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Robust tracking for a class of nonlinear systems using multiple models and switching

Mehmet Akar

Abstract— **This paper presents a multiple model/controller scheme for robust tracking of a class of nonlinear systems in the presence of large plant uncertainties and disturbance. Each model is associated with a sliding mode controller, and a switching logic is designed to pick the model that best approximates the plant at each instant. Theoretically, it is shown that the proposed control scheme achieves perfect tracking despite the existence of disturbance, whereas simulation results verify the improvement in the transient performance in terms of reduced control input, chattering and oscillations.**

I. INTRODUCTION

Sliding mode control is an effective way of controlling uncertain nonlinear systems, because the technique not only stabilizes the system but also provides disturbance rejection and low sensitivity to plant parameter variations [1]. One drawback that has limited the practical applicability of this technique is the chattering phenomenon. Another is the necessity to apply large control inputs to the system in the presence of large plant uncertainties.

The objective of this paper is to reduce these undesired effects by employing a multiple model/controller (MMC) architecture. MMC methods have been previously used in adaptive control of both linear [2], [3], [4], [5], [6], and nonlinear [7], [8], [9] systems. In MMC design, a set of models is identified, one of which is close to the plant in some sense. Each model has its own controller, and depending on the identification errors, the controller for the closest model is chosen, and its output is applied not only to the plant but also to the identification models. What distinguishes our MMC scheme from others [2]–[9] is the consideration of a class of nonlinear systems in the presence of disturbance. The models are chosen so that they cover the allowable parameter space. The controllers are designed using sliding mode techniques. A similar approach was taken by Corradini *et al.* in [7], [8]. However, their work does not take disturbance into account, and moreover, the model states as constructed in [7], [8] may diverge, which we demonstrate in this paper using a simple example.

The rest of the paper is organized as follows. In Section II, the robust tracking problem is stated, and the motivation for the paper is discussed. In Section III, a solution to the tracking problem is proposed using a single sliding mode controller and a single model. Results of Section III are extended to multiple models and controllers in Section IV. Finally, Section V concludes the paper.

The author was with the Communication Sciences Institute, USC, Los Angeles, CA, USA. He is currently visiting the Hamilton Institute, NUI, Maynooth, Ireland. Email: Mehmet.Akar@nuim.ie

II. PROBLEM STATEMENT

Consider the nonlinear continuous–time plant

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(x,\theta) + b(x)(u+d) \end{aligned}\right\} \qquad (1)$$

where $x = [x_1, x_2, \dots, x_n]^T \in \Re^n$ is the measurable state vector, $u \in \Re$ is the control input, $d \in \Re$ is the unmeasurable disturbance, and $\theta \in \Re^p$ is the vector of uncertain parameters. We have the following assumption on the system description in (1).

Assumption 1: (i) $b(x)$ is nonzero for all $x \in \Re^n$.
(ii) The disturbance term is bounded, i.e., $|d(x,t)| < d_0$, for some known positive constant d_0 [1].

Given the above assumptions, the objective of the present paper is to determine a controller for the system in (1) so that its state tracks the state of the reference system

$$\left.\begin{aligned} \dot{x}_{r,1} &= x_{r,2} \\ \dot{x}_{r,2} &= x_{r,3} \\ &\vdots \\ \dot{x}_{r,n} &= k^T x_r + r(t) \end{aligned}\right\} \qquad (2)$$

where $x_r = [x_{r,1}, x_{r,2}, \dots, x_{r,n}]^T \in \Re^n$ is the desired state vector, $r(t) \in \Re$ is a piecewise continuous reference input, and $k \in \Re^n$ is a vector of constants.

A simplified version of the system (1) was considered in [9] with the additional assumptions $d(t) = 0$ (i.e., no disturbance term), $b(x) = 1$, and $f(x,\theta) = \bar{f}^T(x)\theta$, (i.e., $f(x,\theta)$ is linear in θ), and an indirect adaptive controller was designed as $u = k^T x + r - \bar{f}^T(x)\hat{\theta}$, where $\hat{\theta}$ is the estimate of the unknown parameter θ that can be computed adaptively using

$$\dot{\hat{\theta}} = -\bar{f}(x) B^T P (\hat{x} - x)^T. \qquad (3)$$

In (3), $\hat{x}$ is the state variable of the model $\dot{\hat{x}} = A_m \hat{x} + Br$, and P is the solution of the Lyapunov equation $A_m^T P + P A_m = -I$, for the matrix A_m in companion form with the last row consisting of k^T and $B = [0, 0, \dots, 0, 1]^T$.

