2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

On the output regulation for TS fuzzy models using sliding modes

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Abstract— **We address the problem of tracking references generated by an exosystem when the plant is described by a Takagi-Sugeno (TS) fuzzy model. In recent years, some authors have tried to solve this problem by constructing the desired fuzzy regulator based on the design of linear local controllers, unfortunately the regulation carried out in this way does not zero the tracking error in general. Therefore, we propose the inclusion of a discontinuous term into the control law. We demonstrate, that under certain conditions, the behavior of the controller is improved. LMI techniques are employed to make the design process more practical.**

Keywords: Fuzzy control, TS fuzzy models, Regulation theory, Discontinuous control, LMI techniques.

I. INTRODUCTION

The tracking of reference signals, at least asymptotically, is a very important area in system theory. In literature, we can find diverse approaches to perform this task. However, regulation theory provides an elegant frame of work to accomplish both, the stability of the closed-loop system if there is no external signals present, and the tracking of references as well. Basically, the regulator problem for a system affected by perturbation and reference signals, both generated by a known dynamical system named the exosystem, consists in finding a state or error feedback controller such that the equilibrium point of the closed-loop system with no external signals is asymptotically stable, and the tracking error goes to zero when the system is under the influence of the exosystem. In [4], the design of the linear regulator was given in terms of certain matrix equations (Francis equation), whose solution, describing steady state mappings, depens on the property of the exosystem signals to be observable for the system output. For the nonlinear regulator, Isidori and Byrnes have shown that the problem is solvable by means of some partial differential equations, named henceforth, Francis-Isidori-Byrnes (FIB) equations [6]. A relative drawback of this approach could be the task of solving these partial differential equations.

Another additional problem is that a rigorous mathematical model may not be available, but only some local behavior could be obtained. For this situation, Takagi and Sugeno proposed a fuzzy model which describes the dynamics of complex systems under the suitable selection of linear subsystems for some predefined conditions of the original system. Many authors have applied these ideas and have extended well-known stability results from linear systems to the nonlinear field. In some works, the stability properties of the TS fuzzy model depend on the existence of a common definite positive matrix [15], [13]. Other papers relax this condition by searching piece-wise quadratic Lyapunov functions (see e.g. [8]) or designing fuzzy observers-based control laws [7].

Recently in [16], an approach to design the fuzzy regulator based on local controllers was given. Nevertheless, as is mentioned in [10] and [3], this technique only works in very particular cases.

In our work, we suggest a way to overcome this problem. Mainly, the addition of a discontinuous term to the overall fuzzy regulator is proposed. Under certain conditions, it is shown that this scheme guarantees the convergence of the output tracking error to zero.

The paper is organized as follows. In section 2 we review the basic results on output regulation. In section 3 the main result is presented while in section 4 some numerical simulations are carried out. Finally, some concluding remarks are given in section 5.

II. BASIC RESULTS ON REGULATION THEORY

Considering the dynamical system

$$\dot{x} = f(x, u, w) \tag{1}$$
$$\dot{w} = s(w) \tag{2}$$
$$e = h(x, w); \tag{3}$$

with $x \in \mathbb{R}^n$, $w \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, and $e \in \mathbb{R}^q$ as the state of the system, the state of exosystem, the input signal and the error signal, respectively; the *State Feedback Output Regulation Problem* **(SORP)** is defined as the problem of maintaining the closed-loop stability when the plant is not affected by the exosystem, and ensuring the reference tracking when the system is under the influence of the exosystem. More precisely, the **SORP**, consists in finding a controller

$$u(t) = \alpha(x, w) \tag{4}$$

such that, the following conditions hold:

RS) The equilibrium point $x = 0$ of the system

$$\dot{x} = f(x, \alpha(x, 0), 0)$$

is asymptotically stable.

Work supported partially by the Mexican Consejo Nacional de Ciencia y Tecnología, CONACYT, México, under grant No. 37687-A.

