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Optimal Filtering for Linear Systems with State and Observation Delays

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Abstract. In this paper, the optimal filtering problem for linear systems with state and observation delays is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate, error variance, and various error covariances. As a result, the optimal estimate equation similar to the traditional Kalman-Bucy one is derived; however, it is impossible to obtain a system of the filtering equations, that is closed with respect to the only two variables, the optimal estimate and the error variance, as in the Kalman-Bucy filter. The resulting system of equations for determining the filter gain matrix consists, in the general case, of an infinite set of equations. It is however demonstrated that a finite set of the filtering equations, whose number is specified by the ratio between the current filtering horizon and the delay values, can be obtained in the particular case of equal or commensurable ($\tau = qh$, q is natural) delays in the observation and state equations. In the example, performance of the designed optimal filter for linear systems with state and observation delays is verified against the best Kalman-Bucy filter available for linear systems without delays.

I. Introduction

The optimal filtering problem for linear system states and observations without delays was solved in 1960s [1], and this closed form solution is known as the Kalman-Bucy filter. However, the related optimal filtering problem for linear states with delay has not been solved in a closed form, regarding as a closed form solution a closed system of a finite number of ordinary differential equations for any finite filtering horizon. The optimal filtering problem for time delay systems itself did not receive so much attention as its control counterpart, and most of the research was concentrated on the filtering problems with observation delays (the papers [2], [3], [4] could be mentioned to make a reference). A particular case, the optimal filtering problem for linear systems with multiple observation delays, has recently been solved in [5]. A review of the bibliography on dual optimal control problems, as well as robust filtering and control problems, for time delay systems can be found in [5], [6], [7]. Comprehensive reviews of general theory and algorithms for time delay systems are given in [8], [9], [10], [11], [12].

In this paper, the optimal filtering problem for linear systems with state and observation delays is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate, error variance, and various error covariances [13]. As a result, the optimal estimate equation similar to the traditional Kalman-Bucy one is derived. However, it is impossible to obtain a system of the filtering equations, that is closed with respect to the only two variables, the optimal estimate and the error variance, as in the Kalman-Bucy filter. Thus, the resulting system of equations for determining the filter gain matrix consists, in the general case, of an infinite set of equations. It is however demonstrated that a finite set of the filtering equations can be obtained in the particular case of equal or commensurable ($\tau = qh$) delays in the observation and state equations, where τ is the observation delay, h is the state one, and q is a natural number. This finite number of the filtering equations whose number is specified by the ratio between the current filtering horizon and the delay values and increases as the filtering horizon tends to infinity.

The paper is organized as follows. Section 2 and 3 present the filtering problem statement for a linear system with state and observation delays and its solution, respectively. In Section 4, performance of the obtained optimal filter for linear systems with state and observation delays is verified in the illustrative example against the best filter available for linear systems without delays. The simulation results show asymptotic convergence of the estimate given by the obtained optimal filter for linear systems with state and observation delays to the real system state as time tends to infinity, whereas the conventional Kalman-Bucy estimates calculated without delay adjustment do not converge.

II. Filtering Problem for Linear Systems with State and Observation Delays

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes. The partially observed F_t-

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measurable random process $(x(t), y(t))$ is described by a delay differential equation for the system state

$$dx(t) = (a_0(t) + a(t)x(t-h))dt + b(t)dW_1(t), \qquad (1)$$

with the initial condition $x(s) = \phi(s)$, $s \in [t_0 - h, t_0]$, and a delay differential equation for the observation process:

$$dy(t) = (A_0(t) + A(t)x(t-\tau))dt + B(t)dW_2(t), \qquad (2)$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the observation process, $\phi(s)$ is a mean square piecewise-continuous Gaussian stochastic process (see [13] for definition) given in the interval $[t_0 - h, t_0]$ such that $\phi(s)$, $W_1(t)$, and $W_2(t)$ are independent. The system state $x(t)$ dynamics depends on a delayed state $x(t-h)$ and the observations $y(t)$ are collected depending on another delayed state $x(t-\tau)$, which actually make the system state space infinite-dimensional (see, for example, [10]). The vector-valued function $a_0(t)$ describes the effect of system inputs (controls and disturbances). It is assumed that $A(t)$ is a nonzero matrix and $B(t)B^T(t)$ is a positive definite matrix. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The estimation problem is to find the best estimate of the system state $x(t)$ based on the observation process $Y(t) = \{y(s), 0 \le s \le t\}$, which minimizes the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t)) \mid F_t^Y]$$