In order to examine the performance of this adaptive controller numerically, as in [9], we let $f(x,\theta) = [x_2^2, x_1 x_2^2, x_3^3]\theta$, where the initial value of the unknown parameter θ is taken to be $\hat{\theta}(0) = [0, 0, 0]^T$. The other

[1]This assumption can be relaxed by letting d_0 to be a function of the state x.

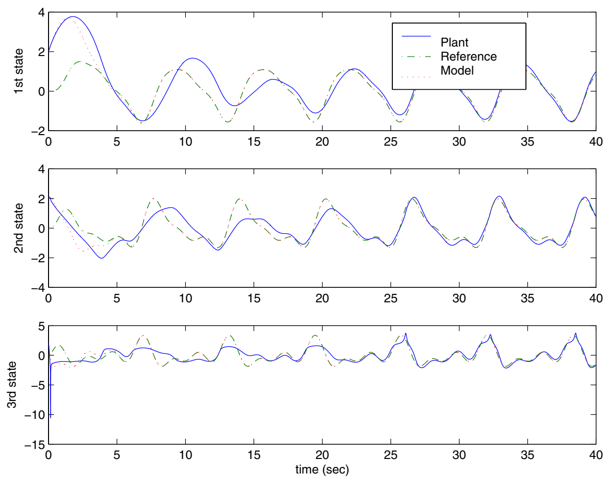

Fig. 1. State variables using a single model and an adaptive controller (no disturbance)

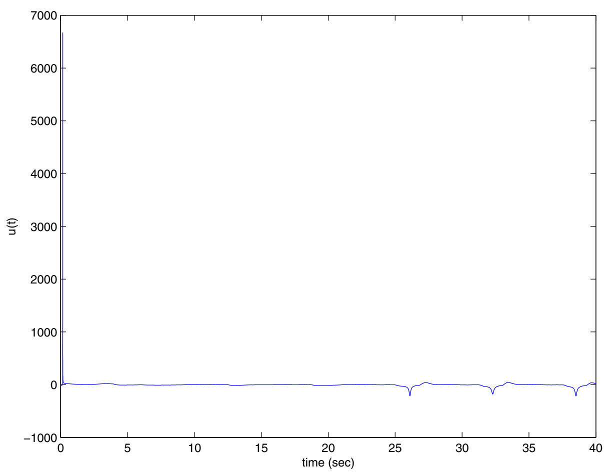

Fig. 2. Input to the plant when using a single model and an adaptive controller (no disturbance)

parameters that are used in the simulation are $k = [-3,-5,-4]^T$, $x(0) = \hat{x}(0) = [2,2,2]^T$, $x_r(0) = [0,0,0]^T$ and $\theta = [2,-3,4]^T$. The reference input is $r(t) = 5\sin(t) + 4\sin(2t) + 4\sin(3t)$.

Figs. 1 and 2 depict the state and the input values respectively, when $d(t) = 0$ (no disturbance). Although the model states track the reference states fairly quickly, a large transient error exists between the states of the reference and those of the plant. Furthermore, as can be noted from Fig. 2, a large control effort is needed initially. We further examined the performance of this controller when the plant is subject to the periodic disturbance

$$d(t) = \sin(4t) + cos(5t). \tag{4}$$

Figs. 3 and 4 summarize the simulation results. Aside that the controlled system will never be able to achieve perfect tracking, we note that a larger control effort is needed initially (compare Figs. 2 and 4).

