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Optimum Cooperative UAV Sensing Based on Cramér-Rao Bound

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Abstract— **We investigate optimal estimation for both the position and the velocity of the ground moving target (GMT) by employing sensors composed of unmanned aerial vehicles (UAVs). The problem is the cooperative sensing by the UAVs, in terms of their location geometries to achieve optimal estimation of the GMT. Based on the Cramér-Rao bound, we are able to derive the minimum achievable error variance in estimation of the position and the velocity of the GMT, and obtain the optimal geometries of the UAV sensors via minimization of the minimum achievable error variance for unbiased estimation commanded by the Cramér-Rao bound. Our solution is complete that encompasses various situations for the GMT, and the number of UAV sensors.**

1. Introduction

Cooperative sensing with UAVs plays an important role in combat ISR (intelligence, surveillance, and reconnaissance), such as GMTI (ground moving target identification) and MSTE (moving surface target engagement). The objective is the early identification and localization of the threat GMT of interest, estimate and track the GMTs in terms of their position and velocities. The ultimate goal is the destruction of the threat GMT. A problem arising from GMTI and MSTE is the sensor geometries: how to cooperate the UAV sensors in terms of their locations to achieve optimal estimation of the position and velocity of the GMT? We assume that the observed data by UAV sensors are radar signals, comprised of noisy azimuth, range, and range rate information. For simplicity we also assume that the observed data are corrupted by Gaussian noise. Based on the well-known Cramér-Rao bound, we provide an answer to the cooperative and optimal sensing problem. We will show that the minimum achievable error variance for estimation of the position and velocity of a given GMT amounts to maximization of certain mathematical quantity that involves the angular positions of the UAV sensors, assuming that all UAV sensors have the same distance to the GMT. The orthogonal, or near orthogonal sensor geometries for UAVs are preferred in most of the applications. We will provide a complete solution to the optimal sensor geometry problem with an arbitrary number of UAV sensors that encompasses various situations for the GMT.

The work to be reported in this paper is related to the general problem of target localization [1], [2], [3], [6], [7]. The emphasis of this paper is placed on the geometries of the UAV sensors in optimal estimation of not only position, i.e., localization, but also velocity of the GMT due to tracking, which is relatively new. We consider the case of two-dimensional plane for the coverage field, which is adequate, if the elevation of the field is known *a priori*, that is true in most of the applications. The problem of cooperative sensing is also related to collaborative signal processing [4] in the sense that sensors need work together to achieve optimal estimation. The notations in this paper are standard. The inner-product and the Euclidean norm of column vectors are denoted by $< \cdot, \cdot >$, and $\|\cdot\|$, respectively, and the transpose by $\cdot^T$. Other notations will be made clear as we proceed.

2. Cooperative Estimation

Let the number of UAV sensors be $m > 1$, with their respective positions and velocities known. For the kth UAV, its position and velocity at time t are denoted by $\vec{p}_k(t)$, and $\vec{v}_k(t)$, respectively. Let $\vec{p}_T(t)$, and $\vec{v}_T(t)$ be the position and velocity of the GMT respectively. Then

$$\vec{p}_T(t) = \vec{p}_k(t) + \delta\vec{p}_k(t), \quad \vec{v}_T(t) = \frac{d\vec{p}_T(t)}{dt} = \vec{v}_k(t) + \delta\vec{v}_k(t),$$

with $\delta\vec{p}_k(t)$ and $\delta\vec{v}_k(t)$ being the respective differences of the position and velocity, respectively, between the GMT and the kth UAV. Let the x-axis be associated with east-west, and y-axis associated with north-south. Then we have the following representations:

$$\vec{p}_k(t) = \begin{bmatrix} x_k(t) \\ y_k(t) \end{bmatrix}, \qquad \vec{v}_k(t) = \begin{bmatrix} \dot{x}_k(t) \\ \dot{y}_k(t) \end{bmatrix},$$
$$\delta\vec{p}_k(t) = \begin{bmatrix} \delta x_k(t) \\ \delta y_k(t) \end{bmatrix}, \qquad \delta\vec{v}_k(t) = \begin{bmatrix} \delta\dot{x}_k(t) \\ \delta\dot{y}_k(t) \end{bmatrix}.$$

Let $\theta_k(t) = \angle\delta\vec{p}_k(t)$ be the argument of $\delta\vec{p}_k(t)$, that is the bearing parameter between the kth UAV and the GMT (cf. Fig. 1).

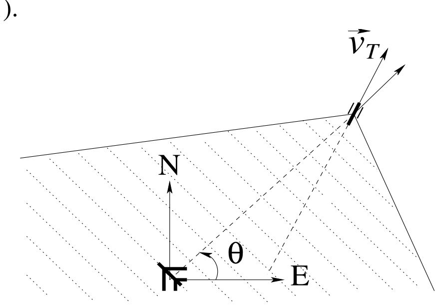

Fig. 1 The GMT and the observable cone by UAV

The radial velocity of the GMT observed by the kth UAV induces the Doppler shift, denoted by $\omega_k(t)$, which is given by

$$\omega_k(t) = \cos(\theta_k(t))\dot{x}_T(t) + \sin(\theta_k(t))\dot{y}_T(t). \quad (1)$$

For simplicity we will skip the time variable t in sequel. We comment that ω_k is the orthogonal projection of $\vec{v}_T$ onto $\delta\vec{p}_k$, i.e., $\omega_k = \|\vec{v}_T\| \cos(\theta_k - \angle\vec{v}_T)$. The greater the velocity $\vec{v}_T$, the easier for radars to capture its movement in measuring its azimuth, range, and range rate. Hence the observation of the range rate by the kth UAV sensor has the form of

$$z_k(\omega) = f(\omega_k)\omega_k + \eta_k(\omega), \quad (2)$$

where $\eta_k(\omega)$ is the Gaussian noise with variance $\sigma^2_{\omega_k}$, and $f(\cdot)$ is a nonlinear function characterizing the GMT indication. As a rule of thumb, the GMT is undetectable if ω_k is smaller than 10 kilometers per hour, which is taken as the critical velocity, denoted by ω_{T_c}.

