A New Approach to the Lur'e Problem for Non-autonomous Systems with Arbitrary Delay

Alexandr A. Zevin and Mark A. Pinsky

Abstract— The classical Lur'e problem consists in finding conditions for absolute stability of a linear system with a nonlinear feedback contained within a prescribed sector. Most of the results obtained on this problem are based on the frequency domain or Lyapunov functions methods which are applied to systems with a time-invariant or periodic linear block. This paper develops a new approach providing a sufficient stability criterion for systems with a non-autonomous linear block and an arbitrary time-varying delay in the feedback. The result is expressed in the transfer function of the linear block and the sector margins of the nonlinear block. It is shown that stability of a system with a sign-constant transfer function is guaranteed by stability of the system with a limit linear feedback (so that, for such systems, the famous Aizerman conjecture is true).

1.INTRODUCTION

The classical Lur'e problem is to find conditions for absolute stability of a control system consisting of a linear block and a nonlinear feedback contained within a prescribed sector [1]. Over the last few decades there has appeared an extensive literature devoted to the problem and its generalization. Most of the known results are obtained by the frequency domain or Lyapunov functions methods and relate to systems with a time-invariant or periodic linear block (e.g., [2]-[10]). The Lyapunov method enables, in principle, to tackle arbitrary time-varying systems; however, finding the Lyapunov function for such systems is, generally, a difficult problem.

In paper [11] sufficient stability conditions for the Lur'e problem which are equally applied to time-invariant and time-varying systems were found. The results are based on a direct analysis of the corresponding integral Volterra equation about the input of the nonlinear block $\sigma(t)$. In this paper we extend this approach to systems with delay in the feedback. Namely, we assume that the corresponding output is of the form $\varphi = \varphi(\sigma(t-\tau(t),t)$ where the function $\tau(t)$ is piecewise continuous, nonnegative and bounded for $t \in [0,\infty)$. The corresponding integral equation becomes

$$\sigma(t) = f(t) + \int_0^t w(t,s)\varphi(\sigma(s-\tau(s)),s)\,\mathrm{d}s$$

(1)

where $w(t,s)$ is the transfer function of the linear block. Note that no other information on the linear block is employed, so the last may be described by time-varying ordinary or partial differential equations with or without delay, integral equations, etc.

The piecewise continuous scalar valued function $f(t)$ describes oscillation of the corresponding linear system in the absence of the feedback caused by nonzero initial conditions and, perhaps, external perturbation disappearing at infinity. We assume that the linear block is asymptotically stable, so

$$|f(t)| \to 0 \text{ as } t \to \infty . \tag{2}$$

The function $\varphi(\sigma,t)$ belongs to the class $\Phi(K_1,K_2)$, i.e. satisfies the inequality

$$K_1\sigma^2 \le \varphi(\sigma,t)\sigma \le K_2\sigma^2 ,\ \sigma \in (-\infty,\infty) \tag{3}$$

We assume that with a given initial function $\sigma(t)$ for $t<0$, the solution $\sigma(t)$ of equation (1) is continuable on $[0,\infty)$.

Definition 1. System (1) is called absolutely stable in the class $\Phi(K_1,K_2)$ if for any functions $f(t)$, $\varphi(\sigma,t)$, satisfying conditions (2), (3), and any piecewise continuous nonnegative bounded for $t \in [0,\infty)$ function $\tau(t)$, the corresponding solution $\sigma(t)$ of (1.1) satisfies the condition

$$|\sigma(t)| \to 0 \text{ as } t \to \infty . \tag{4}$$

If condition (4) is not fulfilled for some $\varphi(\sigma,t)$, $\tau(t)$ and $f(t)$ from the indicated classes, the system is referred to as unstable.