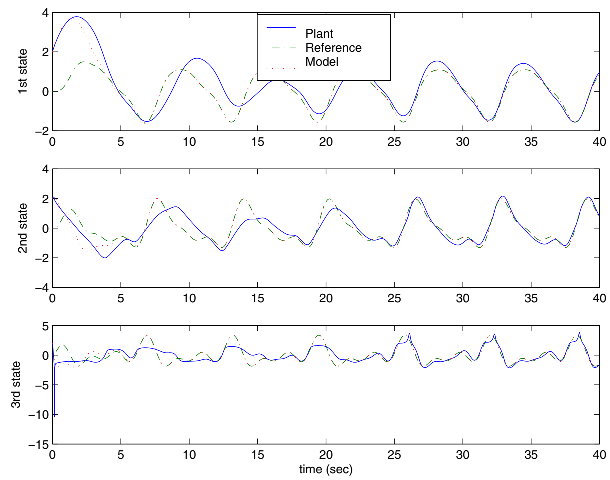

Fig. 3. State variables using a single model and an adaptive controller ($d(t)$ as in (4))

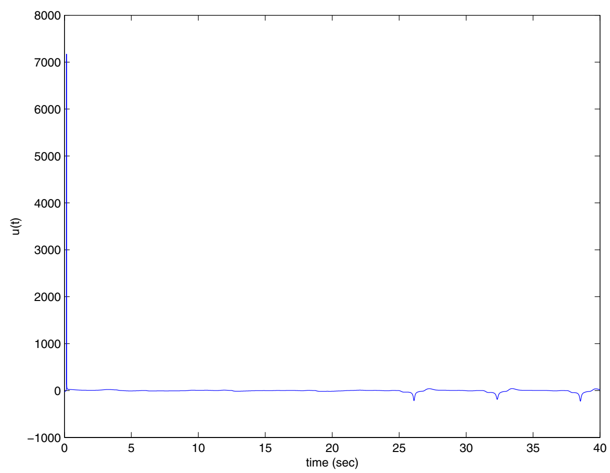

Fig. 4. Input to the plant when using a single model and an adaptive controller ($d(t)$ as in (4))

One method to overcome the disturbance effect is to use a sliding mode controller [1]. With known bounds on the parameter θ, we can design a sliding mode controller to solve the problem discussed in this paper [1]. However, as we demonstrate in the sequel, this controller suffers from chattering and the necessity to apply large control input values in the presence of large plant uncertainties. With the motivation to reduce these undesired effects, Corradini *et al.* recently proposed a multiple model/controller structure [7], [8]. In their proposed scheme, the inputs to the identification models and the plant itself are the same. However, there is no guarantee that their intelligent switching scheme will function properly in general.

To illustrate this fact, consider the same example above. Although reference tracking is not studied in [7], [8], their scheme can be easily modified to incorporate it. As in [9], let the unknown parameter, θ, lie in the compact set $[-2,2] \times [-3,3] \times [-4,4]$. Following [8] and assuming a

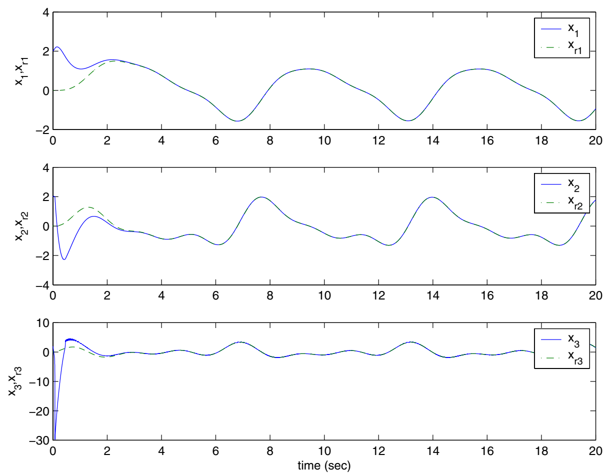

Fig. 5. Plant and reference states when using a single model and the approach in [8].