Suppose that ω_k is large enough such that the GMT is observable, or $f(\omega_k) = 1$. Then

$$\begin{bmatrix} z_k(x) \\ z_k(y) \end{bmatrix} = \begin{bmatrix} x_T \\ y_T \end{bmatrix} + \begin{bmatrix} \eta_k(x) \\ \eta_k(y) \end{bmatrix} \quad (3)$$

are observations of $\vec{p}_T$ by the kth UAV sensor with $\eta_k(x)$ and $\eta_k(y)$ white Gaussian noise of zero mean. If $\theta_k = 0$, then $\eta_k(x)$ and $\eta_k(y)$ are the corruption noises in measuring the range and azimuth respectively. Thus $\eta_k(x)$ and $\eta_k(y)$ are uncorrelated with variances $\sigma^2_{x_k} = \sigma^2_{r_k}$ (range variance) and $\sigma^2_{y_k} = \sigma^2_{a_k}$ (azimuth variance), respectively. Or the joint probability density (JPD) is

$$p[\eta_k(x), \eta_k(y)] = \frac{1}{2\pi\sigma_{r_k}\sigma_{a_k}} \exp\left(-\frac{1}{2}\left[\frac{\eta_k^2(x)}{\sigma^2_{r_k}} + \frac{\eta_k^2(y)}{\sigma^2_{a_k}}\right]\right).$$

That is, the constant probability density contour has an ellipsoidal shape. But if $\theta_k \neq 0$, then $\eta_k(x)$ and $\eta_k(y)$ are correlated, and the covariance matrix is given by

$$\Sigma_{\theta_k} = U(\theta_k) \begin{bmatrix} \sigma^2_{r_k} & 0 \\ 0 & \sigma^2_{a_k} \end{bmatrix} U(\theta_k)^T,$$

where $U(\cdot)$ is a rotation matrix, given by

$$U(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

It follows that the the JPD has the form of

$$p[\eta_k(x), \eta_k(y)] = \frac{\exp\left(-\frac{1}{2}\begin{bmatrix} \eta_k(x) \\ \eta_k(y) \end{bmatrix}^T \Sigma^{-1}_{\theta_k} \begin{bmatrix} \eta_k(x) \\ \eta_k(y) \end{bmatrix}\right)}{2\pi\sqrt{\det(\Sigma_{\theta_k})}}. \quad (4)$$

It should be clear that the unitary matrix $U(\theta)$ performs rotation of angle θ. For simplicity we assume that each UAV sensor has the same noise variances for measuring the azimuth, range, and range rate, respectively, if the distances to the GMT are the same, which are true, provided that identical sensors are used in each UAV.

The position measurement (3) assumes that $\omega_k \geq \omega_c$. In the general case, the observation of the position of the GMT in (3) needs be replaced by

$$\begin{bmatrix} z_k(x) \\ z_k(y) \end{bmatrix} = \begin{bmatrix} x_T \\ y_T \end{bmatrix} f(\omega_k) + \begin{bmatrix} \eta_k(x) \\ \eta_k(y) \end{bmatrix}, \quad (5)$$

where the noise has the JPD as in (4). Denote

$$\chi = \begin{bmatrix} x_T & y_T & \dot{x}_T & \dot{y}_T \end{bmatrix}^T, \quad (6)$$

$$C_k = f(\omega_k) \begin{bmatrix} 0 & 0 & \cos(\theta_k) & \sin(\theta_k) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (7)$$

$$\underline{\eta}_k = \begin{bmatrix} \eta_k(\omega) \\ \eta_k(x) \\ \eta_k(y) \end{bmatrix}, \quad \underline{z}_k = \begin{bmatrix} z_k(\omega) \\ z_k(x) \\ z_k(y) \end{bmatrix}. \quad (8)$$

Combining (2) and (5) yields

$$\underline{z}_k = C_k\chi + \underline{\eta}_k, \qquad \Sigma_k = \mathrm{E}\{\underline{\eta}_k\underline{\eta}_k^T\} = \mathrm{diag}(\sigma^2_{\omega_k}, \Sigma_{\theta_k}), \quad (9)$$

for $1 \leq k \leq m$. Because $f(\cdot)$ and θ_k are nonlinear functions of the parameter vector χ to be estimated, the observed data vector $\underline{z}_k$ is also a nonlinear function of the parameter vector χ in general. However it is a special nonlinear function of the parameter vector. Indeed if θ_k is known, and $\omega_k \geq \omega_c$, then the observed data vector $\underline{z}_k$ becomes a linear function of the parameter vector χ. It is this quasi-linearity, which will be exploited in this paper to derive the Cramér-Rao bound for the underlying estimation problem.