Putting $\varphi_1(\sigma,t) = \varphi(\sigma,t) - K_1\sigma - K\sigma$, $K = (K_2 - K_1)/2$ and returning to the previous notation, we reduce (3) to the form

$$-K\sigma^2 \le \varphi(\sigma,t)\sigma \le K\sigma^2 ,\ \sigma \in (-\infty,\infty) \tag{5}$$

Manuscript received September 19, 2004.
A. A. Zevin is with the Transmag Research Institute, Academy of Sciences of Ukraine, 49005 Dnepropetrovsk, Ukraine (e-mail: zevin@npkista.dp.ua).
M. A. Pinsky is with Mathematics Department, University of Nevada,
Reno, NV 89511 USA (phone: 775-784-6725, fax::775-784-6375, e-mail: pinsky@unr.edu)

Thus, we replace the class $\Phi(K_1,K_2)$ by $\Phi(-K,K)$; therewith we assume that the transfer function $w(t,s)$ in (1) is changed correspondingly.

In Section 2 a value K_* is found such that for $K < K_*$, the system is absolutely stable independent on the delay function $\tau(t)$ (Theorem 1). If the linear block is exponentially stable and the function $f(t)$ exponentially tends to zero, so does the solution $\sigma(t)$; Theorem 2 provides an upper bound for the corresponding Lyapunov exponent. For a class of linear blocks (including, in particular, the autonomous ones) the value K_0 is found such that the system is unstable in the class $\Phi(-K_0,K_0)$ for any $\tau(t)$ (Theorem 3).

In Section 3 systems with a nonnegative transfer function are considered. It is shown (Theorem 4) that asymptotic stability for $\varphi(\sigma,t) = K\sigma$ guarantees absolute stability of the system in the class $\Phi(-K,K)$. Thus, such systems for arbitrary delay $\tau(t)$ in the feedback, satisfy the Aizerman conjecture [12] (note that the known results of such kind [13, 14, 15] relate to time-invariant systems). Under some additional condition, a precise upper bound for the stability sector is found (Theorem 5).

In Section 4 applications of the obtained results to some systems are discussed. It is shown that a closed-loop system consisting of any number of first order time-varying links and arbitrary delay in the feedback satisfies the Aizerman conjecture in the class $\Phi(-K,K)$. For a second order time-invariant system, delay independent bounds K_* and K_0 are found in explicit forms; the bound K_* is precise, because it is reached for some periodic delay $\tau(t)$.

2. ABSOLUTE STABILITY AND INSTABILITY CRITERIA

Suppose that the linear block is exponentially stable, then

$$|w(t,s)| \le C\exp[-\Delta(t-s)], \tag{6}$$

where the constants C and $\Delta > 0$ are independent on t and s.

Let us put

$$W(t) = \int_0^t |w(t,s)|\,\mathrm{d}s, \qquad W_+(t_k) = \sup W(t) \quad \text{for } t \ge t_k,$$

$$W_\infty = \overline{\lim_{t\to\infty}}\, W(t) = \lim_{t_k\to\infty} W_+(t_k)\,. \tag{7}$$

Here W_∞ is the upper limit of $W(t)$ as $t \to \infty$; it coincides with the conventional limit when the last exists. This is certainly the case when the linear block is time-invariant. Really, here $w(t,s) = w(t-s)$, so, setting $t - s = z$, we obtain

$$W(t) = \int_0^t |w(z)|\,\mathrm{d}z\,. \tag{8}$$

The function $W(t)$ in (8) increases monotonically and, therefore, tends to the limit.

The following theorem establishes a sufficient condition for absolute stability of system (1), (5).

Theorem 1. If

$$K < K_* = 1/W_\infty\,, \tag{9}$$

the system is absolutely stable in the class $\Phi(-K,K)$.