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RR) The closed-loop system (1), (2) and (4) satisfies

$$\lim_{t\to\infty} e(t) = 0.$$

On the other hand, the linear approximation for the system (1)-(3) around the equilibrium point $(x, w, u) = (0, 0, 0)$ is

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw && (5)\\ \dot{w} &= Sw && (6)\\ e &= Cx + Qw && (7) \end{aligned}$$

with

$$A = \frac{\partial f(x,w,u)}{\partial x}\mid_{(0,0,0)}, \quad B = \frac{\partial f(x,w,u)}{\partial u}\mid_{(0,0,0)},$$

$$P = \frac{\partial f(x,w,u)}{\partial w}\mid_{(0,0,0)} \quad C = \frac{\partial h(x,w)}{\partial x}\mid_{(0,0)},$$

$$S = \frac{\partial s(w)}{\partial w}\mid_{(0)}, \qquad Q = \frac{\partial h(x,w)}{\partial w}\mid_{(0,0)}.$$

Thus, if the pair (A, B) is stabilizable, the solution for the **SORP** depends on the existence of nonlinear mappings $x_{ss} = \pi(w)$ and $u_{ss} = \gamma(w)$ satisfying the FIB equations [2], [5]

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} s(w) &= f(\pi(w), \alpha(\pi(w), w), w) && (8)\\ 0 &= h(\pi(w), w). && (9) \end{aligned}$$

Mappings $x_{ss} = \pi(w)$ and $u_{ss} = \gamma(w)$ represent the steady state zero output submanifold and the steady state input which ensures the invariance of $\pi(w)$, respectively. This turns out in the following controller

$$u = Kx + \gamma(w) - K\pi(w).$$

with K such that $(A + BK)$ is Hurwitz.

For the linear case, equations (8)–(9) become [5], [9]

$$\begin{aligned} \Pi S =& \quad A\Pi + B\Gamma + P\\ 0 =& \quad C\Pi + Q \end{aligned} \qquad (10)$$

and the controller is $u = Kx + (\Gamma - K\Pi)w$.

III. THE FUZZY OUTPUT REGULATOR PROBLEM

As was described above, the problem of output regulation is to find, if possible, a controller performing two tasks: 1) it must stabilize the closed-loop system, and 2) it must provide the steady state input or internal model, such that the system tracks the reference signal. In this section, we propose an approach to construct an output regulator for a TS fuzzy model, which includes a discontinuous term. Under certain conditions, this controller allows us to reduce the steady state error obtained by the method proposed in [16].

Let us consider the TS fuzzy model described by r rules of the form

Plant rule i:

IF $z_(t)$ is M_{1i} and ..... and z_v is M_{vi}

$$\text{THEN} \quad \textstyle\sum_i : \begin{cases} \dot{x} = A_i x + B_i u + P_i w \\ \dot{w} = S_i w \\ e_i = C_i x + Q_i w, i = 1, ..., r \end{cases}$$

where M_{ji} are the fuzzy sets, $z_1, ..., z_v$ are the corresponding premise variables which may coincide with x or w, or even with a combination of these state vectors. And the linear subsystems can be obtained from some knowledge of the process dynamics [13].

In order to simplify this analysis, we assume that the measurable variables include the whole information of both, the plant and the exosystem, avoiding the use of observers.

So, the aggregate fuzzy system, obtained from singleton fuzzifier, product inference and center average defuzzifier [12], [13], [15], is

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} \mu_i A_i x + \sum_{i=1}^{r} \mu_i B_i u + \sum_{i=1}^{r} \mu_i P_i w, && (11)\\ \dot{w} &= \sum_{i=1}^{r} \mu_i S_i w, && (12)\\ e &= \sum_{i=1}^{r} \mu_i \left[C_i x + Q_i w \right], && (13) \end{aligned}$$

with μ_i as the normalized weight for each rule calculated from the membership functions of z_j in M_{ji} satisfying

$$\begin{aligned} \mu_i &= \mu_i(z) \geq 0 \\ \sum_{i=1}^{r} \mu_i &= 1, \qquad z = [z_1, ..., z_v]^T. \end{aligned}$$

Now, we may attempt to construct the fuzzy controller by designing local regulators, i. e., by solving the following equations [16]

$$\begin{aligned} \Pi_i S_i =& \quad A_i \Pi_i + B_i \Gamma_i + P_i \\ 0 =& \quad C_i \Pi_i + Q_i \end{aligned} \qquad (14)$$

for all $i = 1, \ldots, r$. In this case, local controllers take the form

$$u = K_i x + L_i w,$$

with

$$L_i = \Gamma_i - K_i \left(\sum_{j=1}^{r} \mu_j \Pi_j \right),$$

and the overall nonlinear fuzzy controller would be

$$u = \left(\sum_{i=1}^{r} \mu_i K_i \right) x + \left(\sum_{i=1}^{r} \mu_i L_i \right) w. \qquad (15)$$