at every time moment t. Here, $E[z(t) \mid F_t^Y]$ means the conditional expectation of a stochastic process $z(t) = (x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. As known [13], this optimal estimate is given by the conditional expectation

$$\hat{x}(t) = m(t) = E(x(t) \mid F_t^Y)$$

of the system state $x(t)$ with respect to the observation process $Y(t)$ in the interval $[t_0, t]$.

The matrix functions

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y],$$

that is the estimation error variance, and

$$P(t, t-t_1) = E[(x(t) - m(t))(x(t-t_1) - m(t-t_1))^T \mid F_t^Y],$$

that is the covariance between the estimation error values at different time moments, $P(t,t) = P(t)$, are used to obtain a system of filtering equations.

The proposed solution to this optimal filtering problem is based on the formulas for the Ito differentials of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$, the error variance $P(t)$, and other bilinear functions of $x(t) - m(t)$ (see [13]) and given in the following section.

III. Optimal Filter for Linear Systems with State and Observation Delays

The optimal filtering equations can be obtained using the formula for the Ito differential of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ ([13])

$$dm(t) = E(\varphi(x) \mid F_t^Y)dt + E(x[\varphi_1(x) - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times \big(B(t)B^T(t)\big)^{-1}(dy(t) - E(\varphi_1(x) \mid F_t^Y)dt), \qquad (3)$$

where $\varphi(x)$ is the drift term in the state equation equal to $\varphi(x) = a_0(t) + a(t)x(t-h)$ and $\varphi_1(x)$ is the drift term in the observation equation equal to $\varphi_1(x) = A_0(t) + A(t)x(t-\tau)$. Note that the conditional expectation equality $E(x(t-h) \mid F_t^Y) = E(x(t-h) \mid F_{t-h}^Y) = m(t-h)$ is valid for any $h > 0$, since, in view of a positive delay shift $h > 0$, the treated problem (1),(2) is a filtering problem, not a smoothing one, and, therefore, the formula (3) yields the optimal estimate $m(s)$ for any time s, $t_0 < s \le t$, if the observations (2) are obtained until the current moment t (see [13], [5]). Upon performing substitution of the expressions for φ and φ_1 into (3) and taking into account the conditional expectation equality, the estimate equation takes the form

$$\begin{aligned} dm(t) &= (a_0(t) + a(t)m(t-h))dt + \qquad (4)\\ &E(x(t)[A(t)(x(t-\tau) - m(t-\tau))]^T \mid F_t^Y) \times \\ &(B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-\tau)dt) = \\ &= (a_0(t) + a(t)m(t-h))dt + \\ &E([x(t) - m(t)][x(t-\tau) - m(t-\tau)]^T \mid F_t^Y)A^T(t) \times \\ &(B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-\tau)dt) = \\ &(a_0(t) + a(t)m(t-h))dt + P(t, t-\tau)A^T(t) \times \\ &(B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-\tau)dt) \end{aligned}$$

To compose a system of the filtering equations, the equation for the conditional expectation $E([x(t) - m(t)][(x(t-\tau) - m(t-\tau))]^T \mid F_t^Y)$ should be obtained. This can be done using the equation (1) for the state $x(t)$, the equation (4) for the estimate $m(t)$, and the formula for the Ito differential of a product of two processes satisfying Ito differential equations ([13]):

$$d(z_1 z_2^T) = z_1 dz_2^T + (z_2 dz_1^T)^T + (1/2)[y_1 \nu y_2^T + y_2 \nu y_1^T]dt. \qquad (5)$$

Here, the stochastic process z_1 satisfies the equation

$$dz_1 = x_1 dt + y_1 dw_1,$$

the stochastic process z_2 satisfies the equation

$$dz_2 = x_2 dt + y_2 dw_2,$$

and ν is the covariance intensity matrix of the Wiener vector $[w_1 \ w_2]^T$.