It is noted that C_k is a function of θ_k. Hence if all UAVs have the equal distance to the GMT, then the geometries of the m UAVs are uniquely determined by $\{\theta_k\}_{k=1}^m$. Recall that θ_k is the bearing parameter from the kth UAV to the GMT. Our cooperative sensing problem is to find the optimal geometries of m UAVs so that the minimal error variance of the estimation error is minimized. Denote

$$\psi = \begin{bmatrix} \underline{z}_1 \\ \vdots \\ \underline{z}_m \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} \underline{\eta}_1 \\ \vdots \\ \underline{\eta}_m \end{bmatrix}.$$

It follows that

$$\psi = \mathcal{C}\chi + \mathcal{N}. \quad (10)$$

In light of (9), the Gaussian assumption implies that ψ has a JPD function

$$p(\psi;\chi) = \frac{\exp\left\{-\frac{1}{2}\sum_{k=1}^{m} (\underline{z}_k - C_k\chi)^T \Sigma_k^{-1} (\underline{z}_k - C_k\chi)\right\}}{(2\pi)^{2m} \det(\Sigma_k)^{\frac{m}{2}}}.$$

The error covariance X corresponding to $\hat{\chi}$, any unbiased estimate of χ, based on observations $\psi = \mathcal{C}\chi + \mathcal{N}$, is lower bounded by the well known Cramér-Rao bound matrix $X_{\rm CR}$; Its inverse $X_{\rm CR}^{-1}$ is the Fisher information matrix. That is, $X \geq X_{\rm CR}$ for any unbiased estimation algorithm. Furthermore the Gaussian distribution implies that the Cramér-Rao bound is achievable asymptotically. Our motivation is to minimize $X_{\rm CR}$ by choosing $\{\theta_k\}$ correctly, which will yield the optimal geometries of the UAVs under the assumption of fixed equal distance between the GMT and each UAV. However minimization of $X_{\rm CR}$ is a daunting task. Instead we opt to minimize ${\rm Tr}\{X_{\rm CR}\}$, that is more feasible. We notice that for any unbiased estimate $\hat{\chi}$, there holds

$$\begin{aligned} {\rm E}\{\|\chi-\hat{\chi}\|^2\} &= {\rm Tr}\{{\rm E}[(\chi-\hat{\chi})(\chi-\hat{\chi})^T]\} \\ &= {\rm Tr}\{X\} \geq {\rm Tr}\{X_{\rm CR}\}. \end{aligned}$$

Consequently minimization of ${\rm Tr}\{X_{\rm CR}\}$ makes more sense in searching for the optimal geometries of the UAVs, which will be studied in later sections.

3. THE FISHER INFORMATION MATRIX

In this section we compute the Fisher information matrix, associated with Cramér-Rao bound based on observation $\psi = \mathcal{C}\chi + \mathcal{N}$, by assuming that the distances from UAVs to the GMT are all the same, and ω_k is large enough such that each UAV is inside the observable cone for which $f(\omega_k) \neq 0$, and thus $C_k \neq 0$ for $1 \leq k \leq m$. Let n $(= 4)$ be the dimension of χ. According to the Slepian-Bangs formula [7] (page 289), the Fisher information matrix is given by

$$\begin{aligned} X_{\rm CR}^{-1} &= -{\rm E}\left\{\frac{\partial^2 p(\psi;\chi)}{\partial\chi\partial\chi^T}\right\} = \left[\frac{1}{2}{\rm Tr}\left\{\Sigma^{-1}\frac{d\Sigma}{d\chi_i}\Sigma^{-1}\frac{d\Sigma}{d\chi_j}\right\}\right. \\ &\quad \left. + \left(\frac{d\mathcal{C}\chi}{d\chi_i}\right)^T \Sigma^{-1}\left(\frac{d\mathcal{C}\chi}{d\chi_j}\right)\right]_{i,j=1,1}^{n,n} \end{aligned}$$

where $\Sigma = {\rm diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_m)$, and χ_k is the kth element of χ. Because of the diagonal structure, we can compute each diagonal block separately.

The equal distance assumption for all the UAV to the GMT implies that $\|\delta\vec{p}_k\| = \|\delta\vec{p}\|$, and

$$\omega_k = \omega_T, \qquad \sigma_{\omega_k} = \sigma_\omega, \qquad \sigma_{a_k} = \sigma_a, \qquad \sigma_{r_k} = \sigma_r,$$

where $1 \leq k \leq m$. For the case of single UAV ($m = 1$), $\Sigma = {\rm diag}(\sigma_\omega^2, \Sigma_\theta)$, and

$$\Sigma_\theta = U(\theta){\rm diag}(\sigma_r^2, \sigma_a^2)U(\theta)^T, \tag{11}$$

$$\Sigma_\theta^{-1} = U(\theta){\rm diag}(\sigma_r^{-2}, \sigma_a^{-2})U(\theta)^T. \tag{12}$$

Direct computation gives

$$\frac{d\Sigma_\theta}{d\theta} = \left(\sigma_a^2 - \sigma_r^2\right)\begin{bmatrix} \sin(2\theta) & -\cos(2\theta) \\ -\cos(2\theta) & \sin(2\theta) \end{bmatrix},$$

and $\frac{d\sigma_\omega^2}{d\chi_i} = 0 \;\forall\; i$ in which we have assumed that σ_ω^2 is independent of χ. Since θ is a function of $(\chi_1, \chi_2) = (x_T, y_T)$, but not a function of $(\chi_3, \chi_4) = (\dot{x}_T, \dot{y}_T)$,

$$\begin{aligned} -\frac{d\theta}{d\chi_1} &= \frac{(y_T - y)}{(x_T-x)^2 + (y_T-y)^2} = \frac{\sin(\theta)}{\|\delta\vec{p}\|}, \qquad \frac{d\theta}{d\chi_3} = 0, \\ \frac{d\theta}{d\chi_2} &= \frac{(x_T - x)}{(x_T-x)^2 + (y_T-y)^2} = \frac{\cos(\theta)}{\|\delta\vec{p}\|}, \qquad \frac{d\theta}{d\chi_4} = 0, \end{aligned}$$

in light of $\tan(\theta) = (y_T - y)/(x_T - x)$. Denote

$$\Gamma_\theta = \begin{bmatrix} \sigma_r^{-1} & 0 \\ 0 & \sigma_a^{-1} \end{bmatrix} U(\theta)^T \frac{d\Sigma_\theta}{d\theta} U(\theta) \begin{bmatrix} \sigma_r^{-1} & 0 \\ 0 & \sigma_a^{-1} \end{bmatrix}.$$

By the property of trace, and the expression of Σ_θ, we have, in the case of multiple UAV,

$${\rm Tr}\left\{\Sigma^{-1}\frac{d\Sigma}{d\chi_i}\Sigma^{-1}\frac{d\Sigma}{d\chi_j}\right\} = \sum_{\mu=1}^{m} {\rm Tr}\left\{\Gamma_{\theta_\mu}^2\right\}\frac{d\theta_\mu}{d\chi_i}\frac{d\theta_\mu}{d\chi_j}.$$

We choose to skip the complex expression of ${\rm Tr}\left\{\Gamma_\theta^2\right\}$, and of the derivatives of ψ with respect to χ_i.