Unfortunately, this regulator does not guarantee the asymptotical convergence of the error, in general [10], [3]. In fact, if we substitute

$$\begin{aligned} \hat{\pi}(w) &= \left(\sum_{i=1}^{r} \mu_i \Pi_i \right) w, \\ \hat{\gamma}(w) &= \left(\sum_{i=1}^{r} \mu_i \Gamma_i \right) w. \end{aligned}$$

into equations (8)–(9), we obtain

$$\left[\left(\sum_{i=1}^{r}\dot{\mu}_i\Pi_i\right)+\left(\sum_{i=1}^{r}\mu_i\Pi_i\right)\left(\sum_{i=1}^{r}\mu_i S_i\right)\right]w = \left[\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left(A_i\Pi_j+B_i\Gamma_j+P_i\right)\right]w$$

and

$$0 = \left[\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\left(C_i\Pi_j+Q_i\right)\right]w.$$

From the previous equations, it turns out that $\hat{\pi}(w)$ and $\hat{\gamma}(w)$ solve the fuzzy regulation problem, if and only if the following equations hold

$$\begin{aligned} 0 &= \sum_{i=1}^{r}\dot{\mu}_i\Pi_i+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\mu_i\mu_j\left(\Pi_i S_j+\Pi_j S_i\right)\\ &- \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\mu_i\mu_j\left(A_i\Pi_j+B_i\Gamma_j+P_i\right.\\ &+ \left.A_j\Pi_i+B_j\Gamma_i+P_j\right) \qquad (16)\\ 0 &= \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\mu_i\mu_j\left(C_i\Pi_j+Q_i+C_j\Pi_i+Q_j\right). \qquad (17)\end{aligned}$$

Nevertheless, in general, mappings $\hat{\pi}(w)$, $\hat{\gamma}(w)$ are not the exact solution of the FIB equations (8)–(9).

As we may observe, $\sum_{i=1}^{r}\dot{\mu}_i\Pi_i \neq 0$ in most of the cases, therefore control law (15) is not able to take the tracking error to zero. For the interested reader, the particular cases that are solved by means of $\hat{\pi}(w),\hat{\gamma}(w)$ are analyzed in [3].

The previous analysis is our motivation to formulate the fuzzy output regulation problem in terms of finding a sliding mode controller which reduces the tracking error for the overall fuzzy system while the stability property is achieved. The rationale behind this is that we may suppose the existence of some nominal model for which the aggregate control (15) is exactly the equivalent control [14], in this sense, we consider the TS fuzzy system as the disturbed version of such nominal model. The sliding mode technique is then introduced to obtain the desired performance. The suggested switching function is

$$e(t)=\sum_{i=1}^{r}\mu_i C_i x(t)+\sum_{i=1}^{r}\mu_i Q_i w(t).$$

The rules for the fuzzy regulator have the form
Controller rule i:

IF $z_1(t)$ is M_{1i} and ..... and $z_p(t)$ is M_{pi}

THEN

$$u(t)=K_i(x(t)-\Pi_i w(t))+\Gamma_i w(t),$$

and the final controller is

$$u=u_{eq}+v(e) \qquad (18)$$

where

$$u_{eq}=\left(\sum_{i=1}^{r}\mu_i K_i\right)x+\left(\sum_{i=1}^{r}\mu_i\Gamma_i-\sum_{i,j=1}^{r}\mu_i\mu_j K_i\Pi_j\right)w$$

is the controller proposed in [16], and

$$v(e)=Gsign(e)$$

is the additional discontinuous term.

Thus, the *Fuzzy Output Regulator Problem with Sliding Modes* (**FORPSM**) can be defined as the problem of finding a set of triplets (K_i,Π_i,Γ_i) for $i=1,..,r$ and G such that the following conditions hold:

FRS) The equilibrium point $(x,w)=(0,0)$ of the system

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{r}\mu_i A_i x(t)+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_i K_j x(t)\\ &+ Gsign(e)\end{aligned}$$

is asymptotically stable.

