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Exponential Stability of Discontinuous Dynamical Systems Determined by Differential Equatios in Banach Space

Anthony N. Michel, Ye Sun
Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556
Email: anthony.n.michel.1@nd.edu, ye.sun@csfb.com

Abstract

We present an exponential stability result for a class of discontinuous dynamical systems (DDS) determined by differential equations in Banach space (resp., Cauchy problems on abstract spaces). We demonstrate the applicability of our result in the analysis of several important classes of DDS, including systems determined by functional differential equations and partial differential equations.

1 Introduction

A *dynamical system* is a four-tuple $\{T, X, A, S\}$ where T denotes *time*, X is the *state space* (a metric space with metric d), A is the *set of initial states* and S denotes a *family of motions*. When $T = R^+ = [0, \infty)$ we speak of a *continuous-time dynamical system* and when $T = N = \{0, 1, 2, ...\}$ we speak of a *discrete-time dynamical system*. (For any motion $x(., x_0, t_0) \in S$, we have $x(t_0, x_0, t_0) = x_0 \in A \subset X$ and $x(t, x_0, t_0) \in X$ for all $t \in [t_0, t_1) \cap T$, $t_1 > t_0$, where t_1 may be finite or infinite. The set of motions S is obtained by varying (t_0, x_0) over $(T \times A)$.) When X is a finite-dimensional normed linear space, we speak of *finite-dimensional dynamical systems*, and otherwise, of *infinite-dimensional dynamical systems*. Also, when all motions in a continuous-time dynamical system are continuous with respect to t (relative to the metric d for X), we speak of a *continuous dynamical system* and when one or more of the motions are not continuous with respect to t, we speak of a *discontinuous dynamical system* (DDS). Finite-dimensional dynamical systems may be determined, e.g., by the solutions of ordinary differential equations, ordinary differential inequalities, difference equations, difference inequalities, and the like, while infinite-dimensional dynamical systems may be determined, e.g., by the solutions of differential-difference equations, functional differential equations, Volterra integrodifferential equations, various classes of partial differential equations, and so forth. Additionally, there are dynamical systems whose motions are not determined by classical equations or inequalities of the type enumerated above (e.g., certain classes of discrete event systems whose motions are characterized by Petri nets, Boolean logic elements, and the like). The stability analysis of *discrete-time dynamical systems* and *continuous dynamical systems* of the type enumerated above is a mature subject and is addressed, e.g., in [1]-[3].

Discontinuous dynamical systems (DDS) arise in the modeling process of a variety of systems, including hybrid dynamical systems, discrete event systems, switched systems, intelligent control systems, systems subjected to impulsive effects, and the like (see., e.g., [3]-[10]). The stability analysis of such systems has thus far been concerned primarily with finite dimensional dynamical systems (defined on $X = R^n$ with metric generated by the Euclidean norm) determined by ordinary differential equations; however, stability results for general DDS defined on metric space (i.e., X is an arbitrary metric space) have also been established [3], [4], [6], [7]. In principle, these results provide a general basis for the analysis of DDS determined by the various types of equations and inequalities enumerated earlier. However, the applications of these results to specific classes of DDS, *especially infinite dimensional systems*, are normally not entirely straightforward, and usually require further analysis. (This is also the case for continuous infinite dimensional dynamical systems (see, e.g., [1]-[3])).

In two recent papers, the stability analysis of infinite dimensional DDS determined by a class of *functional differential equations* [21] and by *linear* and *nonlinear semigroups* [20] has been addressed. In the present paper, we establish exponential stability result for infinite dimensional DDS determined by Cauchy problems on abstract spaces (differential equations on Hilbert and Banach spaces). This class of systems includes as special cases DDS determined by the various types of equations discussed earlier (In a companion paper, we address asymptotic stability results for the class of problems considered herein [24].). We apply our results in the analysis of DDS determined by specific classes of *functional differential equations*, *Volterra integrodifferential equations*, and *partial differential equations*.