Proof. Let $\sigma(t)$ be a solution of equation (1). First let us show that for any $t_1 \ge 0$, there exists $t_m \ge t_1$ such that $|\sigma(t_m)| \ge |\sigma(t)|$ for $t \in [t_1,\infty)$. Really, otherwise, there is a sequence $t_1, t_2, \ldots;\ t_k \to \infty$ as $k \to \infty$, such that $|\sigma(t)| \le |\sigma(t_k)|$ for $t \in [t_1, t_k]$. Then from (1) and (5) we have

$$|\sigma(t_k)| \le R(t_k,t_1) + K\int_{t_1}^{t_k} |w(t_k,s)||\sigma(s-\tau(s))|\,\mathrm{d}s \le$$
$$R(t_k,t_1) + KW(t_k)|\sigma(t_k)| \tag{10}$$

where

$$R(t_k,t_1) = |f(t_k)| + K\int_0^{t_1} |w(t_k,s)||\sigma(s-\tau(s))|\,\mathrm{d}s\,.$$

Observing that $W(t_k) \le W_+(t_k)$, $W_+(t_k) \to W_\infty$, $R(t_k,t_1) \to 0$ as $t_k \to \infty$ and, by (9), $KW_\infty < 1$, we find that inequality (10) cannot hold for large k. The contradiction obtained shows that there exists a sequence t_m, $m = 1,2,\ldots$ such that $t_m \to \infty$ as $m \to \infty$ and $|\sigma(t_m)| \ge |\sigma(t)|$ for $t \in [t_m,\infty)$. Evidently, $|\sigma(t_m)| \ge |\sigma(t_{m+1})| \ge 0$, therefore; there exists $\sigma_\infty = \lim|\sigma(t_m)|$ as $t_m \to \infty$. Let us prove that $\sigma_\infty = 0$.

By assumption, $\tau(t) \le h$ for some h. Assuming $t_m - t_i \ge h$, analogously (10) we find

$$|\sigma(t_m)| \le R(t_m,t_i) + K\int_{t_i}^{t_m} |w(t_m,s)||\sigma(s-\tau(s))|\,\mathrm{d}s \le$$
$$R(t_m,t_i) + KW(t_m)|\sigma(t_i)|. \tag{11}$$

Since the sequence $|\sigma(t_m)|$, $m = 1,2,\ldots$ is convergent, then for any $\varepsilon > 0$, there exists such i that $|\sigma(t_i| - |\sigma(t_m| < \varepsilon$ for all $m > i$. Therefore, from (11) we find

$$|\sigma(t_m)|[1 - K\,W(t_m)] < R(t_m,t_i) + \varepsilon K\,W(t_m)\,. \tag{12}$$

Since $R(t_m, t_i) < \varepsilon$ for large $t_m - t_i$, $\lim KW_+(t_m) = KW_\infty < 1$ as $t_k \to \infty$ and $W(t_m) \le W_+(t_m)$, then $KW(t_m) < 1$ for large *m*. Therefore, inequality (12) is true only if $|\sigma(t_m)| \to 0$ as $m \to \infty$, i.e. $|\sigma(t)| \to 0$ as $t \to \infty$. □

Suppose that along with (6),

$$|f(t)| \le C\exp(-\Delta_1 t) \tag{13}$$

for some $\Delta_1 > 0$. In particular, if $f(t)$ is a solution of the linear system in the absence of external perturbations, then $\Delta_1 = \Delta$.

Definition 2. System (1) is called absolutely exponentially stable in the class $\Phi(K_1, K_2)$ if for any functions $\varphi(\sigma, t)$, $f(t)$, satisfying conditions (3), (13) and any piecewise continuous nonnegative bounded for $t \in [0, \infty)$ function $\tau(t)$, there exists an independent on these functions constant $\beta > 0$ such that for some *C*, the corresponding solution $\sigma(t)$ of (1) satisfies the inequality

$$|\sigma(t)| \le C\exp(-\beta t).$$

The infimum of $\lambda = -\beta$ for which the value *C* exists is called the Lyapunov exponent of the function $\sigma(t)$ (see, for example, [16]) of the solution $\sigma(t)$ of system (11), (5).

