2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Explicit Solutions to State-Dependent Scaling Problems for Interconnected iISS and ISS Nonlinear Systems

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Abstract— **This paper addresses the problem of establishing global stability properties of nonlinear interconnected systems. The state-dependent scaling problem is presented as a unified mathematical formulation whose solutions explicitly provide Lyapunov functions proving stability properties of feedback and cascade systems. The framework covers diverse nonlinearities represented by general supply rates. In order to let the state-dependent approach have real usefulness beyond formal applicability, this paper derives explicit formulas for solutions to state-dependent scaling problems for integral input-to-state stable(iISS) systems and input-to-state stable(ISS) systems.**

I. Introduction

The main purpose of the paper is to provide explicit solutions to the state-dependent scaling problems for establishing iISS and ISS properties of nonlinear interconnected systems. The aim of pursuing the problems is to develop a general framework of assessing stability and disturbance rejection properties of interconnected systems which are not limited to the settings of popular classical stability criteria and the ISS small-gain theorem. A stability theorem which brought about the primary version of the state-dependent scaling technique was originally formulated in [3], and that was utilized and modified for solving several control design problems[8], [4]. It was also extended to general stability criteria in the framework of state-dependent scaling problems in [6]. The framework offers several problems having minor differences each other to allow tune-up for reducing conservativeness in individual cases. The idea, however, boils down to a single main problem.

One of benefits from the state-dependent scaling framework is that the ISS small-gain theorem in [10], [16] and the dissipative approach[17], [1], [11] can be explained in a unified language. The ISS small-gain theorem and popular stability criteria such as the $\mathscr{L}_p$ small-gain theorem, the passivity theorems, the circle and Popov criteria can be extracted as special cases[6], [5]. In the framework, Lyapunov functions establishing stability properties of interconnected systems can be obtained in a unified manner, which is useful for controller design. The state-dependent scaling formulation is uniformly applicable to systems whose nonlinearity disagrees with input-to-state stability(ISS) and other classical nonlinearities. In fact, one of research directions has been to expand classes of nonlinearities and systems for which useful answers are obtainable. An achievement presented recently in [7] was the development of small-gain-like theorems for interconnected systems involving integral input-to-state stable(iISS) systems. Until the emergence of the breakthrough[7], in contrast to ISS, the concept of iISS has not yet been fully exploited in analysis and design of interconnected systems although the property of iISS by itself has been investigated deeply[13].

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The state-dependent scaling problems are scalar inequalities we solve for parameters called state-dependent scaling functions. The functions directly lead us to Lyapunov functions of general interconnected systems in a unified manner. It is suspected naturally that the universality may render the applicability only formal since it is typical of general 'nonlinear' problems to have no guarantee of the existence of solutions. We often do not know how to solve them even if solutions exist. Therefore, this paper is devoted to demonstrating that solutions to the state-dependent scaling problems exist actually and are obtainable practically for broader classes of systems by focusing on iISS and ISS properties. The question of the existence was first studied in [5], [7], and primary results of existence criteria obtained there led us to small-gain conditions applicable to iISS systems as well as ISS systems. Those papers were concentrated on proving the existence. The solutions were not derived explicitly so as to be practically useful for obtaining Lyapunov functions. This paper derives explicit formulas for solutions to the state-dependent scaling problems. This paper also refine the results presented in [5], [7] further to remove some restrictions. The explicit formulas directly give us Lyapunov functions describing stability properties of interconnected systems automatically.

All proofs are omitted due to the space limitation. We write $\gamma \in \mathscr{P}$ for a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ if it is a continuous function satisfying $\gamma(0) = 0$ and $\gamma(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$.

II. State-dependent scaling problems

This section presents mathematical problems to be solved in this paper. They are referred to as state-dependent scaling problems. This section also puts system theoretic interpretations on them from the viewpoint of stability properties of nonlinear interconnected systems and construction of Lyapunov functions. The primary idea of the state-dependent scaling framework has been proposed basically in [6].

Problem 1: Given continuously differentiable functions $V_i : (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ and continuous functions $\rho_i : (x_i, x_j, r_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \times \mathbb{R}^{m_i} \to \mathbb{R}$ for $i = 1, 2$ and $j = \{1, 2\} \setminus \{i\}$, find continuous functions $\lambda_i : s \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lambda_i(s) > 0 \quad \forall s \in (0, \infty), \qquad \lim_{s \to 0^+} \lambda_i(s) < \infty \tag{1}$$

$$\int_1^\infty \lambda_i(s) ds = \infty \tag{2}$$

for $i = 1, 2$ such that

$$\lambda_1(V_1(t, x_1))\rho_1(x_1, x_2, r_1) + \lambda_2(V_2(t, x_2))\rho_2(x_2, x_1, r_2) \le \rho_e(x_1, x_2, r_1, r_2), \quad \forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \tag{3}$$

hold for some continuous function $\rho_e : (x_1, x_2, r_1, r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\rho_e(x_1, x_2, 0, 0) < 0 \quad , \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \{(0, 0)\} \tag{4}$$