Let us obtain the formula for the Ito differential of the general expression $P(t, t-t_1) = E([x(t) - m(t)][x(t-t_1) - m(t-t_1)]^T \mid F_t^Y)$, where $t_1 > 0$ is an arbitrary delay, not necessarily equal to τ. Upon representing $P(t,t-t_1)$ as $P(t,t-t_1)=E([x(t)(x(t-t_1))^T]\mid F_t^Y)-m(t)m(t-t_1)$, using first $x(t)$ as z_1 and $x(t-t_1)$ as z_2 and then $m(t)$ as z_1 and $m(t-t_1)$ as z_2 in the formula (5), taking into account independence of the Wiener processes W_1 and W_2 in the equations (1) and (2), and finally subtracting the second derived equation from the first one, the following formula is obtained

$$dP(t,t-t_1)/dt = a(t)P(t-h,t-t_1)+ \tag{6}$$
$$P(t,t-t_1-h)a^T(t-t_1)+$$
$$(1/2)[b(t)b^T(t-t_1)+b(t-t_1)b^T(t)]-$$
$$(1/2)[P(t,t-\tau)A^T(t)(B(t)B^T(t))^{-1}B(t)\times$$
$$B^T(t-t_1)(B(t-t_1)B^T(t-t_1))^{-1}\times$$
$$A(t-t_1)P^T(t-t_1,t-t_1-\tau)-$$
$$P(t-t_1,t-t_1-\tau)A^T(t-t_1)(B(t-t_1)B^T(t-t_1))^{-1}B(t-t_1)\times$$
$$B^T(t)(B(t)B^T(t))^{-1}A(t)P^T(t,t-\tau)].$$

Analysis of the formula (6) in the case $t_1=\tau$ implies that the equation for $P(t,t-\tau)$ includes variables $P(t,t-\tau-h)$, $P(t-h,t-\tau)$ and the same $P(t,t-\tau)$ in its right-hand side. Taking into account that $P(t-h,t-\tau)$ is represented as $P(t,t-\tau+h)$ with the arguments delayed by h, the new variables involved in the equations for $P(t,t-\tau)$ are $P(t,t-\tau-h)$ and $P(t,t-\tau+h)$. This structure is repeated in the equations for $P(t,t-\tau-h)$, $P(t,t-\tau+h)$, etc.

Hence, the system of the optimal filtering equations for the state (1), whose proper dynamics is delayed by h, over the delayed by τ observations (2) is the infinite-dimensional system composed by the equation (4) for the optimal estimate and the equations (6) for the covariances $P(t,t-\tau+kh)$, where $k=\ldots,-2,-1,0,1,2,\ldots$ is an arbitrary integer number.

Using the notation $P_k(t)=P(t,t-\tau-kh)$, the equation (4) can be rewritten as

$$dm(t)=(a_0(t)+a(t)m(t-h))dt+P_0(t)A^T(t)\times \tag{7}$$
$$(B(t)B^T(t))^{-1}(dy(t)-(A_0(t)+A(t)m(t-\tau)dt),$$

and the system (6) can be represented in the following form

$$dP_k(t)/dt=a(t)P_{k-1}(t-h)+P_{k+1}(t)a^T(t-\tau-kh)+ \tag{8}$$
$$(1/2)[b(t)b^T(t-\tau-kh)+b(t-\tau-kh)b^T(t)]-$$
$$(1/2)[P_0(t)A^T(t)(B(t)B^T(t))^{-1}B(t)B^T(t-\tau-kh)\times$$
$$(B(t-\tau-kh)B^T(t-\tau-kh))^{-1}A(t-\tau-kh)P_0^T(t-\tau-kh)-$$
$$P_0(t-\tau-kh)A^T(t-\tau-kh)(B(t-\tau-kh)B^T(t-\tau-kh))^{-1}\times$$
$$B(t-\tau-kh)B^T(t)(B(t)B^T(t))^{-1}A(t)P_0^T(t)].$$

Thus, the preceding conclusion can be formulated in the final form: the system of the optimal filtering equations for the state (1), whose proper dynamics is delayed by h, over the delayed by τ observations (2) is the infinite system composed by the equation (7) for the optimal estimate and the equations (8) for the covariances $P_k(t)=P(t,t-\tau-kh)$, where $k=\ldots,-2,-1,0,1,2,\ldots$ is an arbitrary integer number.