2 Notation and Background Material

Let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$, let R^n denote real *n-space*, and let $|\ .\ |$ denote any one of the equivalent norms on R^n. For a real $n \times n$ matrix C (i.e., $C \in R^{n \times n}$) and $x \in R^n$, let $|C|$ denote the norm of C induced by the vector norm $|x|$.

Let X and Z be Banach spaces and let $\|.\|$ denote norm on Banach space. Let H be a Hilbert space with inner product $< .,. >$. In this case, the norm of $x \in H$ is given by $\|x\| =< x, x >^{1/2}$.

$L_p(G, U)$, $1 \leq p \leq \infty$, denotes the usual space of all Lebesgue measurable functions with domain G and range U. The norm of $L_p(G, U)$ will be denoted by $\|.\|_p$ (or $\|.\|_{L_p}$ if more explicit notation is needed). When the range U does not need emphasis, we utilize the notation $L_p(G)$. In particular, if $G = R^+$ and $U = R^m$, we write $L_p^m = L_p(R^+, R^m)$ and when $m = 1$, we write $L_p = L_p(R^+, R)$. If $1 \leq p < \infty$, we have for $f \in L_p$, $\|f\|_p = (\int_0^\infty |f(t)|^P \, dt)^{1/p}$. Finally, $H^2(\Omega)$ and $H_0^1(\Omega)$ denote the classical Sobolev spaces (see, e.g., [3]).

3 Continuous Dynamical Systems Determined by Abstract Cauchy Problems

A general form of a *system of first order differential equations in a Banach space* X is given by

$$\dot{x} = A(t, x) \qquad (GN)$$

where $t \in R^+$, $x \in C \subset X$, $A : R^+ \times C \to X$ and $\dot{x} = \frac{dx}{dt}$. We say that a function $x : [t_0, t_0 + c) \to C$, $c > 0$ is a solution of (GN) if $x \in C[[t_0, t_0 + c), C]$, if x is differentiable with respect to $t \in [t_0, t_0 + c)$ and if x satisfies the equation $(dx/dt)(t) = A(t, x(t))$ for all $t \in (t_0, t_0 + c)$.

Associated with (GN) we have the *initial value problem*, called a *Cauchy problem on abstract space*, given by

$$\dot{x} = A(t, x), x(t_0) = x_0 \qquad (I_{GN})$$

where $t \in R^+$, $t \geq t_0 \geq 0$ and $x_0 \in C$.

Under appropriate assumptions which ensure the existence of solutions of (GN), the initial value problem (I_{GN}) determines a *continuous dynamical system* (R^+, X, A, S_{GN}), as defined in Section 1, which is determined by the solutions $x(t) = x(t, x_0, t_0)$ of (I_{GN}) with $x(t_0, x_0, t_0) = x_0$ for all $t_0 \in R^+$ and all $x_0 \in C$. For the conditions of existence, uniqueness, continuity with respect to initial conditions, and continuation of solutions of the initial value problem (I_{GN}), refer, e.g., to [17].

Differential equations (GN) include as special cases differential-difference equations, functional differential equations, Volterra-integrodifferential equations, certain classes of partial differential equations, and others. We note, however, that in general, (GN) (resp., (I_{GN})) will not generate semigroups.

A special class of (I_{GN}) are *autonomous initial value problems* given by

$$\dot{x} = A(x), \; x(t_0) = x_0 \qquad (I_N)$$

where $A : C \to X, C \subset X$. If A is continuously differentable (or at least locally Lipschitz continuous), then the theory of existence, uniqueness and continuation of solutions of (I_N) is the same as in the finite-dimensional case [23]. If A is only continuous, then in general (I_N) may not have a solution (see, e.g., [22]). If (I_N) is to include nonlinear partial differential equations, one <u>*must*</u> allow A to be only defined on a dense set $C = \overline{D(A)}$ and to be discontinuous. For such functions A, the accretive property replaces (and generalizes) the Lipschitz property.

