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A deterministic approach for general discrete-time Kalman Filter for singular systems

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***Abstract*— In this paper, the Kalman filter and the corresponding Riccati equation for discrete-time, time-variant descriptor systems are addressed in their most general formulation. A new "9-block" form for the optimal filter is derived using deterministic approach. This new expression, besides including one step delayed state, presents an interesting simple and symmetric structure. Stability results for this filter are also presented.**

***Index Terms*— Descriptor systems, discrete-time, Kalman filtering, singular systems, state estimation, Riccati equation, stability.**

I. INTRODUCTION

The problem of optimal recursive estimation for discrete-time descriptor systems is considered in this paper. The study of estimation and control of descriptor systems (also known as singular systems or implicit systems) is motivated by the fact that systems in descriptor formulation frequently arises naturally in economical systems [10], image modelling [5], and robotics [12]. For discrete-time descriptor systems, the state estimation is presented recursively and the resulting generalized Kalman filter has been intensively studied (see e.g. [1], [2], [3], [4], [7], [9], [13], [14], [15], [16]). Different formulations have been proposed in order to deal with this problem (see more details in [6], [7]). In particular, in [7], we addressed the Kalman filtering problem as a deterministic optimal trajectory fitting problem. With this, we were able to solve the robust filtering problem when there are uncertainties in some system matrices [8].

In this paper, the results of [7] on filtered and one-lag smoothed cases are extended in order to consider the most general linear system which is considered a descriptor system (discrete-time and time-variant) where the measured signal can be depending on the current state and one step delayed state. It is also allowed the correlation between the state and measurement noises. In the literature, the direct treatment of this most general case is usually avoided through a previous change of variables. This is a natural procedure since the final expressions of the filter and Riccati equations for the simplified system are already sufficiently complex.

We develop in this paper a new expression for the filter and corresponding Riccati equation. This new expression, besides including one step delayed state, presents an interesting simple and symmetric structure, which opens, for instance, the possibility of simplify the sensitivity analysis of Riccati equations. The approach of this paper is easy to follow and the filter of [14] appears as a particular case.

The predicted and smoothed filters are more involved and were considered more recently by only few works ([7], [13], [16]). A particular generalization for the predicted filter problem is considered by us in [6].

This paper is organized as follows. The deterministic estimation problem is formulated in Section II. The optimal filtered estimate is derived in Section III. In order to guarantee pratical implementation, some considerations on stability are presented in Section IV. A numerical example is given in Section V to demonstrate the applicability of the result.

II. PROBLEM STATEMENT

Consider a set of measured or observed signals $z = \{z_0,\, z_1, ..., z_k\}$ from a certain real dynamical system. The descriptor Kalman filtering in our context is defined as a deterministic fitting problem (over the entire trajectory). An ideal linear dynamical system which would 'explain' the measurements z, in its more general form, is an implicit or descriptor system

$$\begin{aligned} E_{i+1}x_{i+1} &= F_i x_i,\ i = 0, 1, 2, ... \\ z_i &= H_i x_i \end{aligned} \tag{1}$$

where x_i is the descriptor variable or semi-state which describes the internal behavior of the system; E_i, F_i, and H_i are real rectangular matrices of appropriated dimensions.

With the system matrices fixed, for each state sequence candidate $\{x_{0|k},\ x_{1|k}, ..., x_{k|k}, x_{k+1|k}\}$ we can define the following fitting errors

$$\begin{aligned} &\begin{bmatrix} G_{w,i} & G_{v,i+1} \\ K_{w,i} & K_{v,i+1} \end{bmatrix} \begin{bmatrix} w_{i|k} \\ v_{i+1|k} \end{bmatrix} := \\ &\begin{bmatrix} E_{i+1}x_{i+1|k} - F_i x_{i|k} \\ z_{i+1} - H_{i+1}x_{i+1|k} - J_i x_{i|k} \end{bmatrix},\ i = 0, 1, ..., k-1 \\ &p_{0|k} := E_0 x_{0|k} - F_{-1}\bar{x}_0; \\ &K_{v,0}v_{0|k} := z_0 - H_0 x_{0|k} \end{aligned} \tag{2}$$

where the matrices E_0 and F_{-1} are supposed of appropriate dimensions. These matrices can deal with the *a priori* information on the initial state x_0, and usually it is supposed $E_0 = F_{-1} = I$. Now, the deterministic optimal fitting problem is to find a state sequence which minimizes some