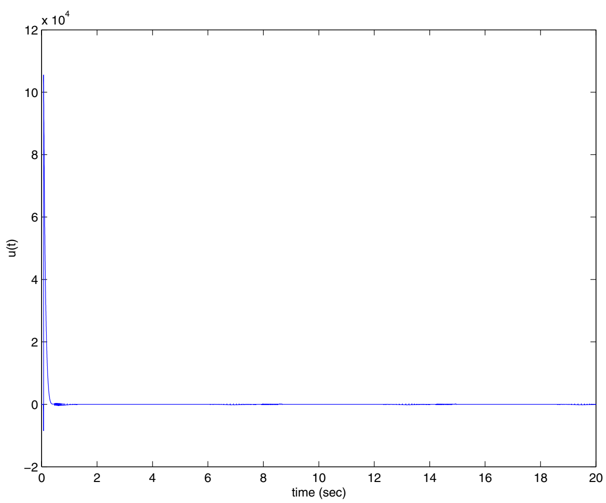

Fig. 6. Plant input when using a single model and the approach in [8].

single model at $\theta = [0, 0, 0]^T$, one can design the sliding mode controller

$$u = -6e_{c,2} - 5e_{c,3} + k^T x_r + r \\ -(d_0 + 2x_2^2 + 3|x_1|x_2^2 + 4|x_3^3|)\mathrm{sgn}(s), \quad (5)$$

where $e_c = x - x_r = [e_{c,1}, e_{c,2}, e_{c,3}]^T$ is the control error; $s = 6e_{c,1} + 5e_{c,2} + e_{c,3} = 0$ is the sliding manifold, and $d_0 = 2$ is the upper-bound for the disturbance term in (4). Since the maximum value of $d(t)$ is approximately equal to 1.97, this further implies that $\dot{s}s < -0.02|s|$, i.e., sliding motion exists on $s = 0$ [1]. Indeed, as shown in Fig. 5, the sliding mode controller in (5) ensures that the plant states track those of the reference system, albeit a large initial error and the necessity to apply large inputs as shown in Fig 6 (despite an unmeasurable disturbance term).

Now, let us examine the behavior of a single model that is constructed as in [8]. Following [8], this single model is

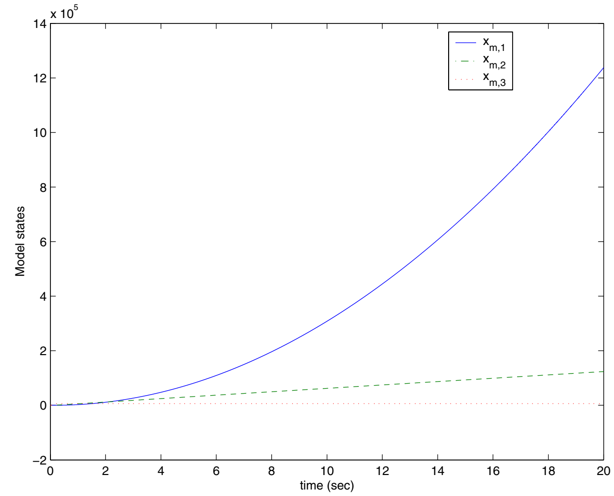

Fig. 7. Model states when using a single model and the approach in [8].

characterized by (for $\theta_m = [0, 0, 0]^T$)

$$\left.\begin{aligned} \dot{x}_{m,1} &= x_{m,2} \\ \dot{x}_{m,2} &= x_{m,3} \\ \dot{x}_{m,3} &= f(x_m, \theta_m) + u = u \end{aligned}\right\} \quad (6)$$

where $x_{(m)} = [x_{m,1}, x_{m,2}, x_{m,3}]^T$ is the model state, and u is the model input as described by (5). Let $\tilde{x} = x_{(m)} - x$ denote the identification error. Since there is a single model, we expect $\tilde{x}$ to converge to the origin. From (1) and (6), the dynamics for the identification error is described by

$$\left.\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \tilde{x}_3 \\ \dot{\tilde{x}}_3 &= -f(x, \theta) - d \end{aligned}\right\}. \quad (7)$$

There is no guarantee that the dynamics in (7) will be stable in general. In fact, by taking $f(x, \theta) = 0$ and $d = 1$ it is easy to note theoretically that it is going to be *unstable*. Indeed, as shown in Fig. 7, the states of the model diverge (for the set of parameters used in earlier simulations). This implies that the intelligent scheme proposed in [8] cannot be reliably used in a multiple model extension.