We notice that by $\omega_T = \omega_T(\theta) = \chi_3\cos(\theta) + \chi_4\sin(\theta)$,

$$[\chi_4\cos(\theta) - \chi_3\sin(\theta)]^2 + [\chi_3\cos(\theta) + \chi_4\sin(\theta)]^2 = \chi_3^2 + \chi_4^2.$$

It is appropriate to define $\omega_{T_\perp}(\theta) = \chi_4\cos(\theta) - \chi_3\sin(\theta)$, which is the orthogonal projection of $\vec{v}_T$ to the vector that is perpendicular to $\delta\vec{p}$, while ω_T is the orthogonal projection of $\vec{v}_T$ to the vector $\delta\vec{p}$. Hence $\omega_T(\theta)^2 + \omega_{T_\perp}(\theta)^2 = \|\vec{v}_T\|^2$. Let

$$\theta_{\rm sc}^{(k)} = \begin{bmatrix} -\sin(\theta_k) \\ \cos(\theta_k) \end{bmatrix}, \quad \theta_{\rm cs}^{(k)} = \begin{bmatrix} \cos(\theta_k) \\ \sin(\theta_k) \end{bmatrix}^T.$$

Then we obtain the expression for the Fisher information matrix associated with the Cramér-Rao bound:

$$\begin{aligned} X_{\rm CR}^{-1} &= \frac{1}{2\|\delta\vec{p}\|^2}\sum_{k=1}^{m}{\rm Tr}\left\{\Gamma_{\theta_k}^2\right\}\begin{bmatrix} \theta_{\rm sc}^{(k)}[\theta_{\rm sc}^{(k)}]^T & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \sum_{k=1}^{m}\begin{bmatrix} \Sigma_{\theta_k}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \sum_{k=1}^{m}\frac{\sigma_\omega^{-2}}{\|\delta\vec{p}\|^2}\begin{bmatrix} \omega_{T_\perp}(\theta_k)\theta_{\rm sc}^{(k)} \\ \|\delta\vec{p}\|\theta_{\rm cs}^{(k)} \end{bmatrix}\begin{bmatrix} \omega_{T_\perp}(\theta_k)\theta_{\rm sc}^{(k)} \\ \|\delta\vec{p}\|\theta_{\rm cs}^{(k)} \end{bmatrix}^T \end{aligned}$$

Remark 3.1: It is worth to pointing out that $\omega_{T_\perp} \leq \|\vec{v}_T\|$, which is the driving speed of the GMT. Hence $\omega_{T_\perp} << \|\delta\vec{p}\|$ holds. For this reason, the Fisher information matrix corresponding to the Cramér-Rao bound has an approximate block diagonal form as $X_{\rm CR} \approx {\rm diag}\left(X_{\rm CR}^{(\rm v)}, X_{\rm CR}^{(\rm p)}\right)$ where $X_{\rm CR}^{(\rm v)}$ and $X_{\rm CR}^{(\rm p)}$ are the Cramér-Rao bound matrices for estimation of velocity and position, respectively. That is, the Cramér-Rao bound matrices for velocity and position are decoupled, and given respectively by

$$\begin{aligned} X_{\rm CR}^{(\rm v)} &= \left(\sigma_\omega^{-2}\textstyle\sum_{k=1}^{m}\theta_{\rm cs}^{(k)}[\theta_{\rm cs}^{(k)}]^T\right)^{-1}, \\ X_{\rm CR}^{(\rm p)} &\approx \left(\textstyle\sum_{k=1}^{m}\Sigma_{\theta_k}^{-1}\right)^{-1} \end{aligned}$$

The approximate expression for $X_{\rm CR}^{(\rm p)}$ is due to $\|\delta\vec{p}\| >> {\rm Tr}\left\{\Gamma_{\theta_k}\right\}$ for $1 \leq k \leq m$. ∎

In the following we consider the Cramér-Rao bounds for estimation of position and velocity separately, in light of Remark 3.1. For velocity only, the Fisher information matrix has an inverse as

$$X_{\rm CR}^{(\rm v)} = \sigma_\omega^2\begin{bmatrix} \|\underline{c}\|^2 & <\underline{c},\underline{s}> \\ <\underline{c},\underline{s}> & \|\underline{s}\|^2 \end{bmatrix}^{-1}$$

where $\underline{s} = \begin{bmatrix} s_1 & \cdots & s_m \end{bmatrix}^T$ and $\underline{c} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix}^T$ with $s_k = \sin(\theta_k)$ and $c_k = \cos(\theta_k)$. It can be verified that

$${\rm Tr}\left(X_{\rm CR}^{(\rm v)}\right) = \frac{m\sigma_\omega^2}{\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2}. \tag{13}$$