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predefined error functional. Once obtained the minimizing sequence $\{\hat{x}_{i|k}\}$, we can define from (2) the corresponding minimum fitting errors $\hat{w}_{i|k}$, $\hat{v}_{i+1|k}$, $\hat{p}_{0|k}$ so that the complete model which 'explains' the set of measured signals $z = \{z_0, z_1, ..., z_k\}$ turns to be

$$\begin{aligned} E_0\hat{x}_{0|k} &= F_{-1}\bar{x}_0 + \hat{p}_{0|k} && (3)\\ E_{i+1}\hat{x}_{i+1|k} &= F_i\hat{x}_{i|k} + G_{w,i}\hat{w}_{i|k} + G_{v,i+1}\hat{v}_{i+1|k} && (4)\\ z_0 &= H_0\hat{x}_{0|k} + K_{v,0}\hat{v}_{0|k} && (5)\\ z_{i+1} &= H_{i+1}\hat{x}_{i+1|k} + J_i\hat{x}_{i|k} + && (6)\\ & \quad K_{w,i}\hat{w}_{i|k} + K_{v,i+1}\hat{v}_{i+1|k}, \\ i &= 0, 1, ..., k-1. \end{aligned}$$

Our fitting problem is to obtain $\widehat{x}_{k|k}$ and $\widehat{x}_{k-1|k}$, i.e., the filtered and smoothed estimates, respectively. The nomenclature 'filtered' is justified by the observation that if $H_k = I$, $J_k = 0$, $K_{w,k} = 0$ and $K_{v,k} = I$ in the model (3)-(7), then we have from (7)

$$z_k = \hat{x}_{k|k} + \hat{v}_{k|k} \tag{7}$$

and so, if the signal $\hat{x}_{k|k}$ was obtained from the actually measured signal z_k, the error signal $\hat{v}_{k|k}$ has been suppressed from z_k.

In the next section we will propose one quadratic functional to obtain the filtered estimate recursion. The intuitive notion of (relative) degree of uncertainty, or how big we allow each error to be, is dealt with the introduction of positive definite weighting matrices. We will suppose known the positive definite weighting matrices Q_j, R_i, P_0 to the errors $w_{j|k}$, $v_{i|k}$, and $p_{0|k}$, respectively, and the weighting cross-term S_i of these errors. As in this paper it is considered the optimal fitting to a given model, we will also suppose known the model parameters E_i, F_i, $G_{w,i}$, $G_{v,i}$, H_i, J_i, $K_{w,i}$, and $K_{v,i}$.

III. The Kalman filter recursion

The deterministic filtered least square fitting problem is to find a sequence $\{\widehat{x}_{0|k}, \widehat{x}_{1|k}, ..., \widehat{x}_{k|k}\}$ which minimizes the following fitting error cost $\mathcal{J}_k\left(\{x_{i|k}\}_{i=0}^k\right)$

$$\mathcal{J}_0(x_{0|0}) := \frac{1}{2}\{\|E_0x_{0|0} - F_{-1}\bar{x}_0\|^2_{P_0^{-1}} + \|v_{0|0}\|^2_{R_0^{-1}}\}, \tag{8}$$

subject to

$$z_0 = H_0x_{0|0} + K_{v,0}v_{0|0} \tag{9}$$

for $k = 0$ and

$$\mathcal{J}_k\left(\{x_{i|k}\}_{i=0}^k\right) := \frac{1}{2}\{\left\|E_0x_{0|k} - F_{-1}\bar{x}_0\right\|^2_{P_0^{-1}} + \left\|v_{0|k}\right\|^2_{R_0^{-1}}$$

$$+\sum_{i=0}^{k-1}\begin{bmatrix} w_{i|k}\\ v_{i+1|k}\end{bmatrix}^T \begin{bmatrix} Q_i & S_i\\ S_i^T & R_{i+1}\end{bmatrix}^{-1}\begin{bmatrix} w_{i|k}\\ v_{i+1|k}\end{bmatrix}\} \tag{10}$$