FRR) The solution of the closed-loop system (11)–(12)–(18) satisfies

$$\lim_{t\to\infty} e(t)=0.$$

A. Mathematical analysis

The following result states the conditions for the existence of such a controller.

Theorem 1: If matrices S_i are neutrally stable for all $i=1,\dots,r$ and

FH1) the pairs (A_i,B_i) are stabilizable for all $i=1,\dots,r$,

FH2) there exist matrices Π_i and Γ_i solving

$$\begin{aligned}\Pi_i S_i &= A_i\Pi_i+B_i\Gamma_i+P_i \qquad (19)\\ 0 &= C_i\Pi_i+Q_i \qquad (20)\end{aligned}$$

for all $i=1,\dots,r$,

FH3) there exists matrices K_i and P such that

$$N_{ii}^T\mathbf{P}+\mathbf{P}N_{ii}<0$$

for $i=1,...,r$ and

$$\left(\frac{N_{ij}+N_{ji}}{2}\right)^T\mathbf{P}+\mathbf{P}\left(\frac{N_{ij}+N_{ji}}{2}\right)<0$$

for all $i,j=1,\dots,r$ satisfying $\mu_i\mu_j\neq 0$ with

$$N_{ij}=\left(A_i+B_iK_j\right), \qquad (21)$$

FH4) there exist four real numbers $\alpha_1>0$, $\alpha_2>0$, $\alpha_3>0$ and G such that $-\alpha_1<G<0$ and $G<-\frac{\alpha_2}{\alpha_3}$

then the **FORPSM** is solvable. Moreover, the controller has the form (18).

Proof: The neutral stability of S_i guarantees that the reference signal neither decays to zero nor tends to infinity.

For the stability of the overall fuzzy system, we make $w = 0$, i. e.

$$e = \sum_{i=1}^{r} \mu_i C_i x,$$

and we rewrite the closed-loop system (11), (18) as

$$\dot{x}(t) = \sum_{i,j=1}^{r} \mu_i \mu_j N_{ij} x + \sum_{i=1}^{r} \mu_i B_i G sign(e). \quad (22)$$

Now, considering the Lyapunov function

$$V = x^T \mathbf{P} x$$

we have

$$\begin{aligned} \dot{V} &= x^T \mathbf{Q} x + 2x^T P \sum_{i=1}^{r} \mu_i B_i G sign(e) \\ &\leq x^T \mathbf{Q} x + x^T P \sum_{i=1}^{r} \mu_i B_i \sum_{i=1}^{r} \mu_i B_i^T P x \\ &+ G^2 sign(e)^T sign(e), \end{aligned}$$

with

$$\mathbf{Q} = \left(\sum_{i,j=1}^{r} \mu_i \mu_j N_{ij}^T \mathbf{P} + \mathbf{P} \sum_{i,j=1}^{r} \mu_i \mu_j N_{ij} \right).$$

By FH3) we know $x^T \mathbf{Q} x < 0$, thus $\dot{V} < 0$ when

$$\left(\|\mathbf{Q}\| - \|\mathbf{P}\|^2 \left(\sum_{i=1}^{r} \|B_i\| \right)^2 \right) \|x\|^2 - G^2 q > 0$$

where q is the dimension of the output signal.

We observe that the equilibrium point $(x, w) = (0, 0)$ is asymptotically stable if

$$|G| < \alpha_1$$

where $\alpha_1 \equiv \sqrt{\frac{\|\mathbf{Q}\| - \|\mathbf{P}\|^2 \left(\sum_{i=1}^{r} \|B_i\| \right)^2 \|x\|^2}{q}} > 0$.

If the latter square-root has no real solution the recalculation of matrices K_i and $\mathbf{P}$ would be needed.