Problem 2: Given continuously differentiable functions $V_2 : (t,x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and continuous functions $\rho_1 : (z_1,x_2,r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}$ and $\rho_2 : (x_2,z_1,r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$, find continuous functions $\lambda_1 : (t,z_1,x_2,r_1,r_2) \in \mathbb{R}_+ \times \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}_+$, $\lambda_2 : s \in \mathbb{R}_+ \to \mathbb{R}_+$, an increasing continuous function $\xi_1 : s \in [0,N] \to \mathbb{R}_+$ and a continuous function $\varphi_1 : (z_1,x_2,r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}_+$ satisfying

$$\lambda_2(s) > 0 \quad \forall s \in (0,\infty), \qquad \lim_{s\to 0^+} \lambda_2(s) < \infty \tag{5}$$

$$\int_1^{\infty} \lambda_2(s)ds = \infty \tag{6}$$

$$\xi_1(s) \geq 0 \quad \forall s \in [0,N] \tag{7}$$

$$\varphi_1(z_1,x_2,r_1) \geq 0, \quad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1} \tag{8}$$

such that

$$\begin{aligned}&\lambda_1(t,z_1,x_2,r_1,r_2)\Big[-\xi_1(\varphi_1(z_1,x_2,r_1)) + \\ &\quad \xi_1(\varphi_1(z_1,x_2,r_1)+\rho_1(z_1,x_2,r_1))\Big] + \\ &\qquad \lambda_2(V_2(t,x_2))\rho_2(x_2,z_1,r_2) \leq \rho_e(x_2,r_1,r_2), \\ &\qquad\qquad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \end{aligned} \tag{9}$$

hold for some continuous function $\rho_e : (x_2,r_1,r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\rho_e(x_2,0,0) < 0 \quad ,\forall x \in \mathbb{R}^{n_2} \setminus \{0\} \tag{10}$$

where $N \in [0,\infty]$ is defined by

$$N = \sup_{(z_1,x_2,r_1)\in\mathbb{R}^{p_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{m_1}} [\varphi_1(z_1,x_2,r_1)+\rho_1(z_1,x_2,r_1))] \tag{11}$$

In this paper, the functions λ_i and ξ_i are referred to as state-dependent scaling functions. The inequalities (3) and (9) are key formulas which will be explained later on. Note that $\lim_{s\to 0^+} \lambda_i(s) < \infty$ is redundant mathematically since each λ_i is supposed to be continuous on $\mathbb{R}_+ = [0,\infty)$. This paper calls a pair of λ_1 and λ_2 a solution to Problem 1 if the pair fulfills all requirements stated in Problem 1. In a similar manner, a quartet of λ_1, λ_2, ξ_1 and φ_1 fulfilling Problem 2 is called a solution to Problem 2. If we take $\xi_1(s) = s$, the inequality (9) becomes

$$\begin{aligned}&\lambda_1(t,z_1,x_2,r_1,r_2)\rho_1(z_1,x_2,r_1) + \lambda_2(V_2(t,x_2))\rho_2(x_2,z_1,r_2) \\ &\quad \leq \rho_e(x_2,r_1,r_2), \\ &\qquad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \end{aligned} \tag{12}$$

Whenever the function $\xi_1(s)$ is affine in s, the function φ_1 disappears from (9). Therefore, in the case of affine $\xi_1(s)$, a triplet of λ_1, λ_2 and ξ_1 is called a solution to Problem 2.

Problem 2 is milder than Problem 1. In other words, Problem 1 has a solution only if so does Problem 2 in the following sense.

Lemma 1: Suppose that a continuous function $\rho_e : (x_1,x_2,r_1,r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfies (4) and

$$\sup_{x_1\in\mathbb{R}^{n_1}} \rho_e(x_1,x_2,r_1,r_2) < +\infty, \quad \forall (x_2,r_1,r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \tag{13}$$

$$\sup_{x_1\in\mathbb{R}^{n_1}} \rho_e(x_1,x_2,0,0) < 0, \quad \forall x_2 \in \mathbb{R}^{n_2} \setminus \{0\} \tag{14}$$

Then, there exists a continuous function $\tilde{\rho}_e : (x_2,r_1,r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ such that

$$\tilde{\rho}_e(x_2,0,0) < 0 \quad ,\forall x_2 \in \mathbb{R}^{n_2} \setminus \{0\} \tag{15}$$

$$\rho_e(x_1,x_2,r_1,r_2) \leq \tilde{\rho}_e(x_2,r_1,r_2), \quad \forall (x_1,x_2,r_1,r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \tag{16}$$

Consider the nonlinear interconnected system Σ shown in Fig.1. Suppose that subsystems Σ_1 and Σ_2 are described by

$$\Sigma_1 : \ \dot{x}_1 = f_1(t,x_1,u_1,r_1) \tag{17}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(t,x_2,u_2,r_2) \tag{18}$$

These two dynamic systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. If Σ_1 is static, we suppose that

$$\Sigma_1 : \ z_1 = h_1(t,u_1,r_1) \tag{19}$$

Then, $u_2 = x_1$ is replaced by $u_2 = z_1$. Assume that $f_1(t,0,0,0) = 0$, $f_2(t,0,0,0) = 0$ and $h_1(t,0,0,0) = 0$ hold for all $t \in [t_0,\infty)$, $t_0 \geq 0$. The functions f_1, f_2 and h_1 are supposed to be piecewise continuous in t, and locally Lipschitz in the other arguments. The exogenous inputs $r_1 \in \mathbb{R}^{m_1}$ and $r_2 \in \mathbb{R}^{m_2}$ are packed into a single vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. The state vector of the interconnected system Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of Σ_i. This paper does not assume that f_i and h_i describing Σ_i in (17), (18) and (19) are known. Instead, we assume the following.