subject to

$$\begin{aligned} E_{i+1}x_{i+1|k} &= F_ix_{i|k} + G_{w,i}w_{i|k} + G_{v,i+1}v_{i+1|k},\\ z_{i+1} &= H_{i+1}x_{i+1|k} + J_ix_{i|k} + K_{w,i}w_{i|k}\\ &+ K_{v,i+1}v_{i+1|k}, \quad i = 0, ..., k-1 \end{aligned} \tag{11}$$

for $k > 0$, where J_i is the matrix of the delayed term. For each $k \geq 0$, it is easy to show by rewriting (8)-(11) that the original optimization problem is equivalent to the following minimization problem

$$\min_{\mathfrak{X}_{k|k},\mathfrak{V}_{k|k}} \frac{1}{2}\mathfrak{V}_{k|k}^T\mathfrak{R}_k^{-1}\mathfrak{V}_{k|k} \tag{12}$$

subject to

$$\mathfrak{A}_k\mathfrak{X}_{k|k} + \mathfrak{K}_k\mathfrak{V}_{k|k} - \mathfrak{B}_k = 0 \tag{13}$$

where

$$\mathfrak{R}_k = \begin{bmatrix} \mathcal{R}_k & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \mathcal{R}_0 \end{bmatrix}, \mathfrak{K}_k = \begin{bmatrix} \mathcal{G}_k & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \mathcal{G}_0 \end{bmatrix},$$

$$\mathfrak{A}_k = \begin{bmatrix} \mathcal{E}_k & \mathcal{A}_{k-1} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathcal{E}_{k-1} & \mathcal{A}_{k-2} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathcal{E}_{k-2} & \ddots & 0 & 0 & 0\\ 0 & 0 & 0 & \ddots & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{A}_1 & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{E}_1 & \mathcal{A}_0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mathcal{E}_0 & \mathcal{A}_{-1} \end{bmatrix},$$

$$\mathfrak{V}_{k|k} := \begin{bmatrix} \mathcal{V}_{k|k}\\ \mathcal{V}_{k-1|k}\\ \vdots\\ \mathcal{V}_{1|k}\\ \mathcal{V}_{0|k} \end{bmatrix}, \mathfrak{X}_{k|k} := \begin{bmatrix} x_{k|k}\\ x_{k-1|k}\\ \vdots\\ x_{1|k}\\ x_{0|k} \end{bmatrix}, \mathfrak{B}_k = \begin{bmatrix} \mathcal{Z}_k\\ \mathcal{Z}_{k-1}\\ \vdots\\ \mathcal{Z}_1\\ \mathcal{Z}_0 \end{bmatrix},$$

$$\mathcal{V}_{j|k} := \begin{bmatrix} w_{j-1|k}\\ v_{j|k}\end{bmatrix}, \mathcal{E}_j := \begin{bmatrix} -E_j\\ H_j\end{bmatrix}, \mathcal{A}_{j-1} := \begin{bmatrix} F_{j-1}\\ J_{j-1}\end{bmatrix},$$

for $0 \leq j \leq k$,

$$\mathcal{R}_i := \begin{bmatrix} Q_{i-1} & S_{i-1}\\ S_{i-1}^T & R_i\end{bmatrix}, \quad \mathcal{G}_i := \begin{bmatrix} G_{w,i-1} & G_{v,i}\\ K_{w,i-1} & K_{v,i}\end{bmatrix},$$

$\mathcal{Z}_i := \begin{bmatrix} 0\\ z_i\end{bmatrix}$, for $1 \leq i \leq k$ and

$$w_{-1|0} := E_0x_{0|0} - F_{-1}\bar{x}_0, \quad \mathcal{R}_0 := \begin{bmatrix} P_0 & 0\\ 0 & R_0\end{bmatrix},$$

$$\mathcal{Z}_0 := \begin{bmatrix} -F_{-1}\bar{x}_0\\ z_0\end{bmatrix}, \ \mathcal{G}_0 := \begin{bmatrix} I & 0\\ 0 & K_{v,0}\end{bmatrix}.$$

In the expressions above, we adopt the following definitions: $Q_{-1} := P_0$, $S_{-1} = S_{-1}^T = 0$, $J_{-1} := 0$, $G_{w,-1} := I$, $G_{v,0} := 0$, $K_{w,-1} := 0$.