For the regulation condition we consider the steady state error given by

$$\tilde{x} = x - \pi(w)$$

so that, the tracking error (13) becomes into

$$e = \sum_{i=1}^{r} \mu_i C_i \tilde{x} + \sum_{i=1}^{r} \mu_i C_i \pi(w) + Qw.$$

On the other hand, from regulation theory we know that the problem has a solution if and only if

$$\frac{\partial \pi}{\partial w} s(w) = \sum_{i=1}^{r} \mu_i A_i \pi(w) + \sum_{i=1}^{r} \mu_i B_i \gamma(w) + \sum_{i=1}^{r} \mu_i P_i w \quad (23)$$

$$0 = \sum_{i=1}^{r} \mu_i C_i \pi(w) + Qw, \quad (24)$$

thus we got

$$e = \sum_{i=1}^{r} \mu_i C_i \tilde{x}$$

whose derivative is

$$\begin{aligned} \dot{e} &= \sum_{i=1}^{r} \dot{\mu}_i C_i \tilde{x} + \sum_{i=1}^{r} \mu_i C_i \dot{\tilde{x}} \\ &= \sum_{i=1}^{r} \dot{\mu}_i C_i \tilde{x} + \sum_{i=1}^{r} \mu_i C_i \left(\sum_{j=1}^{r} \mu_j A_j x + \sum_{j=1}^{r} \mu_j B_j u \right. \\ &+ \left. \sum_{j=1}^{r} \mu_j P_j w - \frac{\partial \pi}{\partial w} s(w) \right). \end{aligned}$$

Adding and subtracting $\gamma(w)$ in (18) and by (23) we obtain

$$\begin{aligned} \dot{e} &= \sum_{i=1}^{r} \mu_i C_i \left(\sum_{j=1}^{r} \mu_j A_j + \sum_{j=1}^{r} \mu_j B_j \sum_{k=1}^{r} \mu_k K_k \right) \tilde{x} \\ &+ \sum_{i=1}^{r} \mu_i C_i \left\{ \sum_{j=1}^{r} \mu_j B_j \left[\sum_{k=1}^{r} \mu_k K_k \pi(w) \right. \right. \\ &- \sum_{k=1}^{r} \mu_k K_k \left(\sum_{\ell=1}^{r} \mu_\ell \Pi_\ell \right) w + \sum_{k=1}^{r} \mu_k \Gamma_k w \\ &- \left. \left. \gamma(w) + v(e) \right] \right\} + \sum_{i=1}^{r} \dot{\mu}_i C_i \tilde{x}. \end{aligned}$$

Now, to show that the tracking error is asymptotically stable we consider the Lyapunov function $V = \frac{1}{2} e^T e$ and its derivative:

$$\begin{aligned} \dot{V} &= e^T \dot{e} \\ &= e^T \sum_{i=1}^{r} \dot{\mu}_i C_i \tilde{x} + e^T \sum_{i=1}^{r} \mu_i C_i \left(\sum_{j=1}^{r} \mu_j A_j \right. \\ &+ \left. \sum_{j,k=1}^{r} \mu_j \mu_k B_j K_k \right) \tilde{x} + e^T \sum_{i=1}^{r} \mu_i C_i \left\{ \sum_{j=1}^{r} \mu_j B_j \right. \\ &\times \left[\sum_{k=1}^{r} \mu_k K_k \pi(w) - \sum_{k=1}^{r} \mu_k K_k \left(\sum_{\ell=1}^{r} \mu_\ell \Pi_\ell \right) w \right. \\ &+ \left. \left. \sum_{k=1}^{r} \mu_k \Gamma_k w - \gamma(w) + G sign(e) \right] \right\} \\ &\leq \|e\| \|M_1\| + \|e\| \|M_2\| + \|e\| \|M_3\| + \|e\| \|M_4\| G, \end{aligned}$$

with

$$M_1 = \sum_{i=1}^{r} \dot{\mu}_i C_i \tilde{x}$$

$$M_2 = \sum_{i=1}^{r} \mu_i C_i \left(\sum_{j=1}^{r} \mu_j A_j + \sum_{j,k=1}^{r} \mu_j \mu_k B_j K_k \right) \tilde{x}$$

$$\begin{aligned} M_3 &= \sum_{i=1}^{r} \mu_i C_i \left\{ \sum_{j=1}^{r} \mu_j B_j \left[\sum_{k=1}^{r} \mu_k K_k \pi(w) \right. \right. \\ &- \sum_{k=1}^{r} \mu_k K_k \left(\sum_{\ell=1}^{r} \mu_\ell \Pi_\ell \right) w + \sum_{k=1}^{r} \mu_k \Gamma_k w \\ &- \left. \left. \gamma(w) \right] \right\} \end{aligned}$$

$$M_4 = \sum_{i,j=1}^{r} \mu_i \mu_j C_i B_j$$

So, in order to ensure the asymptotical stability of the error, we need to satisfy

$$G < -\frac{\alpha_2}{\alpha_3}$$

where $\alpha_2 \equiv (\|M_1\| + \|M_2\| + \|M_3\|) > 0$ and $\alpha_3 \equiv \|M_4\| > 0$.

