta_2^{-1}(\theta_2(s)) = s$ hold for all $s \in \mathbb{R}_+$. The pair of (40) and (42) holds if and only if

$$\frac{\mu^q\tau[\nu_2\alpha_2(\underline{\alpha}_2^{-1}(s))]^{q-1}}{\delta q(\tau-1)} \leq \lambda_2(s), \quad \forall s \in [d,\infty) \tag{43}$$

$$\frac{\mu^q\tau[\nu_2\alpha_2(\underline{\alpha}_2^{-1}(s))]^{q-1}}{\delta q(\tau-1)} \leq \lambda_2(s) \leq \frac{\mu^q[\delta\nu_1\sigma_2\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1\circ\theta_2^{-1}(s)]^q}{q\sigma_2\circ\theta_2^{-1}(s)} \quad , \quad \forall s \in [0,d) \tag{44}$$

There exists an increasing continuous function $\lambda_2(\cdot)$ such that (43) and (44) are achieved if

$$\left(\frac{1}{(\tau-1)\delta^{q+1}}\right)^{1/q}\frac{\nu_2}{\nu_1}\alpha_2\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\tau\sigma_2(s) \leq \sigma_2\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s) \quad \forall s \in \mathbb{R}_+ \tag{45}$$

is satisfied. If there exist $c_1, c_2 > 1$ such that (29) holds, there exist $\tau > 1$, $0 \ll \delta < 1$ and $l > 0$ such that (45) is satisfied for any $q \geq l$. Let the function λ_1 be represented by

$$\lambda_1(s) = \lambda_0(s)\hat{\lambda}(s) \tag{46}$$

The function λ_0 becomes continuous on $\mathbb{R}_+$ and

$$\lambda_i(s) > 0 \quad \forall s \in (0,\infty), \qquad \lim_{s\to 0^+}\lambda_i(s) < \infty \tag{47}$$

are satisfied for $i = 0$ if

$$\lim_{s\to 0^+}\frac{\lambda_1(s)}{\hat{\lambda}(s)} < \infty \tag{48}$$

Due to (41), this inequality is equivalent to

$$\lim_{s\to 0^+}\frac{[\sigma_2(\bar{\alpha}_1^{-1}(s))]^{q-1}}{\hat{\lambda}(s)} < \infty \tag{49}$$

The assumption (33) ensures (49) for any $q \geq m+1$. For any $q \geq m+1$,

$$\liminf_{s\to\infty}\frac{s[\sigma_2(\bar{\alpha}_1^{-1}(s))]^{q-1}}{\hat{\lambda}(s)} > 0 \tag{50}$$

is implied by the assumption (34), which guarantees

$$\int_1^\infty \lambda_i(s)ds = \infty \tag{51}$$

for $i = 0$. The inequalities (43) and (44) allow us to select $\lambda_2(s)$ which is positive for all $s \in (0,\infty)$ and increasing. Then, (47) and (51) are satisfied for $i = 2$. Since

$$\lambda_0(V_1(t,x_1))\rho_1(x_1,x_2) + \lambda_2(V_2(t,x_2))\rho_2(x_2,x_1) \leq \rho_e(x) \quad , \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{52}$$

is implied by the pair of (35) and (46), the $\mathbf{C}^1$ function

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_0(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{53}$$

satisfies $dV_{cl}/dt \leq \rho_e(x) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \forall t \in \mathbb{R}_+$ along the trajectories of the system. The properties (47) and (51) held for $i = 0,2$ guarantee the existence of $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathscr{K}_\infty$. satisfying $\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. ■

If we set $\hat{\lambda} = 1$, the statement of Theorem 2 becomes similar to the ISS small-gain 'theorem'[8], [14]. In fact, the inequality (29) is the ISS small-gain 'condition' for Σ_2 and Σ_1 satisfying (21) and

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1,x_2) \leq -\nu_1\sigma_2(|x_1|) + \nu_2\alpha_2(|x_2|) \tag{54}$$

However, it should be emphasized that the supply rate (54) defined with (28) does not require Σ_1 to be ISS. Thus, ISS small-gain 'theorem' is not applicable to systems Theorem 2 deals with. Theorem 2 demonstrates that the ISS small-gain 'condition' can lead us to the stability even if one of subsystems is only iISS.