If A is *w-accretive* and if A generates a *quasicontractive semigroup* on C, then the solutions of (I_N) allow the estimate (see [3])

$$\|x(t, x_0, t_0) - x(t, y_0, t_0)\| \leq e^{w(t-t_0)} \|x_0 - y_0\| \qquad (3.1)$$

for all $t \in R^+$ and for all $x_0, y_0 \in C$. If in particular, A satisfies the Lipschitz condition

$$\|A(x) - A(y)\| \leq K \|x - y\| \qquad (3.2)$$

for all $x, y \in C$, where $K > 0$ is a constant, then (3.1) assumes the form

$$\|x(t, x_0, t_0) - x(t, y_0, t_0)\| \leq e^{K(t-t_0)} \|x_0 - y_0\| . \qquad (3.3)$$

A special class of (I_N) are *linear initial value problems* given by

$$\dot{x} = Ax, \; x(t_0) = x_0 \in D(A) \qquad (I_L)$$

for $t \in R^+$. Here $A : D(A) \to X$ is assumed to be a linear operator with domain $D(A)$ dense in $C \subset X$ and A is assumed to be closed, or else to have an extension $\bar{A}$ which is closed.

If A generates a $C_0 - semigroup$, then the solutions of (I_L) admit the estimate

$$\|x(t, x_0, t_0)\| \leq M e^{\omega(t-t_0)} \|x_0\| \qquad (3.4)$$

for all $t \geq t_0$ and $x_0 \in D(A)$. If in particular, A is a bounded linear operator, then we have in (3.3) $K = \|A\|$ and (3.4) assumes the form

$$\|x(t, x_0, t_0)\| \leq e^{\|A\|(t-t_0)} \|x_0\| \qquad (3.5)$$

for all $t \geq t_0 \geq 0$, $x_0 \in X$(see [3]).

If A generates a *differentiable* C_0*-semigroup* and if $Re\lambda \leq -\alpha_0$ for any $\lambda \in \sigma(A)$, then given any positive $\alpha < \alpha_0$, there is a constant $K(\alpha) > 0$ such that (see [3])

$$\|x(t, x_0, t_0)\| \leq K(\alpha) e^{-\alpha(t-t_0)} \|x_0\|$$

for all $t \geq t_0 \geq 0$, $x_0 \in X$ ($\sigma(A)$ denotes the spectrum of A).

In the remainder of this section, we consider more specific cases.

Example 3.1. Autonomous first order *retarded functional differential equations* (with delay $-r$) are given by

$$\left.\begin{array}{rcl} \dot{x}(t) & = & f(x_t), t > 0 \\ x(t) & = & \phi(t), -r \leq t \leq 0 \end{array}\right\} \qquad (3.6)$$

where $f : C_r \to R^n$, $X = C_r = C[[-r, 0], R^n]$ is a Banach space with norm defined by

$$\|\phi\| = max\{|\phi(t)| : -r \le t \le 0\} \tag{3.7}$$

and $x_t \in C_r$ is the function determined by $x_t(s) = x(t+s)$ for $-r \le s \le 0$. System (3.6) is clearly a special case of (I_N).

Assume that f satisfies a Lipschitz condition

$$|f(\xi) - f(\eta)| \le K_f \|\xi - \eta\| \tag{3.8}$$

for all $\xi, \eta \in C_r$. Under these conditions, the initial value problem (3.6) has a unique solution for every initial condition $\phi \in C_r$, denoted by $\psi_t(., \phi)$ which exists for all $t \in R^+$ (see, e.g., [14]). In accordance with (3.3), we have for the solutions of (3.6) the estimate

$$\|\psi_t(., \xi) - \psi_t(., \eta)\| \le e^{K_f t} \|\xi - \eta\| \tag{3.9}$$

for all $t \in R^+$ and all $\xi, \eta \in C_r$. □

Example 3.2. If in (3.6), $f = L$ is a linear mapping from C_r to R^n defined by

$$L(\phi) = \int_{-r}^{0} [dB(s)]\phi(s), \tag{3.10}$$

we obtain the initial-value problem

$$\left.\begin{array}{rcl} \dot{x}(t) & = & L(x_t), t > 0 \\ x_t & = & \phi(t), -r \le t \le 0 \end{array}\right\} \tag{3.11}$$