Finally, we can conclude that the fuzzy regulation problem is solvable by means of a discontinuous term if

$$-\alpha_1 < G < 0 \tag{25}$$

and

$$G < -\frac{\alpha_2}{\alpha_3} \tag{26}$$

■

are satisfied.

Notice that the previous analysis agrees with Isidori's result given in [5] since we are assuming the existence of the exact solution for (23)–(24), i. e., $\pi(w)$, and $\gamma(w)$.

B. LMI formulation

The more significative drawback of the latter result is the recalculation of matrices K_i and $\mathbf{P}$ in order to satisfy (25) and (26). Therefore, LMI design is included to provide a numerical way to minimize the tracking error while the stability property is kept.

Next, we introduce suitable LMIs to substitute conditions FH1) and FH3) in Theorem 1. For more details about LMIs, the reader is referred to [1], where a complete analysis of LMIs in control theory is presented.

It can be proved that assumption FH1) is satisfied and at the same time the existence of $\alpha_1 > 0 \in \mathbb{R}$ is guaranteed when the following LMIs are feasible

$$0 > \mathcal{Q}_1 A_i^T + X_i^T B_i^T + A_i \mathcal{Q}_1 + B_i X_i + (\lambda^2 + \beta) I \tag{27}$$

for $i = 1 \ldots r$, where $\mathcal{Q}_1$ and X_i are the unknowns with $\lambda = \sum_{i=1}^{r} \|B_i\|$, $X_i = K_i \mathcal{Q}_1$ and $\mathcal{Q}_1 > 0$. The real number $\beta \geq 0$ is a parameter that may be changed during the design process in order to obtain different values for α_1. These equations ensure the stability for each subsystem, however according with the PDC structure presented in [13], we must also guarantee the stability in the interpolation regions, i. e., we need to satisfy FH3). This can be done by means of

$$\begin{aligned} 0 &> \mathcal{Q}_1 A_i^T + X_j^T B_i^T + \mathcal{Q}_1 A_j^T + X_i^T B_j^T \\ &+ A_i \mathcal{Q}_1 + B_i X_j + A_j \mathcal{Q}_1 + B_j X_i + 2(\lambda^2 + \beta) I \end{aligned} \tag{28}$$

for $i = 1 \ldots r-1$ and $i < j \leq r$. As before, the existence of $\alpha_1 > 0 \in \mathbb{R}$ is guaranteed, $\mathcal{Q}_1$ and X_i are the unknowns with $X_i = K_i \mathcal{Q}_1$ and $\mathcal{Q}_1 > 0$, and the common matrix $\mathbf{P}$ is $\mathcal{Q}_1^{-1}$ [13].

At this point, from (27) and (28), we can easily observe that there exist some G that does not affect the stability property. Besides, in order to allow G's with bigger norm, we only need to increase the value of β.

IV. An Illustrative Example

Let us consider the fuzzy system (11)-(12)-(13) of two rules presented in [16] with

$$A_i = \begin{pmatrix} 0 & 1 \\ a_i & 0 \end{pmatrix}, B_i = \begin{pmatrix} 0 \\ b_i \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \end{pmatrix},$$
$$S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $a_1 = -\frac{Mgl}{Ml^2+I}$, $a_2 = \frac{2}{\pi} a_1$ $b_1 = \frac{1}{Ml^2+I}$, $b_2 = \alpha B_1$, $g = 9.81 m/s^2$, $M = 20kg$, $l = 0.5m$, $I = 0.8kg - m^2$, $\alpha = 2.5$ and membership functions

$$\mu_1 [x_1(t)] = \left[1 - \frac{1}{1 + e^{-7(x_1 - \pi/4)}} \right] \left[\frac{1}{1 + e^{-7(x_1 + \pi/4)}} \right],$$
$$\mu_2 [x_1(t)] = 1 - \mu_1 [x_1(t)].$$