$$\widehat{x}_{k|k} := \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\I \end{bmatrix}^T \begin{bmatrix} P_{k-1|k-1} & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & Q_{k-1} & S_{k-1} & 0 & 0 & 0 & I & 0 & 0 \\ 0 & S_{k-1}^T & R_k & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{k-1} & G_{w,k-1} & G_{v,k} & -E_k \\ 0 & 0 & 0 & 0 & 0 & J_{k-1} & K_{w,k-1} & K_{v,k} & H_k \\ I & 0 & 0 & F_{k-1}^T & J_{k-1}^T & 0 & 0 & 0 & 0 \\ 0 & I & 0 & G_{w,k-1}^T & K_{w,k-1}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & I & G_{v,k}^T & K_{v,k}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_k^T & H_k^T & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \widehat{x}_{k-1|k-1} \\ 0\\0\\0\\z_k\\0\\0\\0\\0 \end{bmatrix} \quad (14)$$

$$P_{k|k} := -\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\I \end{bmatrix}^T \begin{bmatrix} P_{k-1|k-1} & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & Q_{k-1} & S_{k-1} & 0 & 0 & 0 & I & 0 & 0 \\ 0 & S_{k-1}^T & R_k & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{k-1} & G_{w,k-1} & G_{v,k} & -E_k \\ 0 & 0 & 0 & 0 & 0 & J_{k-1} & K_{w,k-1} & K_{v,k} & H_k \\ I & 0 & 0 & F_{k-1}^T & J_{k-1}^T & 0 & 0 & 0 & 0 \\ 0 & I & 0 & G_{w,k-1}^T & K_{w,k-1}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & I & G_{v,k}^T & K_{v,k}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_k^T & H_k^T & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\I \end{bmatrix} \quad (15)$$

To solve the optimization problem (12)-(13) we consider the following assumptions.

A 1: $\begin{bmatrix} -E_i & F_{i-1} & G_{w,i-1} & G_{v,i} \\ H_i & J_{i-1} & K_{w,i-1} & K_{v,i} \end{bmatrix}$ has full row rank for all $0 < i \leq k$;

A 2: $\begin{bmatrix} H_0 & K_{v,0} \end{bmatrix}$ has full row rank;

A 3: $\begin{bmatrix} E_i \\ H_i \end{bmatrix}$ has full column rank for all $0 \leq i \leq k$.

Assumptions $A1$ and $A2$ are related with the consistence of the system parameters and assure that the system does not furnish redundant information. Note that in [14] it was shown that a similar condition assures the use of inverses of a block matrix instead of pseudo-inverses. The assumption $A3$ is related to estimability and assures the existence of the filter [14].

To address the filter problem we first present some auxiliary results.

Lemma 3.1: [13] Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $A \geq 0$. Then $\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$ has inverse if and only if B is a full column rank matrix and $\begin{bmatrix} A & B \end{bmatrix}$ is a full row rank matrix. $\square$

The solvability of above optimization problem is considered in the next lemma.

Lemma 3.2: For a fixed k suppose that it is given an output sequence $\{z_0, z_1, ..., z_k\}$ and that assumptions $A1$-$A3$ hold. Then the data fitting problem $(8)-(11)$ has a unique solution.

Proof: Omitted. It is similar to the proof presented in [6].

Theorem 3.1: Suppose that assumptions $A1$-$A3$ hold and it is given a sequence $\{z_0, z_1, ...\}$. Then the successive optimal estimates $\hat{x}_{k|k}$ resulting from the solution of the data fitting problem $(8)-(11)$ can alternatively be obtained from the following recursive algorithm

Step 0 (Initial Conditions):

$$\hat{x}_{0|0} := \begin{bmatrix} 0\\0\\0\\0\\0\\0\\I \end{bmatrix}^T X \begin{bmatrix} 0\\0\\-F_{-1}\bar{x}_0\\z_0\\0\\0\\0 \end{bmatrix}$$

$$P_{0|0} := -$$

$$\begin{bmatrix} 0\\0\\0\\0\\0\\0\\I \end{bmatrix}^T \underbrace{\begin{bmatrix} P_0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & R_0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I & 0 & -E_0 \\ 0 & 0 & 0 & 0 & 0 & K_{v,0} & H_0 \\ I & 0 & I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & K_{v,0}^T & 0 & 0 & 0 \\ 0 & 0 & -E_0^T & H_0^T & 0 & 0 & 0 \end{bmatrix}^{-1}}_{X} \begin{bmatrix} 0\\0\\0\\0\\0\\0\\I \end{bmatrix}$$

Step k: Update $\{\hat{x}_{k-1|k-1}, P_{k-1|k-1}\}$ to $\{\hat{x}_{k|k}, P_{k|k}\}$ as in (14) and (15). *Proof*: Omitted.