In (3.10), $B(s) = [b_{ij}(s)]$ is an $n \times n$ matrix whose entries are assumed to be functions of bounded variation on $[-r, 0]$. Then L is Lipschitz continuous on C_r with Lipschitz constant K_L less or equal to the variation of B in (3.10). It has been shown [15] that the operator L generates a differentiable C_0-semigroup. The *spectrum* of L consists of all solutions of the equation

$$det(\int_{-r}^{0} e^{\lambda s} dB(s) - \lambda I) = 0. \tag{3.12}$$

If in particular, all the solutions of (3.12) satisfy the relation $Re\lambda \le -\alpha_0$, then for any positive $\alpha < \alpha_0$, there is a constant $M(\alpha) > 0$ such that

$$\|\psi_t(., \xi)\| \le M(\alpha)e^{-\alpha t} \|\xi\| \tag{3.13}$$

$t \ge 0$, $\xi \in C_r$. When the roots of (3.12) have positive real parts, then we obtain, in view of (3.9), the estimate

$$\|\psi_t(., \xi)\| \le e^{K_L t} \|\xi\| \tag{3.14}$$

$t \ge 0$, $\xi \in C_r$. □

Example 3.3. A class of *initial and boundary value problems* determined by the *heat equation* is given by

$$\left.\begin{array}{l} \frac{\partial u}{\partial t} = a\Delta u,\ (t, x) \in [t_0, \infty) \times \Omega \\ u(t_0, x) = \phi(x),\ x \in \Omega \\ u(t, x) = 0,\ (t, x) \in [t_0, \infty) \times \partial\Omega, \end{array}\right\} \tag{3.15}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the *Laplacian* and $a > 0$ is a constant.

It has been shown that for each $\phi \in X = H^2[\Omega, R] \cap H_0^1[\Omega, R]$ there exists a *unique solution* $u = u(t, x)$, $t \ge t_0$, $x \in \Omega$ for (3.15) such that U, defined by $U(t) = u(t, .)$, is a continuously differentiable functions from $[t_0, \infty)$ to X with respect to the H^1-norm (to be specified later) [3]. Then (3.15) can be written as an *abstract Cauchy problem* in the space X with respect to the H^1-norm,

$$U'(t) = AU(t),\ t \ge t_0 \tag{3.16}$$

with initial condition $U(t_0) = \phi \in X$, where the operator A is linear and is defined as $A = \sum_{i=1}^n \frac{d^2}{dx_i^2}$.

In establishing an estimate of the H^1-norm of the solutions of (3.15), we choose the function

$$v(\phi) = \|\phi\|_{H^1}^2 = \int_\Omega (|\nabla\phi|^2 + |\phi|^2)dx \tag{3.17}$$

for any $\phi \in X$. Let $u(t, .)$ denote a solution of (3.15) and let $U(t) = u(t, .) \in X$. Evaluating dv/dt along the solutions of (3.15), we have

$$\begin{aligned} \frac{d[v(U)]}{dt} &= \int_\Omega \frac{\partial}{\partial t}[\sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2 + u^2]dx \\ &= \int_\Omega [\sum_{i=1}^n 2(\frac{\partial u}{\partial x_i})\frac{\partial^2}{\partial x_i \partial t} + 2u\frac{\partial u}{\partial t}]dx \\ &= -\sum_{i=1}^n 2\int_\Omega \frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial t}dx + 2a\int_\Omega u\Delta u dx \\ &= -2a\int_\Omega (\Delta u)^2 dx - 2a\int_\Omega |\nabla u|^2 dx \\ &\le -2a\int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{3.18}$$