Assumption 1: For a dynamic system Σ_i, there exists a $\mathbf{C}^1$ function $V_i : (t,x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(t,x_i) \leq \bar{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, \ t \in \mathbb{R}_+ \tag{20}$$

holds with $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$, and

$$\frac{dV_i}{dt} \leq \rho_i(x_i,u_i,r_i), \quad \forall x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{p_i}, r_i \in \mathbb{R}^{m_i}, t \in \mathbb{R}_+ \tag{21}$$

holds along the trajectories of the system Σ with a continuous function $\rho_i : (x_i,u_i,r_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \times \mathbb{R}^{m_i} \to \mathbb{R}$ satisfying $\rho_i(0,0,0) = 0$.

Assumption 2: For a static Σ_1,

$$\rho_1(z_1,u_1,r_1) \geq 0, \quad \forall u_1 \in \mathbb{R}^{p_2}, r_1 \in \mathbb{R}^{m_1}, t \in \mathbb{R}_+ \tag{22}$$

is satisfied with a continuous function $\rho_1 : (z_1,u_1,r_1) \in \mathbb{R}^{p_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$ satisfying $\rho_1(0,0,0) = 0$.

A dynamic system Σ_i satisfying Assumption 1 is said to be dissipative[17], [1], [11]. The function ρ_i is referred to as a supply rate. When Σ_1 is static, we replace the pair of (20) and (21) by (22). In this paper, for convenience, we call ρ_1 for the static system a supply rate although energy is never stored by any static system.

Proposition 1: Suppose that Σ_1 and Σ_2 are dynamic systems. If there is a solution $\{\lambda_1, \lambda_2\}$ to Problem 1, the equilibrium $x = [x_1^T, x_2^T]^T = 0$ of the interconnected system Σ is globally uniformly asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|), \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{23}$$

is satisfied and

$$\frac{dV_{cl}}{dt} \leq \rho_e(x,r), \quad \forall x \in \mathbb{R}^n, r \in \mathbb{R}^m, t \in \mathbb{R}_+ \tag{24}$$

holds along the trajectories of the system Σ.

A Lyapunov function yielding the above proposition is

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{25}$$

The condition (2) requires certain growth order of the function $\lambda_i(s)$ with respect to s toward ∞. If a system Σ_i in Fig.1 is static, the growth order constraint (2) is unnecessary. In addition, we can employ other flexibilities of functions ξ_i and φ_i. Thereby, Problem 1 can be replaced by a weaker Problem 2.

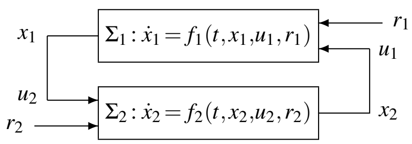

Fig. 1. Feedback interconnected system Σ

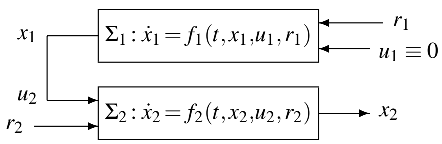

Fig. 2. Cascade connected system Σ_c

Proposition 2: Suppose that Σ_1 is a static system, and Σ_2 is a dynamic system. If there is a solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ to Problem 2, the equilibrium $x = x_2 = 0$ of the interconnected system Σ is globally uniformly asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and class $\mathscr{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that (23) is satisfied and (24) holds along the trajectories of the system Σ.

A Lyapunov function proving the above proposition is

$$V_{cl}(t, x_2) = \int_0^{V_2(t,x_2)} \lambda_2(s) ds \tag{26}$$

According to Proposition 1 and 2, the inequalities of the sum of scaled supply rates (3) and (9) lead directly to Lyapunov functions establishing the stability of interconnected systems. The central inequalities (3) and (9) are not in the form of linear combinations of supply rates. Functional coefficients λ_1, λ_2 and ξ_1 are introduced into the combinations. The use of the functionals λ_1 λ_2 and ξ_1 is contrasted with the early works on Lyapunov stability criteria for interconnected dissipative systems such as [17], [1], [11] where linear combinations of supply rates were employed, i.e., constants λ_1, λ_2 and an identity function $\xi_1(s) = s$.

Solutions to the state-dependent scaling problems also establish stability of cascade systems on the assumption of Assumption 1 and Assumption 2. Indeed, if one of feedback paths $u_1 = x_2$ and $u_2 = x_1$ is disconnected in Fig.1, the interconnection becomes a cascade. When the path of u_i is disconnected, the supply rate $\rho_i(x_i, u_i, r_i)$ becomes $\rho_i(x_i, r_i)$. By the cascade system Σ_c, the paper means that the path of $u_1 = x_2$ is cut, which is depicted in Fig.2.

A solutions to a state-dependent scaling problem exists only if the interconnected system actually possesses a stable property required. This section has not mentioned how easy or difficult it is to find solutions, which is the main issue addressed in this paper. Problem 1 and Problem 2 are jointly affine in λ_1 and λ_2. This property should be helpful in calculating solutions. The next section investigates it deeply.