III. Sliding Mode Control Using a Single Model

In order to extend the multiple model/controller approach in robust tracking of nonlinear control systems, the problem is viewed as an indirect control problem in the sequel with a single model. Let this single model be described by

$$\left.\begin{aligned} \dot{x}_{m,1} &= x_{m,2} \\ \dot{x}_{m,2} &= x_{m,3} \\ &\vdots \\ \dot{x}_{m,n} &= f(x, \theta_m) + b(x)v \end{aligned}\right\} \quad (8)$$

where $x_{(m)} = [x_{m,1}, x_{m,2}, \ldots, x_{m,n}]^T \in \Re^n$ is the state, $v \in \Re$ is the control input, and $\theta_m \in \Re^p$ is the estimate of the parameter for the model. Let $\tilde{x} = x_{(m)} - x$ denote the identification error. Our MMC approach consists of applying different inputs to the plant and the model.

In particular, the model input v is chosen as

$$v = \left(r - f(x, \theta_m) + k^T x_{(m)}\right) / b(x) \tag{9}$$

to ensure that the model tracks the reference. On the other hand, the input to the plant is a sliding mode controller

$$\begin{aligned} u = & \, v + d_0 \text{sgn}(s) \text{sgn}(b(x)) \\ & + \left[\sum_{i=1}^{n-1} c_i \tilde{x}_{i+1} + |f(x, \theta) - f(x, \theta_m)| \text{sgn}(s) \right] / b(x), \end{aligned} \tag{10}$$

where $s = \sum_{i=1}^{n-1} c_i \tilde{x}_i + \tilde{x}_n$, and the coefficients c_i, $i = 1, \ldots, n-1$ are chosen so that the polynomial

$$q^{n-1} + c_{n-1} q^{n-2} + \ldots + c_2 q + c_1 = 0, \tag{11}$$

has roots in the left half plane. Then, we have the following result.

Theorem 1: Under Assumption 1, if we use the controller in (9) for the model in (8) and the sliding mode controller (10) for the plant in (1), then we have the following:

(i) The model tracks the reference system, i.e., $x_{(m)}(t) \to x_r(t)$ as $t \to \infty$.

(ii) The sliding modes are achieved on the manifold $s = 0$, and the plant model tracks the reference system, i.e., $x(t) \to x_r(t)$ as $t \to \infty$.

Proof: Using (9) in (8) leads to $\dot{x}_{(m)} = A_m x_{(m)} + Br$, which can be combined with (2) to obtain

$$\dot{x}_{(m)} - \dot{x}_r = A_m \left(x_{(m)} - x_r\right).$$

Hence, $x_{(m)}(t) \to x_r(t)$ as $t \to \infty$, which establishes (i) in Theorem 1. For the second part, after some algebra, we have $s\dot{s} \leq -\eta|s|$, for some positive constant η given by $\eta = d_0 - \sup_t d(t)$. This implies the existence of sliding modes on the manifold $s = 0$. Furthermore, since the polynomial in (11) has roots in the left half plane, the plant model tracks the reference system, i.e., $x(t) \to x_r(t)$ as $t \to \infty$. (QED) $\square$

We reconsider the example system in the previous section. Figs. 8 and 9 depict the simulation results. Using the control laws proposed in this section, we note that not only the plant but also its model is stabilized despite the existence of the disturbance term. However, as seen from Fig. 9, the control input to the plant takes large values, and also results in a significant amount of chattering and oscillations. The chattering can be reduced using the boundary layer method [10] and higher order sliding mode control [11], [12]. In the sequel, we study how the multiple model/controller scheme can be employed to further improve the transient performance of the controlled system, by reducing chattering, the magnitude of control input values, and oscillations.