Let β_* be the root of the equation

$$W_\infty(\beta, h) = 1/K, \tag{14}$$

where

$$W_\infty(\beta, h) = \overline{\lim_{t\to\infty}} W_+(\beta, h, t),$$

$$W_+(\beta, h, t) = \int_0^t \exp[\beta(t + h - s)] |w(t,s)| \mathrm{d}s.$$

Theorem 2. The Lyapunov exponent of the solution $\sigma(t)$ satisfies the inequality

$$\lambda \le -\beta_*. \tag{15}$$

Proof. Setting in (1)

$$\sigma(t) = \exp(-\beta t) y(t), \tag{16}$$

where $\beta \in (0, \beta_* < \Delta_1)$, we obtain

$$y(t) = \exp(\beta t) f(t) + \exp(\beta t) \cdot \int_0^t w(t,s)\varphi[\exp[(-\beta(s-\tau)) y(s - \tau(s)), s]\,\mathrm{d}s \tag{17}$$

whence analogously (10) we have

$$|y(t_k)| \le R(\beta, t_k, t_1) + K\,W(\beta, h, t_k)|y(t_k)|. \tag{18}$$

Clearly, $W_\infty(\beta, h)$ increases with β, so, in view of (14), $KW_\infty(\beta, h) < 1$ for $\beta < \beta_*$. Therefore, analogously to the proof of Theorem 1, we find that $y(t)$ is bounded on $(0, \infty)$ which along with (16), proves the theorem. □

Let us now obtain a condition guaranteeing instability of the system. To this end, we put

$$W^0(t) = \int_0^t w(t,s)\,\mathrm{d}s. \tag{19}$$

Suppose there exists

$$W_0 = \lim_{t\to\infty} W^0(t) \ne 0. \tag{20}$$

Theorem 3. If

$$K \ge K_0 = 1/|W_0|, \tag{21}$$

then system (1),(2) is unstable.

Proof. Let us put

$$f_0(t) = 1 - W^0(t)/W_0,\ \varphi_0(\sigma) = K_0\sigma\,\mathrm{sgn}\,W_0, \quad \sigma(t) \equiv 1 \text{ for } t < 0. \tag{22}$$

In view of (20) and (22), $|f_0(t)| \to 0$ as $t \to \infty$; by (21), $\varphi_0(\sigma) \in \Phi(-K, K)$. By a direct substitution one can check that $\sigma(t) \equiv 1$ is the corresponding solution of equation (1). Since it does not satisfy condition (4), the system is unstable. □

Let K_b be the value of the constant K such that the system is stable in the class $\Phi(-K, K)$ for $K < K_b$ and unstable for $K \ge K_b$. Then from Theorems 1 and 3 it follows that K_b satisfies the inequality

$$1/W_\infty \le K_b \le 1/|W_0|. \tag{23}$$

3. SYSTEMS WITH SIGN-CONSTANT TRANSFER FUNCTION

Suppose now that the transfer function $w(t,s)$ is sign-constant. Without loss of generality, we assume that

$$w(t,s) \ge 0 \quad \text{for} \quad t \ge s \ge 0, \tag{24}$$

because the case $w(t,s) \le 0$ is reduced to (3.1) by substitution $w_1(t,s) = -w(t,s)$, $\varphi_1(\sigma, t) = -\varphi(\sigma, t)$.

Theorem 4. System (1), (24) is absolutely stable in the class $\Phi(-K, K)$ if it is stable for $\varphi = K\sigma(t - \tau(t))$.

Proof. Let $\sigma_0(t)$ be the solution of the equation

$$\sigma_0(t) = f_0(t) + K\int_0^t w(t,s)\sigma_0(s-\tau(s))\,\mathrm{d}s, \quad (25)$$

where

$$\sigma_0(t) = |\sigma(t)| \quad \text{for } t<0,$$

$$f_0(t) = |f(t)| + \exp(-t) \quad \text{for } t \ge 0. \quad (26)$$

Clearly, $\sigma_0(t-\tau(s)) > |\sigma(t-\tau(s))| > 0$ for sufficiently small $t>0$. Let us show that this inequality cannot break as t increases. Really, let $\sigma_0(t_1-\tau(t_1)) = \sigma(t_1-\tau(t_1))$ for some t_1, then, subtracting (1.1) from (3.2), we find

$$0 = |f(t_1)| - f(t_1) + \exp(-t_1) + \int_0^{t_1} w(t_1,s)[K\sigma_0(s-\tau(s)) - \varphi(\sigma(s-\tau(s),s)]\,\mathrm{d}s, \quad (27)$$

which is impossible, because the right-hand side of (27) is positive ($K\sigma_0 > \varphi(\sigma)$ for $\sigma_0 > |\sigma|$). `