III. Explicit solutions for iISS and ISS supply rates

This paper focuses on ISS and iISS supply rates. For classically popular classes of nonlinear systems, the readers may refer to [6]. This section presents the main results of this paper. Solutions to the state-dependent scaling problems are derived explicitly, and they are related to ISS and iISS properties of the feedback system shown in Fig.1 as well as the cascade system shown in Fig.2. Small-gain rules are obtained as conditions guaranteeing the existence of the solutions for the interconnection of iISS systems as well as ISS systems. This section together with (25) and (26) explicitly provides us with Lyapunov functions establishing the stability of the interconnected systems.

Consider the interconnected system Σ illustrated by Fig.1. For each Σ_i, $i = 1, 2$, suppose that a supply rate function

$$\rho_i(x_i, u_i, r_i) = -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{27}$$

satisfying Assumption 1 is given. It is assumed that α_i, σ_i ,$\sigma_{ri} \in \mathscr{P}$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathscr{K}_\infty$ are known, but information of the differential equations (17) and (18) is not required. The system Σ_i is said to be iISS with respect to input (u_i, r_i) and state x_i if (21) is satisfied for a positive definite function α_i, class $\mathscr{K}$ functions σ_i and σ_{ri}. In the single input case, the second input r_i is null, and the function σ_{ri} vanishes. The function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ iISS Lyapunov function[13]. If α_i is additionally a class $\mathscr{K}_\infty$ function, the system Σ_i is said to be ISS with respect to input (u_i, r_i) and state x_i, and the function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ ISS Lyapunov function[15]. By definition, ISS implies iISS. The converse is not true.

A. Interconnection of systems associated with iISS supply rates

We first consider the interconnected system composed of two systems described by supply rates ρ_i of the iISS type.

Theorem 1: Assume that functions $\rho_i(x_i, u_i, r_i)$, $i = 1, 2$ are in the form of (27) consisting of

$$\alpha_1 \in \mathscr{P}, \quad \sigma_1 \in \mathscr{K}, \quad \sigma_{r1} \in \mathscr{K} \tag{28}$$

$$\alpha_2 \in \mathscr{P}, \quad \sigma_2 \in \mathscr{K}, \quad \sigma_{r2} \in \mathscr{K} \tag{29}$$

Suppose that there exist $c_i > 0$, $i = 1, 2$ and $q \geq 1$ such that

$$[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq c_1 \alpha_1(\bar{\alpha}_1^{-1}(s)), \quad \forall s \in \mathbb{R}_+ \tag{30}$$

$$c_2 \sigma_1(\underline{\alpha}_2^{-1}(s)) \leq [\alpha_2(\bar{\alpha}_2^{-1}(s))]^q, \quad \forall s \in \mathbb{R}_+ \tag{31}$$

$$c_1 < c_2 \tag{32}$$

are satisfied. Then, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \ \alpha_{cl} \in \mathscr{K}, \sigma_{cl} \in \mathscr{K} \tag{33}$$

(ii) In the case of $\alpha_2 \in \mathscr{K}$, a solution to Problem 1 with respect to (33) is given by

$$\lambda_1 = \frac{\nu c_1}{\delta^2}, \quad \lambda_2(s) = \nu q [\delta \alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^{q-1} \tag{34}$$

where ν is any positive constant, and

$$\delta = \left(\frac{c_1}{c_2}\right)^{\frac{1}{q+2}} \tag{35}$$

(iii) In the case of $\alpha_2 \notin \mathscr{K}$, there exists $\hat{\alpha}_2 \in \mathscr{K}$ such that

$$\hat{\alpha}_2(s) \leq \alpha_2(s), \ c\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq [\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s))]^q, \ \forall s \in \mathbb{R}_+ \tag{36}$$

hold, and a solution to Problem 1 with respect to (33) is the same as *(ii)* except that α_2 is replaced by $\hat{\alpha}_2$.

The following is a natural consequence of Proposition 1.

Corollary 1: Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (27), (28) and (29).

(i) If there exist $c_i > 0$, $i = 1, 2$ and $q > 0$ such that (30), (31) and (32) are satisfied, the interconnected system Σ is iISS with respect to input r and state x.

(ii) If there exist $c_1 > 0$ and $q > 0$ such that (30) is satisfied, the cascade Σ_c is iISS with respect to input r and state x.

In the case of $q \geq 1$, the claim follows directly from Theorem 1. To obtain the case of $0 < q < 1$, we switch Σ_1 and Σ_2, and apply Theorem 1 to the systems whose subscripts 1 and 2 are exchanged each other. The cascade case is obtained by letting $\sigma_1 = 0$.