The fuzzy mappings are

$$\hat{\pi}(w) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{29}$$

$$\hat{\gamma}(w) = \begin{pmatrix} 0 & \eta w_2 \end{pmatrix}. \tag{30}$$

with

$$\eta = \mu_1(w_2) \frac{1+a}{b} + \mu_2(w_2) \frac{1 + 2a/\pi}{\alpha b}.$$

We can notice $\hat{\pi}(w)$ and $\hat{\gamma}(w)$ do not solve the fuzzy regulation problem since they do not coincide with the exact solution

$$\pi(w) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$\gamma(w) = \begin{pmatrix} 0 & \frac{1 + a\mu_1(w_2) + 2a\mu_2(w_2)/\pi}{\mu_1 b + \mu_2 \alpha b} w_2 \end{pmatrix},$$

which are clearly different from (29)–(30) when $\alpha \neq 1$. However, discontinuous control will compensate this difference.

Using Matlab LMI toolbox we obtain
$K_1 = \begin{pmatrix} -2.43 & -5.19 \end{pmatrix}$ and $K_2 = \begin{pmatrix} -15 & -2.33 \end{pmatrix}$.

The simulation results are given in figures 1 and 2. Figure 1 shows the behavior of the plant under the action of the

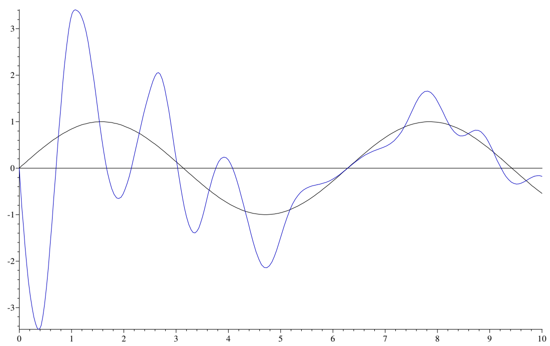

Fig. 1. Reference vs. output without discontinuous term.

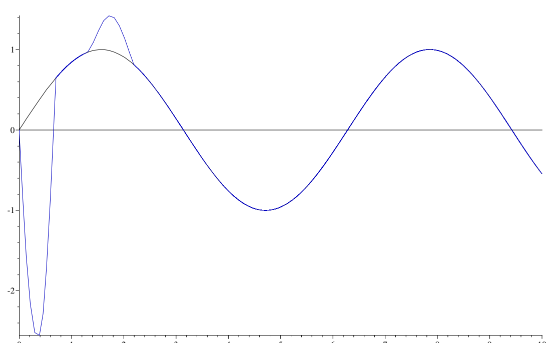

Fig. 2. Reference vs. output with discontinuous term.

fuzzy controller without discontinuous control. Although the tracking error in this case is bounded, it does not decays to zero. Figure 2 shows the results when discontinuous control is included in the regulator, in this case the value for G is -20. As may be observed, the performance of the proposed scheme suggests its validity.

Remark 2: Of course, we can solve the tracking problem using discontinuous control only. However to achieve similar results to those obtained in this example, we need to increase dramatically the absolute value of the sliding mode gain ($G = -40$). A controller of this type demands more effort from the actuator. This could be an inconvenience for some practical cases. On the other hand, when we combine the regulation theory with sliding modes the resulting control signal is smoother.

V. CONCLUSIONS

In this paper we have presented a regulation scheme for nonlinear systems using a combination of regulation theory, Takagi-Sugeno fuzzy models and discontinuous control. Based on the existence of local regulators, the overall tracking error is reduced or even taken to zero by means of a discontinuous term. Under certain conditions, this scheme performs asymptotical tracking for the TS fuzzy model and a bounded-error output tracking when it is applied to the original system, the magnitude of this bound depends on the approximation of the fuzzy model with respect to the original nonlinear system. In that sense, this approach can be viewed as an alternative to the classical nonlinear regulator.

Also, conditions for the existence of the fuzzy regulator are given in a numerical form, which allow us to obtain the controller in a practical way.

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