If $\sigma_0(t_1) = -\sigma(t_1)$, then, summing (25) and (1), we find

$$0 = |f(t_1)| + f(t_1) + \exp(-t_1) + \int_0^{t_1} w(t_1,s)[K\sigma_0(s-\tau(s)) + \varphi(\sigma(s-\tau(s),s)]\,\mathrm{d}s,$$

where the right-hand side is also positive.

The obtained contradiction shows that $\sigma_0(t) > |\sigma(t)|$ for $t>0$ and, therefore, $\sigma(t) \to 0$ as $t\to\infty$. □

Suppose, moreover, that limit (20) exists.

Theorem 5. For absolute stability of system (1), (24), it is necessary and sufficient that

$$K < 1/W_\infty. \quad (28)$$

In fact, by (24), $W(t) = W^0(t)$, $W_\infty = W_0$, so Theorem 5 follows directly from inequality (23).

4. DISCUSSION

The obtained results embrace a wide range of control systems with, generally, time-varying linear block and arbitrary delay $\tau(t)$ in the feedback. In accordance with sufficient stability condition (9), such a system is absolutely stable in the class $\Phi(-W_\infty^{-1}+\varepsilon, W_\infty^{-1}-\varepsilon)$ where $\varepsilon>0$ is an arbitrary small value. If the function $f(t)$ in (1) exponentially tends to zero, the stability is exponential; the corresponding Lyapunov exponent satisfies inequality (15) (Theorem 2). If limit (20) exists, the system is certainly unstable in the wider class $\Phi(-W_0^{-1}, W_0^{-1})$ (Theorem 3).

Note that these results can be extended to the case when the nonlinearity bounds are time-dependent, i.e.,

$$-K(t)\sigma^2 \le \varphi(\sigma,t)\sigma \le K(t)\sigma^2, \quad K(t)\ge 0. \quad (29)$$

Really, as is clear from the proofs, it is only necessary to replace in the above conditions KW_∞ and KW_0 by

$$\overline{\lim_{t\to\infty}} \int_0^t |K(t)w(t,s)|\,\mathrm{d}s \quad \text{and} \quad \lim_{t\to\infty}\int_0^t K(t)w(t,s)\,\mathrm{d}s, \quad (30)$$

respectively.

As is known, the Lur'e problem was first formulated for the system

$$\dot{x} = Ax + b\varphi(cx) \quad (31)$$

where $x\in R^n$, b and c are column and row vectors, correspondingly. The problem is reduced to equation (1) where $\varphi = \varphi(\sigma), \sigma = cx$ and $w(t,s) = w(t-s)$, because equation (31) is time-invariant.

In 1949 Aizerman conjectured [12] that system (31) is absolutely stable in the class $\varphi(\sigma)\in\Phi(K_1,K_2)$, provided that the linear system $\dot{x} = Ax + kbcx$ is stable for any $k\in[K_1,K_2]$. Subsequently counterexamples showed that this conjecture is, in general, false (the history of the Aizerman conjecture can be found in the book by Gil' [13]). So, the problem is to find classes of systems satisfying the Aizerman conjecture. The first result in this direction was obtained by Gil' [13] who proved that if in system (4.3) the transfer function is nonnegative, then its absolute stability in the class $\Phi(0,K)$ is guaranteed by stability of the system $\dot{x} = Ax + Kbcx$. Recently he extended this result to distributed and delay time-invariant systems [14, 15].