Remark 1: When (30)-(32) are imposed simultaneously, at least one system Σ_i of Σ_1 and Σ_2 is ISS with respect to input u_i and state x_i under $r_i(t) \equiv 0$. In order to prevent misunderstanding, two points should be emphasized. First, that system Σ_i does not have to be ISS in the presence of the external input r_i. Secondly, the choice of α_i and σ_i we use in (30) and (31) does not necessarily form a supply rate of the ISS type. In other words, $\alpha_1 \in \mathscr{P} \setminus \mathscr{K}_\infty$ and $\alpha_2 \in \mathscr{P} \setminus \mathscr{K}_\infty$ are allowed in (30) and (31) simultaneously. To verify the statement in the beginning, consider (27) with $\alpha_i \in \mathscr{P}$ and $\sigma_1 \in \mathscr{K}$, and assume $\alpha_i \in \mathscr{K}$ without loss of generality. The conditions (30) and (31) also yield

$$\left[\frac{\sigma_2(\underline{\alpha}_1^{-1}(s))}{\alpha_2(\bar{\alpha}_2^{-1}(s))}\right]^q \leq \frac{c_1\alpha_1(\bar{\alpha}_1^{-1}(s))}{c_2\sigma_1(\underline{\alpha}_2^{-1}(s))}, \quad \forall s \in \mathbb{R}_+ \setminus \{0\}$$

From this inequality and (32), we obtain

$$\lim_{s\to\infty}\left[\frac{\sigma_2(\underline{\alpha}_1^{-1}(s))}{\alpha_2(\bar{\alpha}_2^{-1}(s))}\right]^q \leq \lim_{s\to\infty}\frac{\alpha_1(\bar{\alpha}_1^{-1}(s))}{\sigma_1(\underline{\alpha}_2^{-1}(s))} \tag{37}$$

Suppose that $\alpha_1 \in \mathscr{K} \setminus \mathscr{K}_\infty$ and $\alpha_2 \in \mathscr{K} \setminus \mathscr{K}_\infty$ hold. Then, limiting values of σ_1 and σ_2 toward ∞ are guaranteed to be finite by (37) and $q > 0$ since σ_1 and σ_2 are class $\mathscr{K}$ functions. From (37) and $q > 0$ it also follows that

$$\alpha_j(\infty) < \sigma_j(\infty) \quad \Rightarrow \quad \alpha_i(\infty) > \sigma_i(\infty),\ i \neq j \tag{38}$$

It can be verified that the inequality $\alpha_i(\infty) > \sigma_i(\infty)$ implies the existence of another $\mathbf{C}^1$ ISS Lyapunov function of Σ_i for another pair of $\alpha_i \in \mathscr{K}_\infty$ and $\sigma_1 \in \mathscr{K}$ [15]. Therefore, the property (38) implies that the set of (30)-(32) requires at least one of Σ_1 and Σ_2 to be ISS with respect to input u_i and state x_i under $r_i(t) \equiv 0$. The requirement (38) is intuitively natural in view of 'small gain', and the necessity can be explained. It is, however, beyond the space limit and the scope of this paper.

B. Interconnection of iISS and ISS systems

In this subsection, we suppose that Σ_1 is ISS, and Σ_2 is iISS.

Theorem 2: Assume that functions $\rho_i(x_i, u_i, r_i)$, $i = 1, 2$ are in the form of (27) consisting of

$$\alpha_1 \in \mathscr{K}_\infty, \quad \sigma_1 \in \mathscr{K}, \quad \sigma_{r1} \in \mathscr{K} \tag{39}$$

$$\alpha_2 \in \mathscr{P}, \quad \sigma_2 \in \mathscr{K}, \quad \sigma_{r2} \in \mathscr{K} \tag{40}$$

Suppose that there exist $c_i > 1$, $i = 1, 2$ and $k > 0$ such that

$$\max_{w\in[0,s]}\frac{[c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(w)]^k}{c_1\sigma_1(w)} \leq \frac{[\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)]^k}{c_1\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+ \tag{41}$$

is satisfied. Then, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x, r)$ of the form

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|),\ \alpha_{cl} \in \mathscr{K}, \sigma_{cl} \in \mathscr{K} \tag{42}$$

(ii) In the case of $\alpha_2 \in \mathscr{K}$, a solution to Problem 1 with respect to (42) is given by

$$\lambda_1(s) = \max_{w\in[0,s]} \nu c_1 c_2^q \underline{\delta}^{\frac{q}{q+1}} \frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(w)]^q}{\alpha_1\circ\bar{\alpha}_1^{-1}(w)} \tag{43}$$

$$\lambda_2(s) = \nu q[\underline{\delta}^{\frac{1}{q+1}}\alpha_2\circ\bar{\alpha}_2^{-1}(s)]^{q-1} \tag{44}$$

where ν, $\underline{\delta}$ and q are any constants satisfying

$$\nu > 0, \quad 1 > \underline{\delta} > 0 \tag{45}$$

$$c_2^q > [\underline{\delta}(c_1 - 1)]^{-1}, \quad q \geq k, \quad q > 1 \tag{46}$$

(iii) In the case of $\alpha_2 \notin \mathscr{K}$, there exists $\hat{\alpha}_2 \in \mathscr{K}$ such that

$$\hat{\alpha}_2(s) \leq \alpha_2(s) \tag{47}$$

$$\max_{w\in[0,s]}\frac{[c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(w)]^k}{c_1\sigma_1(w)} \leq \frac{[\hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)]^k}{c_1\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+ \tag{48}$$

$$c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \leq \hat{\alpha}_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s), \quad \forall s \in \mathbb{R}_+ \tag{49}$$

hold, and a solution to Problem 1 with respect to (42) is the same as *(ii)* except that α_2 is replaced by $\hat{\alpha}_2$.