Observe that although we have considered a more general linear system than the usual in the literature, the final expressions (14) and (15) have a central matrix with symmetric structure and are very simple. The results were obtained directly without the need of rewriting the original problem in an augmented system. As one consequence of

$$\widehat{x}_{k-1|k} := \begin{bmatrix} 0\\0\\0\\0\\0\\I\\0\\0\\0 \end{bmatrix}^T \begin{bmatrix} P_{k-1|k-1} & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & Q_{k-1} & S_{k-1} & 0 & 0 & 0 & I & 0 & 0 \\ 0 & S_{k-1}^T & R_k & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{k-1} & G_{w,k-1} & G_{v,k} & -E_k \\ 0 & 0 & 0 & 0 & 0 & J_{k-1} & K_{w,k-1} & K_{v,k} & H_k \\ I & I & 0 & 0 & F_{k-1}^T & J_{k-1}^T & 0 & 0 & 0 \\ 0 & 0 & I & 0 & G_{w,k-1}^T & K_{w,k-1}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & I & G_{v,k}^T & K_{v,k}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -E_k^T & H_k^T & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \widehat{x}_{k-1|k-1} \\ 0\\0\\0\\ z_k \\ 0\\0\\0\\0 \end{bmatrix}. \quad (16)$$

Theorem 3.1, it turns out clear the existence of a filter even when the usual full rank assumption of $\begin{bmatrix} F_i & G_{w,i} \end{bmatrix}$ matrix is weakened to Assumption A1. In order to compare Theorem 3.1 with the results of the literature, consider the particular case where $G_{w,i} = G_i$, $G_{v,i} = 0$, $J_i = 0$, $K_{w,i} = 0$, $K_{v,i} = I$, and $\begin{bmatrix} F_i & G_i \end{bmatrix}$ has full row rank, for all i. Then, using the matrix inversion lemma, we can rewrite (14) and (15) as

$$\widehat{x}_{k|k} := \begin{bmatrix} 0\\0\\I \end{bmatrix}^T \begin{bmatrix} Y & -S_k & E_k \\ -S_k^T & R_k & H_k \\ E_k^T & H_k^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_{k-1}\widehat{x}_{k-1|k-1} \\ z_k \\ 0 \end{bmatrix} \quad (17)$$

$$P_{k|k} := -\begin{bmatrix} 0\\0\\I \end{bmatrix}^T \begin{bmatrix} Y & -S_k & E_k \\ -S_k^T & R_k & H_k \\ E_k^T & H_k^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0\\0\\I \end{bmatrix} \quad (18)$$

(where $Y = F_{k-1}P_{k-1|k-1}F_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T$) or still, if in addition we consider $S_i = 0$, the matrix inversion lemma furnish

$$\widehat{x}_{k|k} = P_{k|k}E_k^T(F_{k-1}P_{k-1|k-1}F_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T)^{-1}$$
$$F_{k-1}\widehat{x}_{k-1|k-1} + P_{k|k}H_k^TR_k^{-1}z_k \quad (19)$$

and

$$P_{k|k} := (E_k^T(F_{k-1}P_{k-1|k-1}F_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T)^{-1}\times$$
$$E_k + H_k^TR_k^{-1}H_k)^{-1}. \quad (20)$$

Note that (17)-(18) with $G_k = I$ is the solution presented in [14]. We also have that (19)-(20) with $E_k = I$ is the usual state space Kalman filter solution.

From the recursive solution obtained in Theorem 3.1, we are lead to conjecture that we could re-state $(8)-(11)$ and consider the following optimization problem: $\min_{x_{k-1},x_k}$ of

$$\left\{ \frac{1}{2}\begin{bmatrix} x_{k-1}-\widehat{x}_{k-1|k-1} \\ w_{k-1} \\ v_k \end{bmatrix}^T \begin{bmatrix} P_{k-1|k-1} & 0 \\ 0 & Q_p \end{bmatrix}^{-1} \begin{bmatrix} \bullet \end{bmatrix} \right\} \quad (21)$$

subject to (11), where $Q_p = \begin{bmatrix} Q_{k-1} & S_{k-1} \\ S_{k-1}^T & R_k \end{bmatrix}$. This is in fact the case as we state in the following lemma.