Clearly the above is minimized, if and only if its denominator is maximized over $\{\theta_k\}_{k=1}^m$, which determine the geometries of the m UAV sensors. For position estimation only, the minimal achievable error variance based on the Cramér-Rao bound is given approximately by

$$\mathrm{Tr}\left\{X_{\mathrm{CR}}^{(\mathrm{p})}\right\} \approx \frac{m(\sigma_r^{-2}+\sigma_a^{-2})}{\mathrm{den}} \tag{14}$$

where with $\delta\sigma^2 = (\sigma_r^{-2}-\sigma_a^{-2})^2$, den is given by

$$\mathrm{den} = m^2\sigma_r^{-2}\sigma_a^{-2} + \delta\sigma^2\left(\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2\right).$$

We come across with $(\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2)$ once more. Hence for $m \geq 2$,

$$\begin{aligned}\mathrm{Tr}\{X_{\mathrm{CR}}\} &= \mathrm{Tr}\left\{X_{\mathrm{CR}}^{(\mathrm{v})}\right\} + \mathrm{Tr}\left\{X_{\mathrm{CR}}^{(\mathrm{p})}\right\} \\ &\approx \frac{m\sigma_\omega^2}{\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2} + \frac{m(\sigma_r^{-2}+\sigma_a^{-2})}{\mathrm{den}}\end{aligned} \tag{15}$$

The approximation is due to the large value of the distance from the UAV sensors to the GMT, which holds true. The next result follows.

Proposition 3.2: Suppose that the m UAVs have equal distance $\|\delta\vec{p}\|$ to the GMT with $\|\delta\vec{p}\|$ sufficiently large compared with azimuth and range variances, and the velocity of the GMT. Let $\{\theta_k\}_{k=1}^m$, which specify the geometries of the m UAV sensors. Then minimization of the achievable minimum error variance $\mathrm{Tr}\{X_{\mathrm{CR}}\}$ over all possible unbiased estimation algorithms is equivalent to maximization of $\left(\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2\right)$ over the sensor geometries $\{\theta_k\}_{k=1}^m$, provided that all m UAV sensors are within the observable cone.

The above proposition states that optimal cooperative sensing is equivalent to the mathematical problem of maximization of $\left(\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2\right)$, which will be investigated in the next section. Before ending this section, we would like to study this maximization issue for small values of m as follows. In the case of $m = 2$,

$$\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2 = \sin^2(\theta_1-\theta_2),$$

leading to the orthogonal geometry $(\theta_1-\theta_2) = 90^o$ easily for the optimal solution, Thus

$$\mathrm{Tr}\,(X_{\mathrm{CR}}) \geq 2\sigma_\omega^2 + \frac{2}{\sigma_r^{-2}+\sigma_a^{-2}}.$$

The right hand side is attained, if the observable cone has an angle no smaller than 90^o.

In the case of $m = 3$, it can be shown that

$$\begin{aligned}&\max_{\theta_k}\left\{\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2\right\} \\ &= \begin{cases} 2.25, & \overline{\Delta} \geq 120^o, \\ 2\sin^2(\overline{\Delta}), & \overline{\Delta} \leq 90^o, \\ 2+2\cos(\overline{\Delta})\cos^2(\overline{\Delta}/2), & 90^o \leq \overline{\Delta} \leq 120^o \end{cases}\end{aligned}$$

where $\overline{\Delta}$ is the angle of the observable cone. In the general case of $m \geq 2$, we have

$$\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2 = \sum_{k=2}^{m}\sum_{i=1}^{k-1}\sin^2(\theta_k-\theta_i).$$

For optimal cooperative sensing, we need maximize the above summation, which will be studied in next section.

4. Optimal UAV Sensing Geometries

As shown in the previous section that the minimization of the minimum achievable error variance among all unbiased estimation algorithms is equivalent to maximization of $(\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2)$ over the UAV sensor geometries $\{\theta_k\}$. This problem will be solved in this section. For convenience, we denote $\Delta_{k,i} = \theta_k - \theta_i$, and $\overline{\Delta}$ as the angle of the observable cone. There holds

$$\|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2 = \sum_{k>i=1}^{m}\sin^2(\Delta_{k,i}). \tag{16}$$

Based on the analysis in the previous section for $m = 2$, and $m = 3$, it is expected that the optimal UAV sensing geometries are easier to determine, if the observable cone has an angle $\overline{\Delta}$ no greater than 90^o. This is indeed true, as stated in the next theorem.

Theorem 4.1: Denote $g_m(\theta_1,\theta_2,\cdots,\theta_m) = \|\underline{c}\|^2\|\underline{s}\|^2 - |<\underline{c},\underline{s}>|^2$. For $\overline{\Delta} \leq \pi/2$, we have

$$\max_{\theta_1,\cdots,\theta_m} g_m(\theta_1,\cdots,\theta_m) = \begin{cases} \frac{m^2-1}{4}\sin^2(\overline{\Delta}), & \text{odd } m \\ \frac{m^2}{4}\sin^2(\overline{\Delta}), & \text{even } m \end{cases} \tag{17}$$

Proof: Consider first the odd m. If we take $\Delta_{k,i} = \overline{\Delta}$ for $1 \leq i \leq \mu$, and $\mu+1 \leq k \leq m$ where $\mu = (m-1)/2$ or $\mu = (m+1)/2$, then

$$g_m(\theta_1,\theta_2,\cdots,\theta_m) = \frac{m^2-1}{4}\sin^2(\overline{\Delta}).$$

It is claimed that the above is the maximum of $g_m(\cdot,\cdots,\cdot)$. We use induction. The claim is true for $m = 3$, as shown in the previous section. For $m > 3$ odd,

$$g_m(\theta_1,\cdots,\theta_m) = g_{m-2}(\theta_2,\cdots,\theta_{m-1}) + h(\theta_1,\cdots,\theta_m)$$

where $h(\theta_1,\theta_2,\cdots,\theta_m)$ has the expression

$$h(\cdots) = \sin^2(\Delta_{m,1}) + \sum_{i=2}^{m-1}[\sin^2(\Delta_{m,i}) + \sin^2(\Delta_{i,1})]. \tag{18}$$