By *Poincaré's inequality* [3], we have

$$\int_\Omega |u|^2 dx \le \gamma^2 \int_\Omega |\nabla u|^2 dx, \tag{3.19}$$

where γ can be chosen as $\delta/\sqrt{n}$ and Ω can be put into a cube of length δ. Hence we have

$$\frac{d[v(U)]}{dt} \le -a(\int_\Omega |\nabla u|^2 dx + \frac{1}{\gamma^2}\int_\Omega |u|^2 dx) \le -c\|U\|_{H^1}^2 \tag{3.20}$$

for all $\phi \in X$, where $c = min\{a, \frac{a}{\gamma^2}\} > 0$. Therefore,

$$\|U(t)\|_{H_1} \le e^{-\frac{c}{2}(t-t_0)} \|U(t_0)\|_{H_1} \tag{3.21}$$

for $t \ge t_0$. □

4 Discontinuous Dynamical Systems Determined by Differential Equations in Banach Space

We first consider *a family of initial-value Cauchy problems in Banach space* X of the form

$$\left.\begin{array}{rcl}\dot{x}(t) & = & A_k(t,x),\ t \geq \tau_k \\ x(\tau_k) & = & x_k\end{array}\right\} \qquad (I_{C_k})$$

for $k \in N$. For each $k \in N$, we assume that $A_k : R^+ \times X \to X$ and that $\dot{x} = dx/dt$. Throughout, we will assume that for every $(\tau_k, x_k) \in R^+ \times X$, (I_{C_k}) possesses a unique solution $x^{(k)}(t, x_k, \tau_k)$ which exists for all $t \in [\tau_k, \infty)$ and which is continuous with respect to initial conditions. We express this by saying that (I_{C_k}) is *well posed*. In addition, for each $k \in N$, we assume that $A_k(t, 0) = 0$, $t \in R^+$. This ensures the existence of the *zero solution* $x^{(k)}(t, x_k, \tau_k) = 0$, $t \geq \tau_k$, with $x_k = 0$, which means that $x_k = 0 \in X$ is an *equilibrium* of

$$\dot{x}(t) = A_k(t,x). \qquad (C_k)$$

We now consider *discontinuous initial-value problems in Banach space* X given by

$$\left.\begin{array}{rcl}\dot{x}(t) & = & A_k(t,x),\quad \tau_k \leq t < \tau_{k+1} \\ x(\tau_{k+1}) & = & g_k(x(\tau_{k+1}^-)), k \in N,\end{array}\right\} \qquad (DC)$$

where for each $k \in N$, A_k is assumed to possess the identical properties given in (C_k), where $g_k \in C[X, X]$, and

$$x(t^-) = lim_{t' \to t, t' < t} x(t'). \qquad (4.1)$$

For each $k \in N$, we assume that $g_k(0) = 0$. The set $E = \{\tau_0, \tau_1, \tau_2, ...\}$, denoting the *set of discontinuities*, is assumed to be an unbounded closed discrete subset of R^+ with $\tau_0 < \tau_1 < \tau_2 < ... < \tau_k < ....$

Under the above assumptions for (DC) and (C_k), it is now clear that for every $(t_0, x_0) \in R^+ \times X$, $t_0 = \tau_0$, (DC) has a unique solution $x(t, x_0, t_0)$ which exists for all $t \in [t_0, \infty)$. This solution is made up of a sequence of solution segments $x^{(k)}(t, x_k, \tau_k)$, defined over the intervals $[\tau_k, \tau_{k+1})$, $k \in N$, with initial conditions (τ_k, x_k), where $x_k = x(\tau_k)$, $k = 1, 2, ...$ and where $(\tau_0 = t_0, x_0)$ are given. Furthermore, (DC) admits the *zero solution* $x(t, x_0, t_0) = 0$ for $t \geq t_0$ (with $x_0 = 0$), and therefore, $x_0 = 0 \in X$ is an *equilibrium* for (DC).

Remark 4.1. Consistent with the characterization of *discontinuous dynamical system* (DDS) given in Section 1, it is clear from the above that (DC) determines a *discontinuous dynamical system* $\{T, X, A, S\}$, where $T = R^+, A = X$, the metric on X is determined by the norm $\|.\|$ defined on X (i.e., $d(x, y) = \|x - y\|$), and S denotes the set of all the piecewise continuous solutions of (DC) corresponding to all possible initial conditions $(t_0, x_0) \in R^+ \times X$. In the interests of brevity, we will refer to this DDS simply as "system (DC)", or simply as "(DC)". □

In finite dimensional spaces all norms are equivalent and therefore, when addressing convergence properties for such systems, such as stability, the choice of norm plays no important role. This is not the case in infinite dimensional systems and the various stability concepts depend intricately on the particular norm (i.e., on the particular Banach space) on hand.