Lemma 3.3: The optimal filtered estimates algorithm of Theorem 3.1 can be obtained by the solution of (21). Furthermore, the optimal one-lag smoothed estimate $\hat{x}_{k-1|k}$ is obtained from the solution of (21) as (16).
□

Considering $G_{w,i} = I$, $G_{v,i} = 0$, $J_i = 0$, $K_{w,i} = 0$, $K_{v,i} = I$, $S_i = 0$ and the matrix inversion lemma, we can rewrite (16) as the smoother expression presented in [7]

$$\hat{x}_{k-1|k} = \hat{x}_{k-1|k-1} - P_{k-1|k-1}F_{k-1}^T(W_{k-1}$$
$$+F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}\Big(I - E_kP_{k|k}E_k^T(W_{k-1}$$
$$+F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}\Big)F_{k-1}\hat{x}_{k-1|k-1} + P_{k-1|k-1}F_{k-1}^T$$
$$(W_{k-1} + F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}E_kP_{k|k}H_k^TV_k^{-1}z_k. \quad (22)$$

Furthermore, for standard state space systems ($E_k = I$), the one-lag smoother (22) is exactly the classical one-lag smoother (see e.g. [11]).

In order to guarantee practical usage of the filter, in the next section we will show results that guarantee the stability of the Kalman filter. In particular, we determine conditions for the existence and unicity of a stabilizing solution P for the Riccati equation.

IV. STABILITY

In order to verify the stability of the steady-state filter, we consider the system (11) time invariant and rewrite it as

$$\mathcal{Z}_{i+1} = \mathcal{E}x_{i+1} + \mathcal{A}x_i + \mathcal{G}\mathcal{V}_{i+1} \quad (23)$$

where

$$\mathcal{Z}_{i+1} := \begin{bmatrix} 0 \\ z_{i+1} \end{bmatrix}, \; \mathcal{E} := \begin{bmatrix} -E \\ H \end{bmatrix}, \; \mathcal{A} := \begin{bmatrix} F \\ J \end{bmatrix},$$

$$\mathcal{G} := \begin{bmatrix} G_w & G_v \\ K_w & K_v \end{bmatrix}, \; \mathcal{V}_{i+1} := \begin{bmatrix} w_i \\ v_{i+1} \end{bmatrix}, \quad i \geq 0.$$

From Section III, the best estimate $\hat{x}_{k|k}$ of x_k is the equation (14), i.e.,

$$\widehat{x}_{k|k}=\begin{bmatrix}0\\0\\0\\0\\0\\I\end{bmatrix}^T\begin{bmatrix}P&0&0&I&0&0\\0&\mathcal{R}&0&0&I&0\\0&0&0&\mathcal{A}&\mathcal{G}&\mathcal{E}\\I&0&\mathcal{A}^T&0&0&0\\0&I&\mathcal{G}^T&0&0&0\\0&0&\mathcal{E}^T&0&0&0\end{bmatrix}^{-1}\begin{bmatrix}\widehat{x}_{k-1|k-1}\\0\\\mathcal{Z}_k\\0\\0\\0\end{bmatrix} \tag{24}$$

where the Algebraic Riccati Equation (ARE) associated with the recursion (15) is given by

$$P=-\begin{bmatrix}0\\0\\0\\0\\0\\I\end{bmatrix}^T\begin{bmatrix}P&0&0&I&0&0\\0&\mathcal{R}&0&0&I&0\\0&0&0&\mathcal{A}&\mathcal{G}&\mathcal{E}\\I&0&\mathcal{A}^T&0&0&0\\0&I&\mathcal{G}^T&0&0&0\\0&0&\mathcal{E}^T&0&0&0\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\0\\0\\0\\I\end{bmatrix} \tag{25}$$

$$\text{where}\quad \mathcal{R}:=\begin{bmatrix}Q&S\\S^T&R\end{bmatrix}.$$

The matrices Q, R, and S are time invariant and represent the weighting matrices to the errors $w_{i|k}$, $v_{i|k}$ and cross-term of these errors, respectively.

Some auxiliary results are presented in the following lemmas which provide conditions to guarantee the filter stability.