For $1 < i < m$, $\Delta_{m,i} + \Delta_{i,1} = \Delta_{m,1}$ and $\Delta_{m,i} - \Delta_{i,1} = \Delta_{m,1} - 2\Delta_{i,1}$. Thus with $d_i(\Delta_{i,1}) = \sin^2(\Delta_{m,i}) + \sin^2(\Delta_{i,1})$,

$$d_i(\Delta_{i,1}) = 1 - \cos(\Delta_{m,1})\cos(\Delta_{m,1} - 2\Delta_{i,1}).$$

Without loss of generality $\Delta_{m,1} \geq 0$ is assumed, which is equivalent to $\theta_m \geq \theta_1$. Since $\Delta_{m,1} \leq \overline{\Delta} \leq \pi/2$, $d_i(\Delta_{i,1})$ is maximized by taking $\Delta_{i,1} = 0$ and $\Delta_{m,1} = \overline{\Delta}$, leading to

$$\begin{aligned}\max_{\Delta_{i,1}} d_i(\Delta_{i,1}) = \sin^2(\Delta_{m,1}) &\leq \sin^2(\overline{\Delta}) \\ \Longrightarrow \max_{\theta_1,\cdots,\theta_m} h(\theta_1,\cdots,\theta_m) &= (m-1)\sin^2(\overline{\Delta}).\end{aligned}$$

By induction assume that $g_{m-2}(\theta_2,\theta_3,\cdots,\theta_{m-1})$ is maximized by taking $\Delta_{k,i} = \overline{\Delta}$ for $2 \leq i \leq \mu$, and $\mu+1 \leq k \leq m-1$ where $\mu = (m-1)/2$ or $\mu = (m+1)/2$ with maximum $[(m-2)^2-1]\sin^2(\overline{\Delta})/4$. With θ_1 and θ_m added in, we obtain

$$\begin{aligned}\max g_m(\theta_1,\theta_2,\cdots,\theta_m) &\leq \max g_{m-2}(\theta_2,\cdots,\theta_{m-1}) \\ &\quad + \max h(\theta_1,\theta_2,\cdots,\theta_m) \\ &= \frac{m^2-1}{4}\sin^2(\overline{\Delta}),\end{aligned}$$

which is indeed achieved. For the case of even m, we assign $\{\theta_k\}_{k=1}^{m/2}$ on one edge of the cone, and $\{\theta_k\}_{k=\frac{m}{2}+1}^{m}$ on the other edge of the cone, leading to

$$g_m(\theta_1,\theta_2,\cdots,\theta_m) = \left(\frac{m}{2}\right)^2\sin^2(\overline{\Delta}) = \frac{m^2}{4}\sin^2(\overline{\Delta}).$$

The same induction procedure as earlier can be used to prove that the above is the maximum, thereby proving (17) for the case of even m, which is omitted. ■

Theorem 4.1 favors clearly the orthogonal assignment for optimal geometries of the UAV sensors, if the observable cone has an angle no smaller than 90^o. Otherwise it favors to assign roughly half of the UAV sensors on one edge of the cone, and the other half or so on the other edge of the cone. While the result in Theorem 4.1 is intuitively appealing, the induction procedure in the proof is not extendible to the case of $\overline{\Delta} > 90^o$. A different solution approach is necessary.

In light of Theorem 4.1, $m^2/4$ seems to be an upper bound for $g_m(\theta_1, \cdots, \theta_m)$, if $\overline{\Delta} \geq 90^o$, which is true for the case of even m. The next result is more general.

Lemma 4.2: Suppose that $\theta_k \in [0,\ \pi)$ are arbitrary for $k = 1, 2, \cdots, n$. Then there holds

$$g_m(\theta_1, \theta_2, \cdots, \theta_m) = \sum_{k>i=1}^{m} \sin^2(\theta_k - \theta_i) \leq \frac{m^2}{4}.$$

In addition the above upper bound inequality is tight.

Proof: Let $\phi_k = 2\theta_k$ for all k. Then

$$\begin{aligned} 2g_m(\theta_1, \theta_2, \cdots, \theta_m) &= \sum_{k=1}^{m}\sum_{i=1}^{m} \sin^2(\theta_k - \theta_i) \quad (19) \\ &= \frac{m^2}{2} - \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{m} \cos(\phi_k - \phi_i). \end{aligned}$$

It is claimed that the following lower bound

$$\sum_{k=1}^{n}\sum_{i=1}^{n} \cos(\phi_k - \phi_i) \geq 0 \quad (20)$$

holds, and is tight. Indeed we have

$$\begin{aligned} 0 &\leq \left(\sum_{k=1}^{n} e^{j\phi_k}\right)^* \left(\sum_{k=1}^{n} e^{j\phi_k}\right) \\ &= n + \sum_{k=2}^{n}\sum_{\ell=1}^{k-1} \left(e^{j(\phi_k - \phi_{k-\ell})} + e^{-j(\phi_k - \phi_{k-i})}\right) \\ &= n + 2\sum_{k=2}^{n}\sum_{i=1}^{k-1} \cos(\phi_k - \phi_i) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} \cos(\phi_k - \phi_i) \end{aligned}$$

which concludes the proof of (20). Furthermore by taking $\phi_k = 2k\pi/n$ for all k, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{n} \cos(\phi_k - \phi_i) = 0,$$

that shows that the lower bound in (20) is tight. It follows from (19) that $g_m(\theta_1, \theta_2, \cdots, \theta_m) \leq m^2/4$, which is tight, by taking $\theta_k = k\pi/n$, in light of (20) again. ■

The upper bound established in Lemma 4.2 assumes that $\{\theta_k\}$ are arbitrary, considering that $\sin^2(\cdot)$ is periodical with period π. In our case, $|\Delta_{k,i}| = |\theta_k - \theta_i|$ are restricted to be within $\overline{\Delta}$, which can be strictly and considerably smaller than π. Hence $m^2/4$ may not be achievable as shown in Theorem 4.1 in the case of odd m, even if $\overline{\Delta} = \pi/2$, that is rather different. However in the case of even m, we obtain the following rather easily in light of Lemma 4.2.