The last step is to establish the initial conditions for the system of equations (7),(8). The initial conditions for (7) are stated as

$$m(s)=E(\phi(s)),\ s\in[t_0-h,t_0)\ \text{and}$$
$$m(t_0)=E(\phi(t_0)\mid F_{t_0}^Y),\ s=t_0, \tag{9}$$

The initial conditions for matrices $P_k(t)=E((x(t)-m(t))(x(t-\tau-kh))^T\mid F_t^Y)$ should be stated as functions in the intervals $[max\{t_0-h,t_0+\tau+(k-1)h\},max\{t_0+\tau+kh,t_0\}]$, since the equations (8) corresponding to non-negative k depend on coefficients with arguments delayed by $\tau+kh$, which are not defined for $t<t_0$. Thus, the initial conditions for the matrices $P_k(t)$ are stated as

$$P_k(s)=E((x(s)-m(s))(x(s-\tau-kh)-$$
$$m(s-\tau-kh))^T\mid F_s^Y), \tag{10}$$
$$s\in[max\{t_0-h,t_0+\tau+(k-1)h\},max\{t_0+\tau+kh,t_0\}].$$

Unfortunately, the system (7),(8) cannot be reduced to a finite system for any fixed filtering horizon t, as it can be done in the case of only state delay in the equations (1),(2) (see [14]), since the infinite number of the equations (8) for $P_k(t)$ with negative k are always needed to compose a closed system for any time t. However, this reduction is possible for some particular cases, for example, in the case of equal, $\tau=h$, (or commensurable, $\tau=qh$, q is natural) delays in the equations (1),(2), which is considered in details in the next subsection.

A. Optimal Filter for Linear Systems with Commensurable State and Observation Delays

An important and frequently encountered in practical applications particular case of commensurable delays in state and observation equations is recovered assuming $\tau=qh$, $q=1,2,\ldots$ is a natural number. In doing so, the state and observation equations (1),(2) take the form

$$dx(t)=(a_0(t)+a(t)x(t-h))dt+b(t)dW_1(t), \tag{11}$$

with the initial condition $x(s)=\phi(s)$, $s\in[t_0-h,t_0]$,

$$dy(t)=(A_0(t)+A(t)x(t-qh))dt+B(t)dW_2(t). \tag{12}$$

Accordingly, the optimal filtering equation (7) for the optimal estimate $m(t)$ turns to

$$dm(t)=(a_0(t)+a(t)m(t-h))dt+P_0(t)A^T(t)\times \tag{13}$$
$$(B(t)B^T(t))^{-1}(dy(t)-(A_0(t)+A(t)m(t-qh)dt),$$

and the system (8) is given by

$$dP_k(t)/dt=a(t)P_{k-1}(t-h)+P_{k+1}(t)a^T(t-(q+k)h)+ \tag{14}$$

$$(1/2)[b(t)b^T(t-(q+k)h)+b(t-(q+k)h)b^T(t)]-$$
$$(1/2)[P_0(t)A^T(t)(B(t)B^T(t))^{-1}B(t)B^T(t-(q+k)h)\times$$
$$(B(t-(q+k)h)B^T(t-(q+k)h))^{-1}\times$$
$$A(t-(q+k)h)P_0^T(t-(q+k)h)-$$
$$P_0(t-(q+k)h)A^T(t-(q+k)h)\times$$
$$(B(t-(q+k)h)B^T(t-(q+k)h))^{-1}\times$$
$$B(t-(q+k)h)B^T(t)(B(t)B^T(t))^{-1}A(t)P_0^T(t)].$$

Using the equality

$$P_{-q-1}(t-h)=P(t-h,t-(-q-1)h-qh-h)=$$
$$P(t-h,t)=P^T(t,t-h)=P^T_{-q+1}(t),$$

the equation for P_{-q} in (14) can be rewritten as

$$dP_{-q}(t)/dt=a(t)P^T_{-q+1}(t)+P_{-q+1}(t)a^T(t)+ \quad (15)$$
$$b(t)b^T(t)-P_0(t)A^T(t)(B(t)B^T(t)^{-1}A(t)P_0^T(t).$$

Note that $P_{-q}(t)=E((x(t)-m(t))(x(t))^T \mid F_t^Y)$ is the estimation error variance.