Lemma 4.1: The estimate $\widehat{x}_{k|k}$ and its corresponding Riccati equation given by (24) and (25) can be rewritten as

$$\widehat{x}_{k|k}=L_{31}\widehat{x}_{k-1|k-1}+L_{33}\mathcal{Z}_k \tag{26}$$

$$\text{and}\quad P=L_{31}PL_{31}^T+L_{32}\mathcal{R}L_{32}^T \tag{27}$$

respectively, with

$$L:=\begin{bmatrix}L_{11}&L_{12}&L_{13}\\L_{21}&L_{22}&L_{23}\\L_{31}&L_{32}&L_{33}\end{bmatrix}=\begin{bmatrix}0&0&0&I&0&0\\0&0&0&0&I&0\\0&0&0&0&0&I\end{bmatrix}$$
$$\begin{bmatrix}P&0&0&I&0&0\\0&\mathcal{R}&0&0&I&0\\0&0&0&\mathcal{A}&\mathcal{G}&\mathcal{E}\\I&0&\mathcal{A}^T&0&0&0\\0&I&\mathcal{G}^T&0&0&0\\0&0&\mathcal{E}^T&0&0&0\end{bmatrix}^{-1}\begin{bmatrix}I&0&0\\0&I&0\\0&0&I\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}. \tag{28}$$

□

Definition 4.1: P is a stabilizing solution of Algebraic Riccati Equation (ARE) (27) if P satisfies (27) and L_{31} is stable.

Lemma 4.2: Consider the ARE (27) with L given by (28). Then, we have that $-L_{33}\mathcal{A}=L_{31}$. □

Lemma 4.3: For given matrices $A\in C^{m\times m}$, $B\in C^{n\times n}$ and $\Gamma\in C^{n\times m}$, the Stein equation

$$S-BSA=\Gamma \tag{29}$$

has a unique solution, if and only if, $\lambda_r\mu_s\neq 1$ for any $\lambda_r\in\sigma(A)$ and $\mu_s\in\sigma(B)$. □

Now, we are in position to present the main result of this section. We will show that the ARE has a stabilizing semi-definite solution P.

Theorem 4.1: Suppose that $\begin{bmatrix}\lambda\mathcal{E}+\mathcal{A} & \mathcal{G}\end{bmatrix}$ has full row rank for $|\lambda|\geq 1$, $\mathcal{R}>0$ and $\mathcal{E}$ has full column rank. Let P a solution of the ARE (25). If $P\geq 0$, then P is the stabilizing solution for (25).

Proof: Suppose that there exists a positive semi-definite solution P for the ARE (27). We must show that L_{31} is stable. From Lemma 4.2 we have that $L_{31}=-L_{33}\mathcal{A}$ and

$$\widehat{x}_{k|k}=-L_{33}\mathcal{A}\widehat{x}_{k-1|k-1}+L_{33}\mathcal{Z}_k \tag{30}$$

$$P=L_{33}(\mathcal{A}P\mathcal{A}^T+\mathcal{G}\mathcal{R}\mathcal{G}^T)L_{33}^T \tag{31}$$

Thus, it is enough to show that $-L_{33}\mathcal{A}$ is stable.

Suppose, for contradiction, that $-L_{33}\mathcal{A}$ is not stable. Then, there exists a complex number λ and a vector v ($v\neq 0$) such that $|\lambda|\geq 1$ and $-v^TL_{33}\mathcal{A}=\lambda v^T$. However, we have that

$$L_{33}\mathcal{E}=I.$$

Multiplying the equation above by λv^T we get

$$\lambda v^TL_{33}\mathcal{E}=\lambda v^T$$

and as $\lambda v^T=-v^TL_{33}\mathcal{A}$ we have that

$$\lambda v^TL_{33}\mathcal{E}=-v^TL_{33}\mathcal{A}$$
$$v^TL_{33}(\lambda\mathcal{E}+\mathcal{A})=0. \tag{32}$$

However, pre-multiplying (31) by v^T and pos-multiplying by v, we obtain

$$v^TPv=v^T\left(L_{33}\mathcal{A}\right)P\left(L_{33}\mathcal{A}\right)^Tv+v^TL_{33}\mathcal{G}\mathcal{R}\mathcal{G}^TL_{33}^Tv.$$