Theorem 2 provides us with the flexibility of the function $\hat{\lambda}$ in the supply rate of Σ_1. We can use any function $\hat{\lambda}$ in (31) whenever (32)-(34) are met. The set of supply rates parametrized by the free function $\hat{\lambda}$ allows us to use supply rate functions which do not appear in the the small-gain condition. There will be more chance to come at a supply rate that fit Σ_1 under a single fixed small-gain condition. Compared with Theorem 1, we have additional constraints (33)-(34) on $\hat{\lambda}$ since Σ_1 is allowed to be only iISS.

Remark 2: For the set of functions $\hat{\lambda}$ allowed by Theorem 2, the choice $\int_0^{V_1} 1/\hat{\lambda}(s)ds$ is not guaranteed to be radially unbounded, so that it is *not* qualified to be a Lyapunov function proving the *global* asymptotic stability. Indeed, Theorem 2 employs the Lyapunov function (53) which is totally different from $\int_0^{V_1} 1/\hat{\lambda}(s)ds$. Therefore, the parametrization is fundamentally beyond the technique in [13], [2], and enables us to use a broader class of free functions. The integrability and radially unboundedness of $\int_0^{V_1} 1/\hat{\lambda}(s)ds$ are not implied by (33)-(34) unless we intentionally limiting the freedom to $m = 0$.

Remark 3: If we replace $|x_i|$ by $V_i(x_i)$ for $i = 1,2$ in (21) and (31), the functions $\underline{\alpha}_i$ and $\bar{\alpha}_i$ vanish in all arguments of Theorem 2. In fact, the conditions (29), (33) and (34) are replaced by

$$\nu_2 < \nu_1, \quad \lim_{s\to 0^+} \frac{[\sigma_2(s)]^m}{\hat{\lambda}(s)} < \infty, \quad \liminf_{s\to\infty} \frac{s[\sigma_2(s)]^m}{\hat{\lambda}(s)} > 0$$

respectively. The same occurs in the case of $\underline{\alpha}_i = \bar{\alpha}_i$ $i = 1,2$.

Remark 4: It is worth noting that (33) and (34) are satisfied for all $m > l$ if they are fulfilled for $m = l$. If $\lim_{s\to\infty} \sigma_2(s) < \infty$ holds, the existence of $m \geq 0$ fulfilling (34) is equivalent to

$$\liminf_{s\to\infty} s/\hat{\lambda}(s) > 0 \tag{55}$$

This stronger constraint is reasonable since the supply rate assigned to Σ_1 in (31) with $\sigma_2 \in \mathscr{K}\setminus\mathscr{K}_\infty$ is 'weaker' than the one with $\sigma_2 \in \mathscr{K}_\infty$. Note that (55) is not necessary for (34) when $\sigma_2 \in \mathscr{K}_\infty$.

Theorem 3: The system Σ_2 is supposed to accept an iISS supply rate (21) defined with

$$\alpha_2 \in \mathscr{K} \setminus \mathscr{K}_\infty, \quad \sigma_2 \in \mathscr{K}_\infty \tag{56}$$

Suppose that real numbers $\nu_i > 0$, $c_i > 1$, $i = 1,2$ satisfy

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \sigma_2^{-1} \circ c_1 \frac{\nu_2}{\nu_1} \alpha_2(s) \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \quad , \forall s \in \mathbb{R}_+ \tag{57}$$

If there exists a $\mathbf{C}^1$ function $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+$ and a continuous function $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\underline{\alpha}_1(|x_1|) \leq V_1(t,x_1) \leq \bar{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}, \forall t \in \mathbb{R}_+ \tag{58}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1,x_2) \leq \hat{\lambda}(V_1(t,x_1))[-\nu_1\sigma_2(|x_1|) + \nu_2\alpha_2(|x_2|)] \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, \forall t \in \mathbb{R}_+ \tag{59}$$

$$\hat{\lambda}(s) > 0, \quad \forall s \in (0,\infty) \tag{60}$$

$$\lim_{s\to 0^+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^m}{\hat{\lambda}(s)} < \infty \tag{61}$$

hold with some $\underline{\alpha}_1, \bar{\alpha}_1 \in \mathscr{K}_\infty$ and some $m \geq 0$, the equilibrium $x = 0$ of the interconnected system (18)-(19) is globally uniformly asymptotically stable.