Definition 4.1. a) The zero solution of (DC) is *exponentially stable* if there exists $\alpha > 0$, and for every $\epsilon > 0$ and every $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x(t, x_0, t_0)\| < \epsilon e^{-\alpha(t-t_0)}$ for all $t \geq t_0$, whenever $\|x_0\| < \delta$.

b) The zero solution of (DC) is *exponentially stable in the large* if there exists $\alpha > 0$, $\gamma > 0$ and for every $\beta > 0$, there exists $k(\beta) > 0$ such that

$$\|x(t, x_0, t_0)\| \leq k(\beta) \|x_0\|^\gamma e^{-\alpha(t-t_0)}$$

for all $t \geq t_0$, whenever $\|x_0\| < \beta$. □

The above definitions are adaptations of corresponding stability and boundedness definitions for (GN) (resp., (C_k)) (see, e.g., [3], [11]-[12]).

5 Main Stability Results

Theorem 5.1. Assume that there exists a function $V : X \times R^+ \to R^+$ and three positive constants c_1, c_2 and b such that

$$c_1 \|x\|^b \leq V(x, t) \leq c_2 \|x\|^b \qquad (5.1)$$

for all $x \in X$ and $t \in R^+$ (resp., for all x in some neighborhood of the origin $0 \in X$ and $t \in R^+$).

i) Assume that for every $x(., x_0, t_0)$, $V(x(t, x_0, t_0), t)$ is continuous for all $t \geq t_0 \geq 0$ except on a set of discontinuities $E = \{t_0 = \tau_0, \tau_1, \tau_2, ...\}$. Furthermore, assume that there exists a function $h \in C[R^+, R^+]$, independent of $x(., x_0, t_0)$, such that $h(0) = 0$ and such that for any $x(., x_0, t_0)$,

$$\begin{array}{rcl}V(x(t, x_0, t_0), t) & \leq & h(V(x(\tau_k, x_0, t_0), \tau_k), \\ & & t \in (\tau_k, \tau_{k+1}), k \in N\end{array} \qquad (5.2)$$

and such that for some positive constant q, h satisfies

$$h(r) = o(r^q) \quad as\ r \to 0. \qquad (5.3)$$

ii) Assume that there exists a constant $c_3 > 0$ such that

$$DV(x(\tau_k, x_0, t_0), \tau_k) \leq -c_3 \|x(\tau_k, x_0, t_0)\|^b, \quad k \in N \qquad (5.4)$$

for all $x(., x_0, t_0)$ and all $x_0 \in X$ (resp., for all $x(., x_0, t_0)$ with x_0 in some neighborhood of the origin $0 \in X$), where $DV(x(\tau_k, x_0, t_0), \tau_k)$ is defined by

$$\begin{array}{rl} & DV(x(\tau_k, x_0, t_0), \tau_k) \\ = & \frac{1}{\tau_{k+1} - \tau_k}[V(x(\tau_{k+1}, x_0, t_0), \tau_{k+1}) \\ & -V(x(\tau_k, x_0, t_0), \tau_k)],\end{array} \qquad (5.5)$$

Then the zero solution of (DC) is *exponentially stable in the large* (resp., *exponentially stable*).