In paper [11] time-variable systems with a nonnegative transfer function were considered via direct analysis of the corresponding Volterra equation. As a result, it was shown that stability of such a system for $\varphi(\sigma,t) = K\sigma$ guarantees absolute stability in the class $\Psi(-K,K)$. Theorem 4 of the present paper extends this result to systems with arbitrary delay $\tau(t)$ in the feedback. Namely, if the transfer function $w(t,s)$ is nonnegative, then for absolute stability of the system in the class $\Phi(-K,K)$, it is necessary and sufficient that it is asymptotically stable for $\varphi = K\sigma(t-\tau(t))$. If in (20) the limit W_0 exists (in particular, if the linear block is time-invariant), the precise bound for the stability sector equals $K = 1/W_\infty$ (Theorem 5) for any delay $\tau(t)$. Note that at first sight the invariance of the stability sector on $\tau(t)$ looks surprising, however, this is due to the fact that for $K = 1/W_\infty$ and $f(t)$, determined by (22), equation (1) admits the 'unstable' solution $\sigma(t)\equiv 1$ for any $\tau(t)$.

When analyzing the Lur'e problem, the nonlinearity class $\Phi(K_1,K_2)$ is usually reduced to $\Phi(0,K)$ $(K = K_2 - K_1)$ by substitution $\varphi_1(\sigma,t) = \varphi(\sigma,t) - K_1\sigma$. It can be shown that on the transformation $\varphi_1(\sigma,t) = \varphi(\sigma,t) - c\sigma$, the corresponding transfer function $w(t,s,c)$ increases with c, provided that $w(t,s,c) > 0$ for $t \ge s \ge 0$. So, if for $c = K_1$ ($\Phi = \Phi(0,K)$) condition (24) is satisfied, then it certainly holds for $c = 0.5(K_1 + K_2) > K_1$ ($\Phi = \Phi(-K,K)$. The converse is not, in general, true, i.e., condition (24) valid in the class $\Phi(-K,K)$ may be lost on the transformation to the class $\Phi(0,K)$.

Clearly, on the transformation $\varphi_1(\sigma,t) = \varphi(\sigma,t) - c\sigma$, the lower bound of the stability sector, provided by Theorem 4, becomes $K(c) = -K + 2c$. Thus, the largest stability sector corresponds to the minimal value $c < 0$ for which the transfer function $w(t,s,c)$ is still nonnegative.

Let $[K_1^0, K_2^0]$ be the Hurwitz angle of linear system (1.1) with $\varphi(\sigma,t) = k\sigma$ (i.e. it is asymptotically stable for $K_1^0 < k < K_2^0$ and unstable or not asymptotically stable for $k = K_1^0$ and $k = K_2^0$). If the transfer function is nonnegative and the limit exists, then from Theorem 5 it follows that. Since, by Theorem 4, stability for $\varphi(\sigma,t) = K\sigma$ provides stability in the class $\Phi(-K,K)$, then $K_1^0 \le -K_2^0$. A more detailed analysis shows that $K_1^0 < -K_2^0$.

Let the linear block be a closed-loop system consisting of n (generally, time-varying) links. Suppose that the individual transfer functions $w_i(t,s), i = 1,...,n$ are sign-constant. For a sign-constant input, the output of each link is sign-constant as well, therefore, so is the transfer function $w(t,s)$ of the entire linear block.

Suppose, in particular, that the links are of the first order, i.e. the linear block is described by the equations

$$\dot{x}_1 + a_1(t)x_i = 0,$$
$$\dot{x}_i + a_i(t)x_i = k_i x_{i-1}, \; i = 2,...,n. \tag{32}$$

There exists an extensive literature devoted to an analysis of feasibility of the Aizerman conjecture to closed-loop systems with the first order time-invariant links and time-invariant feedback $\varphi(\sigma)$. In particular, Bergen and Williams proved [17] that systems of the third order satisfy the Aizerman conjecture. Trukhan extended this result on systems with up to five stable links of the first order [18]. For an arbitrary number of links, the transfer function is positive, so stability in the class $\Phi(0,K)$ follows from Gil' theorem [13]. Let us show that the above findings enable us to essentially generalize these results in some respects.