Proof: Define a class $\mathscr{K}$ function by

$$\theta_1(s) = \bar{\alpha}_1 \circ \sigma_2^{-1} \circ \frac{\nu_2}{\nu_1} \tau\alpha_2(s), \quad \tau > 1 \tag{62}$$

Let λ_1, $\lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing continuous functions. Calculation for $\nu_1\sigma_2(|x_1|) \geq$ (and $<$) $\tau\nu_2\alpha_2(|x_2|)$ lead to

$$\lambda_1(V_1(t,x_1))\{-\nu_1\sigma_2(|x_1|) + \nu_2\alpha_2(|x_2|)\} \leq -\frac{\tau-1}{\tau}\lambda_1(V_1(t,x_1))\nu_1\sigma_2(|x_1|) + \lambda_1(\theta_1(|x_2|))\nu_2\alpha_2(|x_2|)$$

For arbitrary $\mu > 0$ and $q > 1$, Young's inequality yields

$$\lambda_2(V_2(t,x_2))\{-\alpha_2(|x_2|) + \sigma_2(|x_1|)\} \leq -\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + \frac{1}{p\mu^p}\lambda_2(V_2(t,x_2))^p + \frac{\mu^q}{q}\sigma_2(|x_1|)^q$$

where $(1/p) + (1/q) = 1$. Pick $\rho_e(x)$ as

$$\rho_e(x) = -(1-\delta)\left[\frac{\tau-1}{\tau}\lambda_1(\underline{\alpha}_1(|x_1|))\nu_1\sigma_2(|x_1|) + \lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right], \quad 0 < \delta < 1 \tag{63}$$

Then, a sufficient condition for (35) is obtained as

$$-\delta\frac{\tau-1}{\tau}\lambda_1(s)\nu_1 + \frac{\mu^q}{q}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^{q-1} \leq 0, \ \forall s \in \mathbb{R}_+ \tag{64}$$

$$\frac{1}{p\mu^p}\lambda_2(s)^p - \delta\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s)) + \lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\nu_2\alpha_2(\underline{\alpha}_2^{-1}(s)) \leq 0, \ \forall s \in \mathbb{R}_+ \tag{65}$$

The inequality (64) holds if and only if

$$\lambda_1(s) \geq \frac{\mu^q\tau}{\delta q(\tau-1)\nu_1}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^{q-1}, \quad \forall s \in \mathbb{R}_+ \tag{66}$$

is achieved by λ_1. Take

$$\lambda_2(s) = \mu^{p/(p-1)}[\delta\alpha_2(\bar{\alpha}_2^{-1}(s))]^{1/(p-1)} \tag{67}$$

which is an increasing function of s. The inequality (65) holds if and only if $\lambda_1(\cdot)$ satisfies

$$\lambda_1(\theta_1((s)) \leq \frac{\mu^q}{q}\frac{[\delta\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q}{\nu_2\alpha_2(s)}, \quad \forall s \in \mathbb{R}_+ \tag{68}$$

Let $d = \lim_{s\to\infty}\theta_1(s) \in (0,\infty)$. Let $\theta_1^{-1}(\cdot)$ denote a continuous function such that $\theta_1^{-1}(\theta_1(s)) = s$ hold for all $s \in \mathbb{R}_+$. The pair of (66) and (68) holds if and only if

$$\frac{\mu^q \tau[\sigma_2(\underline{\alpha}_1^{-1}(s))]^{q-1}}{\delta q(\tau-1)v_1} \le \lambda_1(s), \quad \forall s \in [d,\infty) \tag{69}$$

$$\frac{\mu^q \tau[\sigma_2(\underline{\alpha}_1^{-1}(s))]^{q-1}}{\delta q(\tau-1)v_1} \le \lambda_1(s) \le \frac{\mu^q[\delta\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\theta_1^{-1}(s)]^q}{qv_2\alpha_2\circ\theta_1^{-1}(s)}, \quad \forall s \in [0,d) \tag{70}$$

There exits an increasing continuous function $\lambda_1(\cdot)$ such that (69) and (70) are achieved if

$$\left(\frac{1}{(\tau-1)\delta^{q+1}}\right)^{1/q}\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\sigma_2^{-1}\circ\frac{v_2}{v_1}\tau\alpha_2(s) \le \alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s), \quad \forall s \in \mathbb{R}_+ \tag{71}$$

is satisfied. If there exist $c_1, c_2 > 1$ such that (57) holds, there exist $\tau > 1$, $0 \ll \delta < 1$ and $l > 0$ such that (71) is satisfied for any $q \ge l$. Let the function $\lambda_1(\cdot)$ be represented by (46). The function $\lambda_0(\cdot)$ becomes continuous on $\mathbb{R}_+$ and (47) are satisfied if (48) holds. Due to (69) and (70), this inequality holds if we have