Proof. The proof is omitted due to space limitations. □

6 Applications

Example 6.1. (*Time-invariant differential equations in Banach space*)
If in (C_k) we let $A_k(t,x) \equiv A_k(x)$, then (I_{C_k}) takes the form

$$\left.\begin{array}{rcl} \dot{x}(t) & = & A_k(x) \\ x(\tau_k) & = & \phi_k \end{array}\right\} \qquad (I'_{C_k})$$

$k \in N$, $t \in [\tau_k, \infty)$, and (DC) assumes the form

$$\left.\begin{array}{rcl} \dot{x}(t) & = & A_k(x), \quad \tau_k \le t < \tau_{k+1} \\ x(\tau_{k+1}) & = & g_k(x(\tau_{k+1}^-)) \end{array}\right\} \qquad (DC')$$

$k \in N$. Assuming that for all $k \in N$, $A_k(0) = 0$ and that A_k satisfies the Lipschitz condition

$$\|A_k(x) - A_k(y)\| \le K_k \|x - y\| \tag{6.1}$$

for all $x, y \in X$, we obtain, in accordance with (3.3), the estimate

$$\left\|x^{(k)}(t, \phi_k, \tau_k)\right\| \le e^{K_k(t-\tau_k)} \|\phi_k\| \tag{6.2}$$

for all $t \ge \tau_k$ and all $\phi_k \in X$. In system (DC') we assume that for all $k \in N$, $g_k(0) = 0$ and that

$$\|g_k(x)\| \le \gamma_k \|x\| \tag{6.3}$$

for some $\gamma_k > 0$ and for all $x \in X$ and we let $\tau_{k+1} - \tau_k = \lambda_k$

Proposition 6.1. Let K_k,γ_k and λ_k be the parameters for system (DC') given in (6.1)-(6.3). If for all $k \in N$, $\gamma_k e^{K_k \lambda_k} \le \alpha < 1$, where $\alpha > 0$ is a constant, then the zero solution of (DC') is *exponentially stable*, in fact, *exponentially stable in the large*.

Proof. The proof is omitted due to space limitations. □

Example 6.2. (*Time-invariant linear functional differential equations*)
If in (C_k) we let $X = C_r$ and $A_k(t,x) = A_k(x) = L_k x_t$, where C_r, x_t and L_k are defined as in Examples 3.1 and 3.2, then (I_{C_k}) takes the form

$$\left.\begin{array}{rcl} \dot{x}(t) & = & L_k x_t \\ x_{\tau_k} & = & \phi_k, \end{array}\right\} \tag{6.4}$$

$k \in N$, $t \in [\tau_k, \infty)$. If in (DC) we let $g_k(\eta) = G_k \eta$, then (DC) assumes the form

$$\left.\begin{array}{rcl} \dot{x}(t) & = & L_k x_t, \quad \tau_k \le t < \tau_{k+1} \\ x_{\tau_{k+1}} & = & G_k x_{\tau_{k+1}^-}, \end{array}\right\} \tag{6.5}$$

$k \in N$. For each $k \in N$, L_k is defined, as in (3.10), by

$$L_k(\phi) = \int_{-r}^{0} [dB_k(s)]\phi(s). \tag{6.6}$$

We suppose that all assumptions that we made for L given in (3.10) hold as well for L_k. Then L_k is Lipschitz continuous on C_r with Lipschitz constant K_k less or equal to the variation of B_k, and as such, condition (6.1) still holds for (6.4). As in (3.12), the spectrum of L_k consists of all solutions of the equation

$$det(\int_{-r}^{0} e^{\lambda_k s} dB_k(s) - \lambda_k I) = 0. \tag{6.7}$$

In accordance with (3.13), when all solutions of (6.7) satisfy the relation $Re\lambda_k \le -\alpha_0$, then for any positive $\alpha_k < \alpha_0$, there is a constant $M_k(\alpha_k) > 0$ such that the solutions of (6.4) allow the estimate

$$\left\|x^{(k)}(t, \phi_k, \tau_k)\right\| \le M_k(\alpha_k) e^{-\alpha_k(t-\tau_k)} \|\phi_k\| \tag{6.8}$$

for all $t \ge \tau_k \ge 0$ and $\phi_k \in C_r$. When the above assumption is not true, then in accordance with (3.14), the solutions of (6.4) still allow the estimate

$$\left\|x^{(k)}(t, \phi_k, \tau_k)\right\| \le e^{K_k(t-\tau_k)} \|\phi_k\| \tag{6.9}$$

for all $t \ge \tau_k$ and $\phi_k \in C_r$. Thus, in all cases we have

$$\left\|x^{(k)}(t, \phi_k, \tau_k)\right\| \le Q_k e^{w_k(t-\tau_k)} \|\phi_k\| \tag{6.10}$$

for all $t \ge \tau_k \ge 0$ and $\phi_k \in C_r$, where $Q_k = 1$ and $w_k = K_k$ when (6.9) applies and $Q_k = M_k(\alpha_k)$ and $w_k = -\alpha_k$, $\alpha_k > 0$, when (6.8) applies.