Evidently, the individual transfer functions of a link,

$$w_i(t,s) = \exp[-\int_s^t a_i(s)\,\mathrm{d}s], \; i = 1,...,n, \tag{33}$$

is positive, hence, the transfer function of time-varying system (32) is positive as well. So, from Theorem 4 it follows that system (32) with the feedback $\varphi(\sigma(t-\tau(t)),t)$ is absolutely stable in the class $\Phi(-K,K)$, provided that it is stable for $\varphi = K\sigma(t-\tau(t),t)$. If in (20) the limit W_0 exists, then for any prescribed delay $\tau(t)$, the obtained bound for the stability sector coincides with the upper bound of the Hurwitz angle, i.e. $K_* = 1/W_\infty = K_2^0$ (Theorem 5). However, as is mentioned above, the obtained lower bound of the stability sector, $-1/W_\infty > K_1^0$ (note that the results [17,18] provide stability of the particular systems in the whole Hurwitz angle).

Consider now the second order system

$$\ddot{x}(t) + 2h\dot{x}(t) + (1+h^2)x(t) + \varphi(x(t-\tau(t),t) = 0, \tag{34}$$
$$-Kx^2 \le \varphi(x,t)x \le Kx^2, \quad \varphi(0,t) = 0.$$

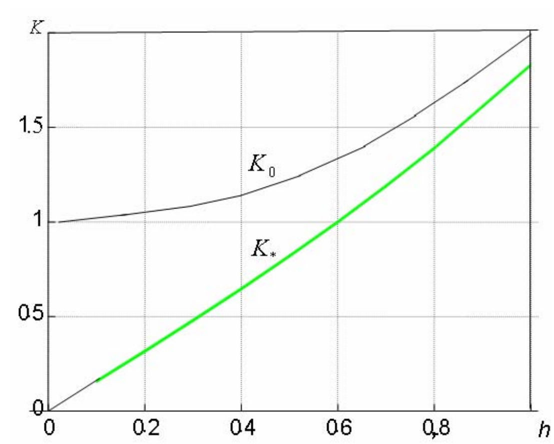
Here

$$w(t,s) = \exp[-h(t-s)]\sin(t-s),$$
$$W^0(t) = \frac{1}{1+h^2}\exp(-ht)[h\sin t - \cos t],$$

so $W_0 = 1/(1+h^2)$. Integrating (7), we find, see [11], $W_\infty = [1+\exp(-\pi h)]/[1+\exp(-\pi h)]$.

By Theorems 1,2 and 3, for any $\tau(t)$, system (34) is absolutely exponentially stable if $K < K_* = 1/W_\infty = \coth(-\pi h/2)$ and unstable for $K \ge K_0 = 1 + h^2$.

The functions $K_*(h)$ and $K_0(h)$ are plotted in the Figure; as is seen, they approach each other as h increases. This, in particular, testifies that the impact of a delay $\tau(t)$ in the feedback on the system stability decreases with an increase of stability of the linear part.



Figure

Note that the obtained delay independent stability condition $K < K_* = \coth(-\pi h/2)$ is precise, because for

$K = K_*$, the function $\tau(t)$ can be found such that equation (34) is unstable. Really, let us put

$$\begin{aligned} &\varphi(x(t-\tau(t),t) = K_* x(t-\tau(t)), \\ &\tau(t) = t \text{ for } t \in [0,\pi), \quad \tau(t+\pi) = \tau(t). \end{aligned} \tag{35}$$

Putting $x(0) = -1, \dot{x}(0) = 0$ and observing that $\varphi(x(t-\tau(t),t) = K_*$ for $t \in [0,\pi)$, we find that the corresponding solution is

$$x(t) = -\exp(-ht)\left(\frac{K_*}{1+h^2}+1\right)(\cos t + h\sin t) + \frac{K_*}{1+h^2}$$

.