Furthermore, the statements (i), (ii) and (iii) are true even in the case of $\alpha_1 \in \mathscr{K}$ fulfilling

$$\lim_{s\to\infty}\alpha_1(s) = \bar{\eta}\lim_{s\to\infty}\{\sigma_1(s) + \sigma_{r1}(s)\} \tag{50}$$

for some $\bar{\eta} > 1$ if the constants c_1, $\underline{\delta}$ and q satisfy

$$\frac{(1-\underline{\delta}^{\frac{1}{q+1}})\bar{\eta}(\bar{\nu}+1)}{(1-\underline{\delta}^{\frac{1}{q+1}})\bar{\eta}(\bar{\nu}+1) - \bar{\nu}} < c_1 \tag{51}$$

$$\frac{\bar{\nu}}{\bar{\nu}+1} < (1-\underline{\delta}^{\frac{1}{q+1}})\bar{\eta} \tag{52}$$

where $\bar{\nu} \geq 0$ is given by

$$\bar{\nu}\lim_{s\to\infty}\sigma_1(s) = \lim_{s\to\infty}\sigma_{r1}(s) \tag{53}$$

Note that there always exist ν, $\underline{\delta}$ and q fulfilling (45) and (46). The function $\lambda_1(s)$ given in (43) fulfills $\lim_{s\to 0^+}\lambda_1(s) < \infty$, which is guaranteed by (41). In fact, the left hand side of (41) is a non-decreasing continuous function due to the maximization. The right hand side of (41) takes finite positive value at all $s \in (0, \infty)$. In this situation, the inequality of (41) implies $\lim_{s\to 0+}[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^k/[\alpha_1\circ\bar{\alpha}_1^{-1}(s)] < \infty$. Hence, the function $\lambda_1(s)$ given in (43) is a non-decreasing continuous function and $\lim_{s\to 0^+}\lambda_1(s) < \infty$ is satisfied.

Remark 2: The existence of $\bar{\eta} > 1$ satisfying (50) implies that the system Σ_1 is ISS with respect to input (u_1, r_1) and state x_1 even in the case of $\alpha_1 \in \mathscr{K}$ [15]. It is verified that there is another function $\tilde{V}_1(t, x_1)$ qualified as a $\mathbf{C}^1$ ISS Lyapunov function with $\tilde{\alpha}_1 \in \mathscr{K}_\infty$. Furthermore, it is worth mentioning that if the exogenous signal r_1 is absent, the two cases of $\alpha_1 \in \mathscr{K}$ and $\alpha_1 \in \mathscr{K}_\infty$ can be treated exactly in the same way. Indeed, the inequalities (51)-(52) are automatically satisfied when $\bar{\nu} = 0$ holds.

Remark 3: It is verified that each of simpler conditions

$$\exists k > 0 \text{ s.t. } \frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^k}{\alpha_1\circ\bar{\alpha}_1^{-1}(s)} \text{ is non-decreasing} \tag{54}$$

$$\exists k > 0 \text{ s.t. } \frac{[\alpha_2\circ\bar{\alpha}_2^{-1}(s)]^k}{\sigma_1\circ\underline{\alpha}_2^{-1}(s)} \text{ is non-decreasing} \tag{55}$$

implies (41) if there exist $c_i > 1$, $i = 1,2$ such that

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \le \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s),\ \forall s \in \mathbb{R}_+ \quad (56)$$

holds.

Corollary 2: Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (27), (39) and (40).

(i) If there exist $c_i > 0$, $i = 1,2$ and $k > 0$ such that (41) is satisfied, the interconnected system Σ is iISS with respect to input r and state x.

(ii) If there exists $k > 0$ such that

$$\lim_{s\to 0+} [\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k / [\alpha_1 \circ \bar{\alpha}_1^{-1}(s)] < \infty \quad (57)$$

holds, the cascade system Σ_c is iISS with respect to input r and state x.

The condition *(ii)* is met if each of $\sigma_2(s)$, $\underline{\alpha}_1^{-1}(s)$ and $1/\alpha_1(s)$ satisfies a Lipschitz condition of some order at $s = 0$.

C. Interconnection of ISS systems

This subsection deals with ISS systems.

Theorem 3: Assume that functions $\rho_i(x_i, u_i, r_i)$, $i = 1,2$ are in the form of (27) consisting of

$$\alpha_1 \in \mathcal{K}_\infty, \quad \sigma_1 \in \mathcal{K}, \quad \sigma_{r1} \in \mathcal{K} \quad (58)$$

$$\alpha_2 \in \mathcal{K}_\infty, \quad \sigma_2 \in \mathcal{K}, \quad \sigma_{r2} \in \mathcal{K} \quad (59)$$

Suppose that there exist $c_i > 1$, $i = 1,2$ such that

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \le s,\ \forall s \in \mathbb{R}_+ \quad (60)$$

is satisfied. Then, the following hold.