Finally, for each $k \in N$, G_k in (6.5) is assumed to be a linear operator, $G_k : C_r \to C_r$. We have

$$\|G_k \eta\| \le \|G_k\| \|\eta\| \tag{6.11}$$

for all $\eta \in C_r$, where $\|G_k\|$ is the norm of G_k induced by the norm $\|.\|$ defined on C_r.

Proposition 6.2. Let $w_k, \|G_k\|, Q_k, \lambda_k$ be the parameters for system (6.5) defined above. If for all $k \in N$, $\|G_k\| Q_k e^{w_k \lambda_k} \le \alpha < 1$, where $\alpha > 0$ is a constant, then the zero solution of (6.5) is *exponentially stable*, in fact, *exponentially stable in the large*.

Proof. The proof is omitted due to space limitations. □

Example 6.3. (*Heat equation*)
We consider a family of initial and boundary value problems determined by the heat equation

$$\left.\begin{array}{l} \frac{\partial u}{\partial t} = a_k \Delta u, \ (t,x) \in [\tau_k, \infty) \times \Omega \\ u(\tau_k, x) = \phi_k(x), \ x \in \Omega \\ u(t,x) = 0, \ (t,x) \in [\tau_k, \infty) \times \partial\Omega, \end{array}\right\} \tag{6.12}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $a_k \in R^+$ are constants. Next we consider a DDS determined by

$$\left.\begin{array}{l} \frac{\partial u}{\partial t} = a_k \Delta u, \ (t,x) \in [\tau_k, \tau_{k+1}) \times \Omega \\ u(\tau_{k+1}, .) = g_k(u(\tau_{k+1}^-, .)), \\ u(t,x) = 0, \ (t,x) \in R^+ \times \partial\Omega \end{array}\right\} \tag{6.13}$$

where all symbols are defined similarly as in (6.12), $g_k: X \to X$, $X = H^2[\Omega, R] \cap H_0^1[\Omega, R]$ with the H^1-norm (see, (3.17)), $k \in N$. We assume that $g_k(0) = 0$ and there exists γ_k such that $\|g_k(\phi)\|_{H^1} \leq \gamma_k \|\phi\|_{H^1}$ for all $\phi \in X$, $k \in N$.

Along any solution $u^{(k)}$ of (6.12), similarly as in Example 3.3 (see, (3.21)), we obtain the estimate

$$\left\| U^{(k)}(t) \right\|_{H_1} \leq e^{-\frac{c_k}{2}(t-\tau_k)} \left\| U^{(k)}(\tau_k) \right\|_{H_1} \tag{6.14}$$

for $t \geq \tau_k$, where $c_k = min\{a_k, \frac{a_k}{\gamma^2}\}$, where γ is a constant determined by Ω (see, (3.19)). Each solution $u(t, x, \phi, t_0)$ of (6.13) is made up of a sequence of solution segments $u^{(k)}(t, x, \phi_k, \tau_k)$, defined on $[\tau_k, \tau_{k+1})$ for $k \in N$, which are determined by (6.12) with $\phi_k = u(\tau_k, .)$. □

Proposition 6.3. For system (6.13), let $\omega_k = -\frac{c_k}{2}$ and $\lambda_k = \tau_{k+1} - \tau_k$, $k \in N$. If for all $k \in N$, $\gamma_k e^{\omega_k \lambda_k} \leq \alpha < 1$, where $\alpha > 0$ is a constant, then the zero solution of (6.13) is *exponentially stable*, in fact, *exponentially stable in the large.*

Proof. The proof is omitted due to space limitations. □

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