Corollary 4.3: For $\overline{\Delta} \geq \pi/2$ and even m, $g_m(\theta_1, \theta_2, \cdots, \theta_m) = \|\underline{c}\|^2\|\underline{s}\|^2 - | < \underline{c}, \underline{s} > |^2 \leq m^2/4$, which is attained.

The above result follows easily from the fact that $m^2/4$ is an upper bound, that is achieved for the case $\overline{\Delta} = \pi/2$ and even m.

Remark 4.4: There are more than one assignment of $\{\theta_k\}$ in achieving the upper bound $m^2/4$, if $\overline{\Delta} > \pi/2$ and m is even. Indeed by grouping $\{\theta_k\}_{k=1}^{m}$ into $m/2$ pairs, with each pair satisfying $\Delta_{k,i} = \pi/2$, the upper bound $m^2/4$ can be achieved, regardless of the location of each pair so long as they are within the observable cone, which is illustrated in Fig. 2 with $m = 4$. This fact has its root in the necessary condition of optimality. Setting its partial derivatives with respective to $\{\theta_k\}$ to zero yields

$$\begin{aligned} \frac{\partial g_m}{\partial \theta_k} = \frac{\partial g_{m-2}}{\partial \theta_k} - 2\cos(\Delta_{m,1})\sin(\Delta_{m,k} - \Delta_{k,1}) &= 0, \\ \frac{\partial g_m}{\partial \theta_1} = \sin(2\Delta_{m,1})\left[1 + \sum_{k=2}^{m-1}\cos(\Delta_{m,k} - \Delta_{k,1})\right] &= 0, \\ \frac{\partial g_m}{\partial \theta_m} = \cos(\Delta_{m,1})\sum_{k=2}^{m-1}\sin(\Delta_{m,k} - \Delta_{k,1}) &= 0. \end{aligned}$$

Hence with $\{\theta_k\}_{k=2}^{m-1}$ optimally assigned, then the enforcement of $\Delta_{m,1} = \theta_m - \theta_1 = 90^o$ will satisfy the above necessary condition, regardless of the values of θ_1 and θ_m, that happen to be globally optimal for even m when $\overline{\Delta} \geq \pi/2$. This is advantageous in applications, because the geometries of the first $(m-2)$ UAVs need not be changed, if an additional pair of UAVs is added in. ■

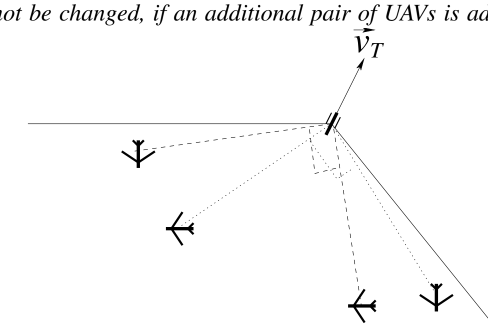

Fig. 2 Optimum sensing geometry for $m = 4$

The difficult case is clearly odd m, for which $g_m(\theta_1, \theta_2, \cdots, \theta_m)$ has the maximum $(m^2-1)/4$, if $\overline{\Delta} = \pi/2$ as shown in Theorem 4.1. On the other hand, when $\overline{\Delta}$ is close to π, $g_m(\theta_1, \theta_2, \cdots, \theta_m)$ has $m^2/4$ as the maximum, by Lemma 4.2. The following result is constructive.

Theorem 4.5: Let $g_m(\theta_1, \theta_2, \cdots, \theta_m)$ be as in Theorem 4.1 with m odd. If $\overline{\Delta} \geq 2\pi/3$, then

$$\max_{\theta_1, \cdots, \theta_m} g_m(\theta_1, \theta_2, \cdots, \theta_m) = m^2/4.$$

Proof: We need only show that $m^2/4$ is achievable, because it is an upper bound by Lemma 4.2. For $m = 3$, the result from the previous section can be used to show that

$$\max_{\theta_1, \theta_2, \theta+3} \sum_{k>i=1}^{m=3} \sin^2(\theta_k - \theta_i) = 3\sin^2(\pi/3) = \frac{3^2}{4},$$

which holds by taking $\Delta_{3,2} = \Delta_{2,1} = \pi/3$ and $\Delta_{3,1} = 2\pi/3$. For $m > 3$ odd, we set $\{\theta_i\}_{i=1}^{m}$ as illustrated in Fig. 3 with $n = (m-1)/2$. It is claimed that $g_m(\theta_1, \cdots, \theta_m) = m^2/4$.