(i) Problem 1 is solvable with respect to a continuous function $\rho_e(x,r)$ of the form

$$\rho_e(x,r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|),\ \alpha_{cl} \in \mathcal{K}_\infty, \sigma_{cl} \in \mathcal{K} \quad (61)$$

(ii) In the case of $\sigma_1 \in \mathcal{K}_\infty$, a solution to Problem 1 with respect to (61) is given by

$$\lambda_1(s) = \left[\nu_1 \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \times \left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \left[\frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right]^m \quad (62)$$

$$\lambda_2(s) = \frac{c_2}{\delta(c_2-1)} \left[\nu_1 \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right] \left[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{m+1} \quad (63)$$

where $\nu_1 : s \in \mathbb{R}_+ \to \mathbb{R}_+$ is any non-decreasing continuous function satisfying

$$\nu_1(s) > 0, \quad \forall s \in (0,\infty) \quad (64)$$

and δ, τ_1 and m are any real numbers satisfying

$$0 \le m, \quad 0 < \delta < 1, \quad 1 < \tau_1 \le c_1 \quad (65)$$

$$\frac{\tau_1}{\left[\delta^2(\tau_1 - 1)(c_2 - 1)\right]^{\frac{1}{m+1}}} \le c_1 \quad (66)$$

(iii) In the case of $\sigma_1 \notin \mathcal{K}_\infty$, there exists $\hat{\sigma}_1 \in \mathcal{K}_\infty$ such that

$$\sigma_1(s) \le \hat{\sigma}_1(s), \quad \forall s \in \mathbb{R}_+ \quad (67)$$

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\hat{\sigma}_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \quad (68)$$

hold, and a solution to Problem 1 with respect to (61) is the same as *(ii)* except that σ_1 is replaced by $\hat{\sigma}_1$.

There always exist m, δ, τ_1 such that (65) and (66) hold.

Corollary 3: Assume that Σ_1 and Σ_2 accept supply rates ρ_1 and ρ_2 in the form of (27), (58) and (59).

(i) If there exist $c_i > 1$, $i = 1,2$ such that (60) is satisfied, the interconnected system Σ is ISS with respect to input r and state x.

(ii) The cascade Σ_c is ISS with respect to input r and state x.

The inequality (60) guaranteeing the existence of a solution to Problem 1 for ISS supply rates agrees with the ISS small-gain condition derived in [10], [16]. The statement of Corollary 3(i) by itself is the same as the ISS small-gain theorem presented in [10], [16]. This paper, however, takes a completely different approach of Theorem 3 in order to provide useful information about how to construct a Lyapunov function establishing the ISS property of the feedback. It contrasts with the original form of the ISS small-gain theorem[10], [16], [2] stated and proved in terms of trajectories of systems. The study [9] focuses on the equivalence between the trajectory-based criterion and the existence of a Lyapunov function. On the other hand, this paper aims at the development of explicit formulas of Lyapunov functions for systems which are more general than ISS systems, and obtains the ISS small-gain theorem as a special case.

It is known that the cascade of ISS systems is ISS, and another Lyapunov-type proof can be found in [14], [2]. In ISS analysis of open-loop and cascade systems, Lyapunov functions have been used successfully by [12], [15], [14], [2]. This paper has demonstrated how to extend their techniques to a framework covering feedback systems and other types of supply rates. Regardless of the difference between feedback and cascade and the difference between ISS supply rate and iISS supply rate, the construction of the Lyapunov function falls within the same single framework of state-dependent scaling which can be solved explicitly.

Remark 4: Theorem 3 answers an important problem that remained unsolved in previous papers [5]. The previous papers require a technical assumption $(c_1 - 1)(c_2 - 1) > 1$ in addition to (60). This additional condition has rendered the state-dependent scaling formulation in [5] slightly more conservative than trajectory-based ISS small-gain theorem in [10], [16]. This paper not only has succeeded in removing the additional condition risen technically in those previous papers, but also has provided a new formula of scaling functions λ_i.

D. Relation between solutions

Conditions in Theorem 1, 2 and 3 are related as follows:

Theorem 4: Suppose that σ_1 and σ_2 are class $\mathcal{K}$ functions.

(i) Assume that $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{P}$ hold. If there exist a pair of $c_1 > 0$, $c_2 > 0$ and $q \ge 1$ such that (30)-(32) are satisfied, there exist another pair of $c_1 > 1$, $c_2 > 1$ and $k > 0$ such that (41) holds.

(ii) Assume that $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K}_\infty$ hold. If there exist a pair of $c_1 > 1$, $c_2 > 1$ and $k > 0$ such that (41) holds, the inequality (60) is satisfied.

The broader the class of systems covered by a theorem is, the more restrictive the condition for existence is. Naturally, solutions to state-dependent scaling problems are not unique. For example, the pair $\{\lambda_1, \lambda_2\}$ given in Theorem 1 is a solution to the problem for systems considered in Theorem 2 and Theorem 3. In the same manner, the pair $\{\lambda_1, \lambda_2\}$ given in Theorem 2 is also a solution to Theorem 3.