IV. Multiple Model and Controller Approach

The general architecture of our MMC scheme is shown in Fig. 10. In our case, the identification models are described

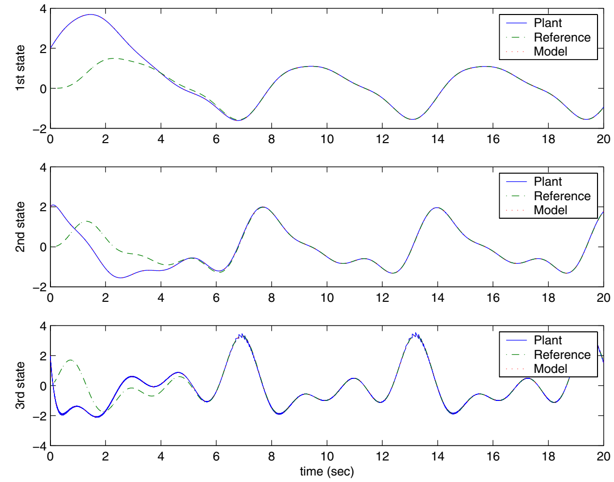

Fig. 8. State variables using a single model and a sliding mode controller ($d(t)$ as in (4))

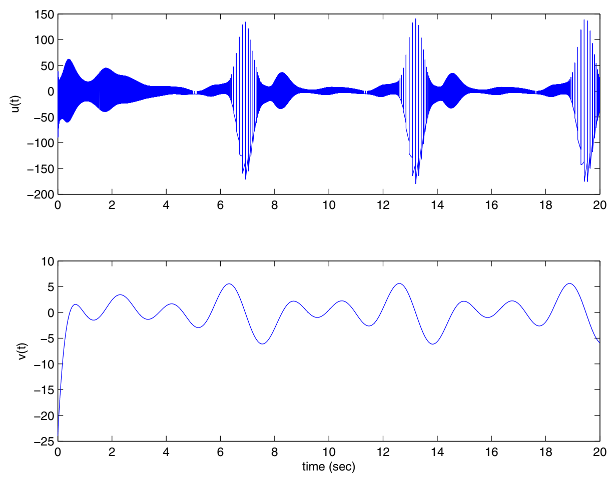

Fig. 9. Input variables when using a single model and a sliding mode controller ($d(t)$ as in (4))

by the same form given in (8), and can be represented as

$$\left.\begin{aligned} \dot{x}_{m,1} &= x_{m,2} \\ \dot{x}_{m,2} &= x_{m,3} \\ &\vdots \\ \dot{x}_{m,n} &= f(x, \theta_m) + b(x) v \end{aligned}\right\} \tag{12}$$

where $x_{(m)} = [x_{m,1}, x_{m,2}, \ldots, x_{m,n}]^T \in \Re^n$ is the state, $v \in \Re$ is the control input, and $\theta_m \in \Re^p$ is the estimate of the parameter for the m–th model, $m \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$. These M models are usually chosen so that they are uniformly distributed in the parameter space, and at least one of them with the corresponding controller stabilizes the plant. Note that the disturbance term cannot be included in the identification model description. Because of this constraint, an alternate approach to MMC design is necessary in contrast to previously proposed [2]–[9]. Before we proceed any further, we state the following assumption.

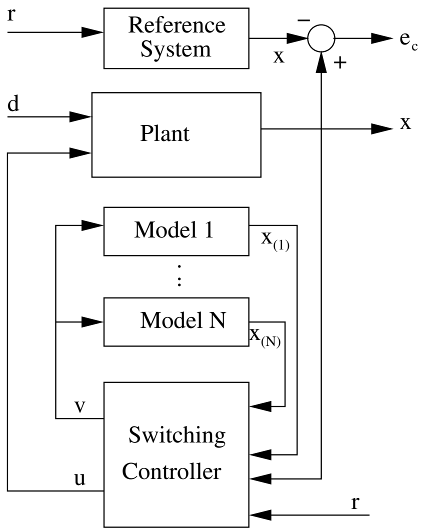

Fig. 10. MMC architecture

Assumption 2: There exists at least one model/controller pair which achieves the objectives in Theorem 1.