Using the induction consider first $m = 5$. We have

$$\sum_{k>i=1}^{m=5} \sin^2(\theta_k - \theta_i) = 6.25.$$

Thus the result is true for the case $m = 5$. For $m \geq 5$ that is odd, the decomposition in the proof of Theorem 4.1 can be used to arrive at

$$g_m(\theta_1, \theta_2, \cdots, \theta_m) = \frac{(m-2)^2}{4} + h(\theta_1, \theta_2, \cdots, \theta_m),$$

where $g_{m-2}(\theta_2, \cdots, \theta_{m-1}) = (m-2)^2/4$ is assumed by the induction argument, and

$$h(\theta_1, \theta_2, \cdots, \theta_m) = (m-1),$$

by $\Delta_{m,i} + \Delta_{i,1} = \Delta_{m,1} = \pi/2$. It follows that

$$g_m(\theta_1, \theta_2, \cdots, \theta_m) = \frac{(m-2)^2}{4} + (m-1) = \frac{m^2}{4}$$

that concludes the proof. ■

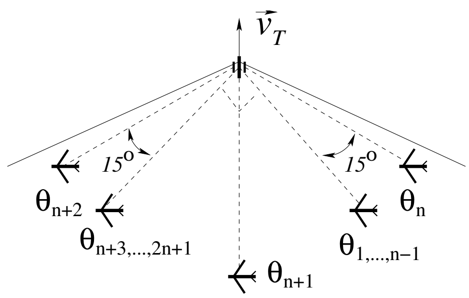

Fig. 3 Optimum sensing geometry for $m = 2n + 1$

Fig. 3 shows that there are $(n-1) = (m-3)/2$ orthogonal pairs. In fact these $(n-1)$ orthogonal pairs need not be in the same location as illustrated in Fig. 3 for which the optimality still holds, that follows from the same argument as in Remark 4.4. For the case $\pi/2 < \overline{\Delta} < 2\pi/3$ and odd m, the situation is more delicate. Nevertheless we have the following result.

Theorem 4.6: Suppose that m is odd and $\varphi_o \in [\pi/4,\ \pi/2)$ satisfies $1 + (m-1)\cos(2\varphi_o) = 0$. Then for $\overline{\Delta} \geq \Delta_o = 2\varphi_o$, there holds

$$\max_{\theta_1, \cdots, \theta_m} g_m(\theta_1, \theta_2, \cdots, \theta_m) = m^2/4$$

Proof: We choose $\{\theta_k\}_{k=1}^m$ such that

$$\theta_{2\ell} - \theta_{2\ell-1} = 2\varphi, \qquad \theta_{2\ell} - \theta_m = \theta_m - \theta_{2\ell-1} = \varphi \tag{21}$$

for $\ell = 1, 2, \cdots, (m-1)/2$. Then it can be verified that

$$g_m(\theta_1, \cdots, \theta_m) = \frac{(m-1)}{4}\left[4\sin^2(\varphi) + (m-1)\sin^2(2\varphi)\right],$$

which is denoted by $w(\varphi)$. Setting

$$\frac{dw(\varphi)}{d\varphi} = (m-1)\left[\sin(2\varphi) + (m-1)\sin(2\varphi)\cos(2\varphi)\right] = 0$$

yields $\sin(2\varphi) = 0$ or $1 + (m-1)\cos(2\varphi) = 0$. We are interested in the case $0 \neq 2\varphi < \pi$, by the fact that $2\varphi \leq \overline{\Delta} < \pi$. Hence $\varphi = \varphi_o$ is the maximal, yielding

$$\begin{aligned} \frac{w(\varphi_o)}{m-1} &= \sin^2(\varphi_o) + \left(\frac{m-1}{4}\right)\sin^2(2\varphi_o) \\ &= \frac{1-\cos(2\varphi_o)}{2} + \left(\frac{m-1}{4}\right)\left(1-\cos^2(2\varphi_o)\right) \\ \Longrightarrow &\quad w(\varphi_o) = \frac{m^2}{4}, \end{aligned}$$

by $\cos(2\varphi_o) = -1/(m-1)$. ■

It is noted that $2\varphi_o = 2\pi/3 = 120^o$ for $m = 3$, that coincide with the result in the previous section, and that in Theorem 4.5. For $m = 5$, we have $2\varphi_o = 104.5^o$. Hence as m increases, $2\varphi_o$ approaches to $\pi/2 = 90^o$, consistent with the result for even m. The next result deals with the case of odd m with $90^o < \overline{\Delta} < 2\varphi_o$, which is the last scenario uncovered.

Corollary 4.7: Suppose that m is odd, and $\varphi_o \in [\pi/4,\ \pi/2)$ satisfies $1 + (m-1)\cos(2\varphi_o) = 0$. If $90^o < \overline{\Delta} < 2\varphi_o < 120^o$, then with $n = (m-1)$ there holds

$$\max_{\theta_1, \cdots, \theta_m} g_m(\theta_1, \cdots, \theta_m) = \frac{n}{4}\left[4\sin^2(\overline{\Delta}/2) + n\sin^2(\overline{\Delta})\right].$$

The proof of this corollary is skipped due to page limit.

5. Conclusion

We studied the cooperative sensing problem for estimation of the position and velocity of the GMT by employing multiple UAV sensors. The measured data are radar signals of azimuth, range, and range rate corrupted by Gaussian noise. Assuming equal distance from UAV sensors to the GMT, the geometries of the UAVs are uniquely determined by their angular positions. We have employed the Cramér-Rao bound to obtain the minimum achievable error variance based on the assumption that the measurement noise is Gaussian to which the Slepian-Bangs formula applies. The minimum achievable error variance is a function of the UAV sensors geometries, that favors orthogonal, or near orthogonal assignment for the UAV sensors. We comment that except in the case of odd m and $2\varphi_o \leq \overline{\Delta} < 120^o$, the optimal UAV geometries admit the property that no change needs be made for the existing UAVs, if an additional pair of UAV sensors is added into the existing ones, that is why induction argument is used in the proof, as commented in Remark 4.4. This is advantageous in applications. However in the case of odd m with $90^o < \overline{\Delta} < 120^o$, this advantage is lost. The geometries of all the UAV sensors need be reconfigured, when new ones are added in. We would like to point out that the optimal cooperative sensing problem does not take the trajectories of the UAV sensors into account, which is currently under study.

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