E. Interconnection of iISS and static systems

This subsection considers interconnection of dynamic and static systems. Suppose that the system Σ_1 is static in the interconnected system Σ shown by Fig.1 and satisfies Assumption 2. Let Σ_2 be a dynamic system satisfying Assumption 1. Assume that supply rate functions are given as

$$\rho_1(z_1,u_1,r_1) = -\alpha_i(|z_1|)+\sigma_i(|u_1|)+\sigma_{r1}(|r_1|) \quad (69)$$

$$\rho_2(x_2,u_2,r_2) = -\alpha_i(|x_2|)+\sigma_i(|u_2|)+\sigma_{r2}(|r_2|) \quad (70)$$

Suppose that α_i, σ_i, $\sigma_{ri} \in \mathscr{P}$ are known, but exact information of (19) and (18) is not required. For the static system Σ_1, we assume

$$\limsup_{s\to\infty}\{\sigma_1(s)+\sigma_{r1}(s)\} \le \liminf_{s\to\infty}\alpha_1(s) \quad (71)$$

without loss of generality since $h_1(t,u_1,r_1)$ is locally Lipschitz with respect to u_1 on $\mathbb{R}^{n_{u1}}$ and r_1 on $\mathbb{R}^{m_1}$.

Theorem 5: Assume that functions ρ_1 and ρ_2 are in the form of (69) and (70), respectively, and consist of

$$\alpha_1 \in \mathscr{K}_\infty, \quad \sigma_1 \in \mathscr{K}, \quad \sigma_{r1} \in \mathscr{K} \quad (72)$$

$$\alpha_2 \in \mathscr{P}, \quad \sigma_2 \in \mathscr{K}, \quad \sigma_{r2} \in \mathscr{K} \quad (73)$$

Suppose that there exist $c_i > 1$, $i = 1,2$ such that

$$c_2\sigma_2 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \quad (74)$$

is satisfied. Then, the following hold.

(i) Problem 2 is solvable with respect to a continuous function $\rho_e(x,r)$ of the form

$$\rho_e(x_2,r) = -\alpha_{cl}(|x_2|)+\sigma_{cl}(|r|)), \ \alpha_{cl} \in \mathscr{P}, \sigma_{cl} \in \mathscr{K} \quad (75)$$

(ii) A solution to Problem 2 with respect to (75) is given by

$$\lambda_1 = \lambda_2 = \nu, \ \xi_1(s) = \sigma_2 \circ \alpha_1^{-1}(s), \ \varphi_1(s) = \alpha_1(s) \quad (76)$$

where ν is any positive constant.

Furthermore, the statements (i) and (ii) are true even in the case of $\alpha_1 \in \mathscr{K}$ if the constant c_1 satisfies

$$\frac{\bar{\eta}(\bar{\nu}+1)}{\bar{\eta}(\bar{\nu}+1)-\bar{\nu}} \le c_1 \quad (77)$$

where $\bar{\eta} \ge 1$ and $\bar{\nu} \ge 0$ denote constants which fulfill

$$\lim_{s\to\infty}\alpha_1(s) = \bar{\eta}\lim_{s\to\infty}\{\sigma_1(s)+\sigma_{r1}(s)\} \quad (78)$$

$$\bar{\nu}\lim_{s\to\infty}\sigma_1(s) = \lim_{s\to\infty}\sigma_{r1}(s) \quad (79)$$

Corollary 4: Assume that Σ_1 is a static system accepting a supply rate ρ_1 in the form of (69) (72), and Σ_2 is a dynamic system accepting a supply rate ρ_2 in the form of (70) and (73).

(i) If there exist $c_i > 1$, $i = 1,2$ such that (74) is satisfied, the interconnected system Σ is iISS with respect to input r and state x_2.

(ii) The cascade Σ_c is iISS with respect to input r and state x_2.

Furthermore, the cascade Σ_c is ISS if $\alpha_2 \in \mathscr{K}_\infty$ holds additionally.

We can reach similar consequences by using Theorem 2 instead of Theorem 5 due to the inclusive relation between Problem 1 and Problem 2. In other words, we can prove the iISS property of the closed loop by using λ_1 and λ_2 given by (43), (44) and $\xi_1(s) = s$. Note that we set $\underline{\alpha}_1(|z_1|) = V_1(z_1) = \bar{\alpha}_1(|z_1|)$ for the static system Σ_1 in Problem 1. Compared with (74), the condition (41) is conservative. In the case of $\alpha_2 \in \mathscr{K}_\infty$, i.e., Σ_2 is ISS, we can also invoke Theorem 3 to obtain the ISS property in Corollary 4. An important point of Corollary 4 derived from Theorem 5 is that Σ_2 is not required to be ISS. The small-gain condition (74) by itself is sufficient for establishing the stability even when the dynamic system Σ_2 is only iISS. It contrasts with the case where the interconnected system contains only dynamic systems.

Remark 5: When Σ_1 does not have r_1, the constant c_1 in Theorem 5 and Corollary 4(i) is required to satisfy only $c_1 \ge 1$ in both the cases of $\alpha_2 \in \mathscr{K}_\infty$ and $\alpha_2 \in \mathscr{K}$. It is verified with $\bar{\nu} = 0$ in (77) and (79).

IV. Conclusion

This paper has derived explicit solutions to the state-dependent problems for interconnected iISS and ISS systems, and it has provided formulas for Lyapunov functions establishing stability properties of the interconnected systems. The explicit formulas are considered as new evidences that the state-dependent problems are tractable and have practical solutions for sufficiently large classes of systems compared with the existing stability criteria. These points are new although the facts presented in corollaries, aside from the explicit solutions, have been obtained basically in previous papers. This paper has also removed technical assumptions and restrictions on the existence conditions, the supply rate functions and the scaling functions used in previous studies.

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