If $q=1,2,\ldots$, the right-hand side of (15) does not include variables P_k corresponding to negative $k<-q$. Hence, a closed system of the filtering equations is formed by the equations (13),(15) and the equations (14) with $k\geq -q$ only. This enables one to obtain a finite system of the filtering equations for any fixed filtering horizon t, as follows.

Namely, for every fixed t, the number of equations corresponding to $k\geq -q$ in (14), that should be taken into account to obtain a closed system of the filtering equations, is not equal to infinity, since the matrices $a(t)$, $b(t)$, $A(t)$, and $B(t)$ are not defined for $t<t_0$. Therefore, if the current time moment t belongs to the semi-open interval $(t_0+(k+q)h, t_0+(k+q+1)h]$, where h is the delay value in the equations (1),(2), the number of equations in (14) is equal to $k+q$.

The last step is to establish the initial conditions for the system of equations (13),(15),(14). The initial conditions for (13) and (15) are stated as

$$m(s)=E(\phi(s)),\ s\in[t_0-\tau,t_0) \text{ and}$$
$$m(t_0)=E(\phi(t_0)\mid F^Y_{t_0}),\ s=t_0, \quad (16)$$

and

$$P(t_0)=E[(x(t_0)-m(t_0)(x(t_0)-m(t_0))^T\mid F^Y_{t_0}]. \quad (17)$$

The initial conditions for matrices $P_k(t)=E((x(t)-m(t))(x(t-(q+k)h))^T\mid F_t^Y)$ should be stated as functions in the intervals $[t_0+(k+q-1)h, t_0+(k+q)h]$, since the kth of the equations (14) depends on functions with the arguments delayed by $(k+q)h$ and the definition of $P_k(t)$ itself assumes dependence on $x(t-(k+q)h)$. Thus, the initial conditions for the matrices $P_k(t)$ in (14) are stated as

$$P_k(s)=E((x(s)-m(s))(x(s-(q+k)h)-$$
$$m(s-(q+k)h))^T\mid F_s^Y), \quad (18)$$
$$s\in[t_0+(q+k-1)h, t_0+(q+k)h].$$

The obtained system of the filtering equations (13),(15),(14) with the initial conditions (16)–(18) presents the optimal solution to the filtering problem for the linear state with delay (11) over the linear observations (12). A considerable advantage of the designed filter is a finite number of the filtering equations for any fixed filtering horizon, although the state space of the delayed system (11) is infinite-dimensional.

Remark. The convergence properties of the obtained optimal estimate (7) are given by the standard convergence theorem (see, for example, [15]): if in the system (1),(2) the pair $(a(t)\Psi(t-h,t), b(t))$ is uniformly completely controllable and the pair $(a(t)\Psi(t-h,t),\ A(t)\Psi(t-\tau,t)$ is uniformly completely observable, where $\Psi(t,\tau)$ is the state transition matrix for the equation (1) (see [10] for definition of matrix Ψ), then the error of the obtained optimal filter (7),(8) is uniformly asymptotically stable. As usual, the uniform complete controllability condition is required for assuring non-negativeness of the error variance matrix $P_{-q}(t)$ and may be omitted, if the matrix $P_{-q}(t)$ is non-negative in view of its intrinsic properties. The uniform complete controllability and observability conditions for a linear system with delay (1) and observations (2) can be found in [10].

IV. Example

This section presents an example of designing the optimal filter for linear systems with state and observation delays and comparing it to the best filter available for linear systems without delay, that is the Kalman-Bucy filter [1].