Since $\lambda v^T=-v^TL_{33}\mathcal{A}$, the above equation becomes

$$v^TPv=\lambda v^TP\left(\lambda v^T\right)^T+v^TL_{33}\mathcal{G}\mathcal{R}\mathcal{G}^TL_{33}^Tv. \tag{33}$$

Thus,

$$(|\lambda|^2-1)v^TPv+v^TL_{33}\mathcal{G}\mathcal{R}\mathcal{G}^TL_{33}^Tv=0. \tag{34}$$

We know that $|\lambda|\geq 1$, $P\geq 0$ and $\mathcal{R}\geq 0$. Then

$$(|\lambda|^2-1)v^TPv\geq 0 \tag{35}$$

$$\text{and}\quad v^TL_{33}\mathcal{G}\mathcal{R}\mathcal{G}^TL_{33}^Tv\geq 0 \tag{36}$$

From (34), (35), and (36) it follows

$$v^TL_{33}\mathcal{G}=0. \tag{37}$$

From (32) and (37) we get

$$v^T L_{33}(\begin{bmatrix}\lambda\mathcal{E} + \mathcal{A} & \mathcal{G}\end{bmatrix}) = 0 \quad \text{with} \quad |\lambda| \geq 1. \tag{38}$$

Since L_{33} has full row rank, we have that $v \neq 0 \Longleftrightarrow v^T L_{33} \neq 0$. Thus, in (38), $\begin{bmatrix}\lambda\mathcal{E} + \mathcal{A} & \mathcal{G}\end{bmatrix}$ has not full row rank for $|\lambda| \geq 1$, i.e., we get a contradiction. Then, $L_{33}\mathcal{A}$ is stable.

Lemma 4.4: Consider the ARE (27). If there exist a stabilizing solution, it is unique.

Proof: Omitted.

V. Numerical Example

Let us consider the descriptor system described by (11), where the system matrices are given by

$$E_{i+1} = E = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix};\ F_i = F = \begin{bmatrix}0.9 & 0\\0.2 & 0.2\end{bmatrix};$$

$$G_{w,i} = G_w = \begin{bmatrix}0.4 & 0.1\\0.1 & 6\end{bmatrix};\ G_{v,i+1} = G_v = \begin{bmatrix}1\\1\end{bmatrix};$$

$$H_i = H = \begin{bmatrix}1.4 & 0.8\end{bmatrix};\ J_i = J = 0;$$

$$K_{w,i} = K_w = \begin{bmatrix}1.4 & 1.4\end{bmatrix}; \quad K_{v,i+1} = K_v = 1$$

and the weighting matrices of the errors w_k and v_k (with the cross-term) given respectively by

$$Q = \begin{bmatrix}7 & 2\\2 & 1\end{bmatrix};\quad R = 0.1;\quad S = \begin{bmatrix}0.001\\0.05\end{bmatrix}. \tag{39}$$

The simulation results based on the filter presented in Theorem 3.1, are presented in Figures 1 and 2.

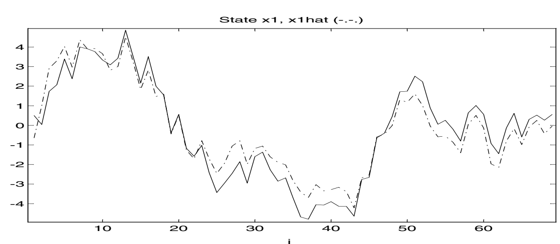

Fig. 1. True value of state x1(k) and filtered estimate.

VI. Conclusion

We have considered the Kalman filtering problem for linear systems in its more general formulation where all system parameters are considered at once. We have introduced a "9-block" form for the filter and Riccati equation which present an interesting simple and symmetric structure. In this case, we observed that it is natural, in this framework, to solve this problem with a one step delayed state. We are interested in this form mainly to analyze the effect of perturbations on all system matrices in order to generalize the results of [8].

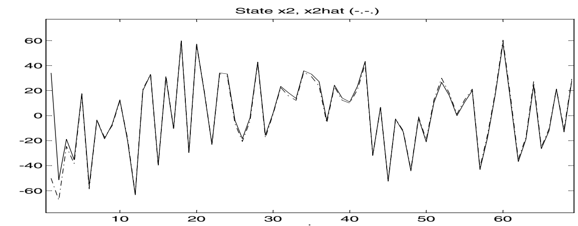

Fig. 2. True value of state x2(k) and filtered estimate.

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