Remark 1: Assumption 2 can be full–filled as follows. Given the bounds on the parameter vector θ, the parameter space is partitioned into M non-overlapping regions. Although non–uniform and overlapping partitions are also possible so long as they cover up the whole parameter space, it is advantageous to have uniformly partitioned regions to maximize the benefits of the MMC approach, and to minimize unnecessary computations. Once the partitions are determined, the models can be placed at the center of each partition. Note that in the case of non-overlapping covering of the parameter space, there is exactly one model/controller pair that achieves the objectives in Theorem 1. Hence Assumption 2 is satisfied.

The following algorithm summarizes our MMC scheme.

MMC Algorithm

S1. (Initialization) Set $N = 1$, $\mathcal{M}_u = \{1\}$, and $t_s = 0$ [2].

S2. Compute the control v to be applied to the models as

$$v = \left(r(t) - f(x, \theta_N) + k^T x_{(N)}\right) / b(x). \tag{13}$$

S3. Compute the plant input u from

$$u = v + d_0 \text{sgn}(s)\text{sgn}(b(x)) + \frac{1}{b(x)} \sum_{i=1}^{n-1} c_i (x_{N,i} - x_i) + \frac{1}{b(x)} |f(x, \theta) - f(x, \theta_N)| \text{sgn}(s), \tag{14}$$

where

$$s = \sum_{i=1}^{n-1} c_i (x_{N,i} - x_i) + x_{N,n} - x_n, \tag{15}$$

and c_i, $i = 1, 2, \ldots, n-1$, are constant coefficients chosen such that the polynomial in (11) has roots in the left half plane.

S4. Compute the next state and the value of s.

S5. If $|s|$ is non–increasing, go to **S2**. Otherwise, choose the model which best approximates the plant using

$$N = \arg \min_{i \in \mathcal{M} \setminus \mathcal{M}_u} J(x, x_{(i)}), \tag{16}$$

where

$$J(x, x_{(i)}) = \|x_{(i)}(t) - x(t)\|^2 + \alpha \int_{t_s}^{t} \|x_{(i)}(\tau) - x(\tau)\|^2 d\tau, \tag{17}$$

where α is a non–negative design parameter[3]. Let $\mathcal{M}_u = \mathcal{M}_u \cup \{N\}$, $t_s = t$, increment t, and go to step **S2**. □

Theorem 2: The MMC Algorithm converges under Assumptions 1 and 2, and the plant tracks the reference system, i.e., $x(t) \to x_r(t)$ as $t \to \infty$.

Proof: In the MMC algorithm, M models are assumed to cover the parameter space (Assumption 2); therefore it follows from Theorem 1 that there is one model/controller pair which makes the plant to track the reference system. MMC algorithm starts with the first model, $N = 1$ (step S1).

If $N = 1$ is the correct model, then the choice of the controllers in steps S2 and S3 will ensure that the value of s defined in step S4 will be non–increasing (from Theorem 1). Hence, there won't be any switching, and the plant tracks the reference system, i.e., $x(t) \to x_r(t)$ as $t \to \infty$.

If $N = 1$ is not the closest model, then the controllers used in steps S2 and S3 cannot ensure sliding motion, i.e., $|s|$ is increasing, and this will be determined in step S5. Therefore, the model that is currently in use will be discarded, and a new model will be chosen according to (16) using the performance metric given in (17). In the algorithm, $\mathcal{M}_u$ denotes the set of models that have been tried up to the present time; hence cyclic switching is not possible. Therefore, the correct model will be identified in finite time, and sliding modes occur. This implies that $x(t) \to x_r(t)$ as $t \to \infty$. (QED) □

A. Performance evaluation of the MMC algorithm

We consider the same example discussed previously. The compact set $[-2, 2] \times [-3, 3] \times [-4, 4]$, in which the unknown parameter θ is assumed to lie, is partitioned uniformly into 1000 sub–cubes whose centers are the fixed parameters for the identification models. Hence the set of θ_m, $m = 1, \ldots, 1000$, is given by $\{[-2 + 0.2(2i-1), -3 + 0.3(2j-1), -4 + 0.4(2k-1)]^T : i, j, k = 1, \ldots, 10\}$. Note that there is at least one model/controller pair which will ensure that the plant tracks the reference system.