$$\lim_{s\to 0^+}\frac{[\sigma_2(\underline{\alpha}_1^{-1}(s))]^{q-1}}{\hat{\lambda}(s)} < \infty \tag{72}$$

assured by (61). It follows from (69) that we can select λ_1 so that

$$\liminf_{s\to\infty}\frac{s\lambda_1(s)}{\hat{\lambda}(s)} > 0 \tag{73}$$

is fulfilled to guarantee (51). The function $\lambda_2(s)$ given in (67) is positive for all $s \in (0,\infty)$ and increasing, so that (47) and (51) are satisfied. Finally, since (52) follows from (35) and (46), the Lyapunov function (53) proves the global asymptotic stability. ■

Remark 3 applies to Theorem 3.

Remark 5: The set of continuous functions $\hat{\lambda}(s)$ satisfying (61) is quite broad. Note that the existence of m is guaranteed whenever $\hat{\lambda}(s)$ is bounded from below by $[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^k$ for a finite $k \ge 0$ in an arbitrarily small neighborhood of $s = 0$. Therefore, the parametrization offered by Theorem 3 is much more flexible than the technique in [13], [2]. Theorem 2 requires (34) in addition to (61). Nevertheless, the existence of m in (34) is guaranteed whenever $\hat{\lambda}(s)$ is bounded from above by $s[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^k$ for a finite $k \ge 0$ and all s beyond an arbitrarily large number.

Theorem 4: The system Σ_2 is supposed to accept an iISS supply rate (21) defined with

$$\alpha_2 \in \mathcal{K}, \quad \sigma_2 \in \mathcal{K} \tag{74}$$

Suppose that real numbers $v_i > 0$, $i = 1,2$ satisfy

$$v_2 < v_1 \tag{75}$$

If there exists a $\mathbf{C}^1$ function $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+$ and a continuous function $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\underline{\alpha}_1(|x_1|) \le V_1(t,x_1) \le \bar{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}, \forall t \in \mathbb{R}_+ \tag{76}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1(t,x_1,x_2) \le \hat{\lambda}(V_1(t,x_1))\left[-v_1\sigma_2(|x_1|) + v_2\alpha_2(|x_2|)\right] \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, \forall t \in \mathbb{R}_+ \tag{77}$$

$$\hat{\lambda}(s) > 0, \quad \forall s \in \mathbb{R}_+ \tag{78}$$

$$\liminf_{s\to\infty}\frac{s}{\hat{\lambda}(s)} > 0 \tag{79}$$

hold with some $\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty$, the equilibrium $x = 0$ of the system (18)-(19) is globally uniformly asymptotically stable.

The constraint (79) is more restrictive than (34). Theorem 3 does not need any constraint like (79) and (34) on $\hat{\lambda}$ with respect to $s \to \infty$. It is natural because Theorem 4 is applicable to a class of systems broader than Theorem 2 and Theorem 3. Theorem 3 dealing with $\sigma_2 \in \mathcal{K}_\infty$ requires Σ_1 to have stronger stability than Theorem 2, so that (34) is not necessary in Theorem 3. On the other hand, Theorem 1 does not require any constraints on $\hat{\lambda}$ with respect to $s \to \infty$ and $s \to 0^+$. The assumption (23) is more restrictive than (28), (56) and (74), so that Theorem 1 is applicable to the smallest class of systems, while it allows the largest class of the free function $\hat{\lambda}$. It should be stressed that the three conditions (24), (29) and (57) are identical when α_2 and σ_2 satisfy (23). They are also the same as (75) except the small amount of difference arising from $\underline{\alpha}_i(s) \le \bar{\alpha}_i(s)$. The slight discrepancy is inevitable as far as we derive contractive conditions from Lyapunov-based properties. Three conditions (24), (29) and (57) reduce to (75) precisely if $|x_i| = V_i(x_i)$ or $\underline{\alpha}_i = \bar{\alpha}_i$ holds for $i = 1,2$.

V. Conclusion

This paper has developed a new "parametrization of supply rates" in order to characterize a set of supply rates with which stability of a nonlinear interconnected system can be established under a fixed single small-gain condition. The result covers iISS systems which include more general nonlinearities than ISS systems considered in a previous paper[5]. The results in this paper are more useful than the small-gain type of theorem given for fixed supply rates[6]. The parametrization offers flexibility which provides more chances to come at supply rates establishing stability of interconnected systems.

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