Setting here $K_* = \coth(-\pi h/2)$, we obtain $x(\pi) = 1, \dot{x}(\pi) = 0$. Clearly, $\varphi(x(t-\tau(t),t) = -K_*$ for $t \in [\pi, 2\pi)$, so analogously we find $x(2\pi) = -1, \dot{x}(2\pi) = 0$. Thus, the corresponding solution of equation (34) is 2π - Putting $x(0) = -1, \dot{x}(0) = 0$ and observing that $\varphi(x(t-\tau(t),t) = K_*$ for $t \in [0,\pi)$, we find that the corresponding solution is

$$x(t) = -\exp(-ht)\left(\frac{K_*}{1+h^2}+1\right)(\cos t + h\sin t) + \frac{K_*}{1+h^2}.$$

(35)

Setting here $K_* = \coth(-\pi h/2)$, we obtain $x(\pi) = 1, \dot{x}(\pi) = 0$. Clearly, $\varphi(x(t-\tau(t),t) = -K_*$ for $t \in [\pi, 2\pi)$, so analogously we find $x(2\pi) = -1, \dot{x}(2\pi) = 0$. Thus, the corresponding solution of equation (34) is 2π-periodic; therefore, in accordance with the above definition, the system is unstable.

References

[1] A.I. Lur'e, *Some Nonlinear Problems in the Theory of Automatic Control.* London: H.M.Stationery Off., 1957.

[2] V.M. Popov, "Absolute stability of nonlinear systems of automatic control," *Autom. Remote Contr.*, vol. 22, pp. 857-875, 1962.

[3] R.E. Kalman, "Ljapunov functions for the problem of Lur'e in automatic control," *Proc.Nat.Acad.Sci.USA*, vol.49, pp. 201-205, 1963.

[4] M.A. Aizerman and F.R. Gantmaher, *Absolute Stability of Regulator Systems.* San Francisko: CA: Holden-Day, 1964.

[5] S. Lefschetz, *Stability of Nonlinear Control Systems*, New York: Academic Press, 1965.

[6] E.S. Pyatnitsky, "New studies in absolute stability of automatic control systems. Survey," *Autom.Remote Contr.*, vol. 6, pp.5-36, 1968.

[7] J.C. Willems, *The Analysis of Feedback Systems.* Cambridge: MIT Press, 1971.

[8] K.S. Narendra and J.F. Taylor, *Frequency Domain Criteria for Absolute Stability*. New York: Academic Press, 1973.

[9] M.G. Safonov, *Stability and Robustness of Multivariable Feedback Systems*. Cambridge: MIT Press, MA, 1980.

[10] V.A. Yakubovich, "Dichotomy and absolute stability of nonlinear systems with periodically non-stationary linear part," *Syst.Control Lett.*, vol. 11, pp.221-228, 1988.

[11] A.A. Zevin and M.A. Pinsky, "A new appproach to the Lur'e problem in the theory of absolute stability," *SIAM J.Control Optim.*, vol.42, pp.1895-1904, 2003.

[12] M.A. Aizerman, "On a conjecture from absolute stability theory," *Uspechi Mat. Nauk*, vol. 4 , pp.25-49, (1949) (In Russian).

[13] M.I. Gil', "On one class of absolutely stable systems," *Soviet Phys. Dokl.*, vol. 280, pp. 811-816, 1983.

[14] M.I. Gil', *Stability of Finite and Infinite Dimentional Systems*. Boston: Kluwer Academic Publishers, 1998.

[15] M.I. Gil', "Boundedness of solutions of nonlinear differential delay equations with positive Green functions and the Aizerman-Myshkis problem," *Nonlinear Analysis*, vol. 49, pp. 1065-1078, 2002.

[16] E.A. Barbashin, *Introduction to the Theory of Stability.* Groningen: Wolters-Noordhoff, 1970.

[17] A.R. Bergen and S.Williams, "Verification of Aizerman's conjecture," IRE Trans. Automat. Control, AC-7, pp.42-46,1962.

[18] N.M. Trukhan, "On single-loop systems absolutely stable in the Hurwitzian angle," *Autom. Remote Contr.*, vol. 11, pp.5-8, 1966.