Figs. 11 and 12 depict the simulation results for the case when the actual value of the unknown parameter is $\theta =$

[2] N denotes the model in use at time t; $\mathcal{M}_u$ is the set of models that have been tried up to time t; and t_s is the time instant at which the last switching occurred.

[3] For a detailed discussion on the tradeoffs involved in the choice of α, please see [3]. In this paper, $\alpha = 1$ is used for the simulations.

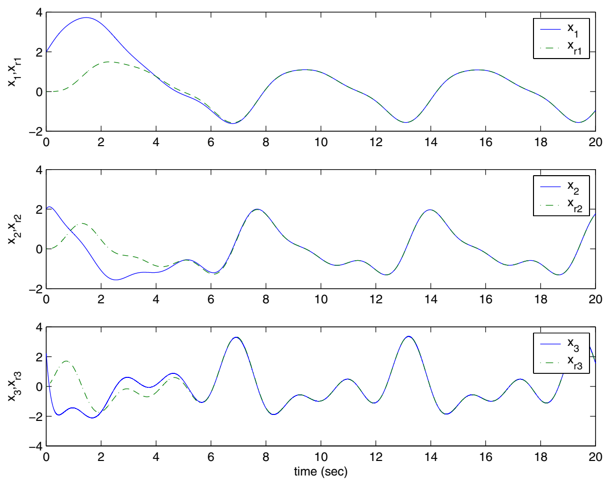

Fig. 11. State variables using the MMC approach

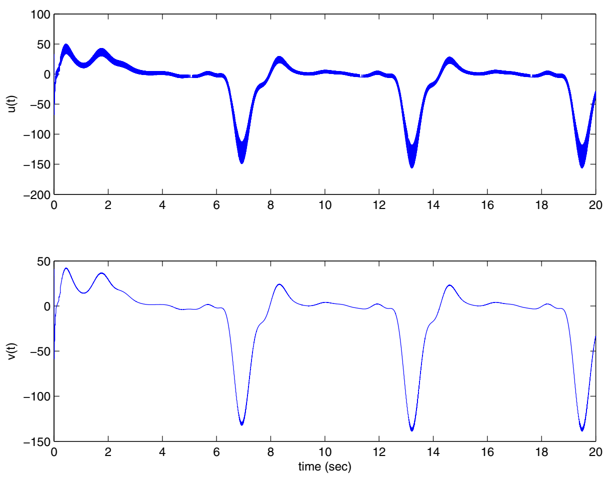

Fig. 12. Input variables using the MMC approach

$[2, -3, 4]^T$. From Figs. 8 and 11, we note that chattering is reduced (in particular, compare the third states). Moreover, from Figs. 9 and 12, we observe that the magnitude of the input to the plant and the magnitude of the oscillations are also reduced substantially. In our extensive simulations, we have also confirmed that the MMC algorithm successfully chose the model closest to the plant.

The reported performance gains by the MMC algorithm can be better understood by examining the term $|f(x, \theta) - f(x, \theta_N)|$ in (14) more closely. With a single model at $\theta_N = [0, 0, 0]^T$, the aforementioned term is equal to

$$|f(x, \theta) - f(x, \theta_N)| = |f(x, \theta)| = |2x_2^2 - 3x_1x_2^2 + 4x_3^3|.$$

With the 1000 models considered above, the closest parameter vector to the actual value of θ is $\theta_N = [1.8, -2.7, 3.6]^T$. Hence, we obtain

$$|f(x, \theta) - f(x, \theta_N)| = |0.2x_2^2 - 0.3x_1x_2^2 + 0.4x_3^3|,$$

which is one-tenth of its value for the single model case. This simple computation demonstrates the source of achievable gains via the MMC algorithm.

V. Conclusion

A switching controller using multiple models and sliding mode controllers has been studied for robust tracking of a class of nonlinear systems with disturbance. Simulation results indicate that the closest model can indeed be identified, and the proposed scheme leads to improvement in performance in terms of reduced control effort, chattering and oscillations. The design procedure studied in this paper can be extended to multi–input nonlinear systems by following the two–step design procedure developed in Section III. Hence the proposed methodology has wide applications in robust control of nonlinear systems under large parametric uncertainties and disturbance.

VI. Acknowledgements

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