Let the unobserved state $x(t)$ with delay be given by

$$\dot{x}(t)=x(t-5),\quad x(s)=\phi(s),\ s\in[-5,0], \quad (19)$$

where $\phi(s)=N(0,1)$ for $s\leq 0$, and $N(0,1)$ is a Gaussian random variable with zero mean and unit variance. The observation process is given by

$$y(t)=x(t-5)+\psi(t), \quad (20)$$

where $\psi(t)$ is a white Gaussian noise, which is the weak mean square derivative of a standard Wiener process (see [13]). The equations (12) and (13) present the conventional form for the equations (1) and (2), which is actually used in practice [16]. Since the observation delay is equal to the state one, the system (19),(20) satisfies the conditions of Subsection 3.1 with $q=1$.

The filtering problem is to find the optimal estimate for the linear state with delay (19), using the linear observations with delay (20) confused with independent and identically distributed disturbances modeled as white Gaussian noises. Let us set the filtering horizon time to $T=80$. Since $80\in(15\times 5, 16\times 5]$, where 5 is the delay value in the equations

(19),(20), the first 15 of the equations (14), along with the equations (13) and (15), should be employed.

The filtering equations (13),(15), and the first 15 of the equations (14) take the following particular form for the system (19),(20)

$$\dot{m}(t) = m(t-5) + P_0(t)[y(t) - m(t-5)], \quad (21)$$

with the initial condition $m(s) = E(\phi(s)) = 0$, $s \in [-5,0)$ and $m(0) = E(\phi(0) \mid y(0)) = m_0$, $s = 0$;

$$\dot{P}_i(t) = P_{i-1}(t-5) + P_{i+1}(t) - P_0(t)P_0(t-5(i+1)), \quad (22)$$

with the initial condition $P_i(0) = E((x(s)-m(s))(x(s-5(i+1)) - m(s-5(i+1))) \mid F_s^Y)$, $s \in [5i, 5(i+1)]$, $i = 0,\ldots,14$; and

$$\dot{P}_{-1}(t) = 2P_0(t) - P_0^2(t), \quad (23)$$

with the initial condition $P_{-1}(0) = E((x(s) - m(s))^2 \mid y(0)) = R_0$; note that $P_{-1}(t)$ is the error variance. The particular forms of the equations (19) and (21) and the initial condition for $x(t)$ imply that $P_i(s) = R_0$, $i = 0,\ldots,15$, for $s \in [5i, 5(i+1)]$.

The estimates obtained upon solving the equations (21)–(23) are compared to the conventional Kalman-Bucy estimates satisfying the following filtering equations for the linear state with delay (19) over linear observations with delay (20), where the variance equation is a Riccati one and the equation for matrix $P_0(t)$ is not employed:

$$\dot{m}_K(t) = m_K(t-5) + P_K(t)[y(t) - m_K(t-5)], \quad (24)$$

with the initial condition $m_K(s) = E(\phi(s)) = 0$, $s \in [-5,0)$ and $m_K(0) = E(\phi(0) \mid y(0)) = m_0$, $s = 0$;

$$\dot{P}_K(t) = 2P_K(t) - P_K^2(t), \quad (25)$$

with the initial condition $P_K(0) = E((x(0) - m(0))^2 \mid y(0)) = R_0$.

Numerical simulation results are obtained solving the systems of filtering equations (21)–(23) and (24)–(25). The obtained values of the estimates $m(t)$ and $m_K(t)$ satisfying (21) and (24) respectively are compared to the real values of the state variable $x(t)$ in (19).

For each of the two filters (21)–(23) and (24)–(25) and the reference system (19) involved in simulation, the following initial values are assigned: $x_0 = 1$, $m_0 = 10$, $R_0 = 100$. Gaussian disturbance $\psi(t)$ in (20) is realized using the built-in MatLab white noise function.

The following graphs are obtained: graphs of the reference state variable $x(t)$ for the system (19); graphs of the Kalman-Bucy filter estimate $m_K(t)$ satisfying the equations (24)–(25); graphs of the optimal filter estimate for linear systems with state and observation delays $m(t)$ satisfying the equations (21)–(23). The graphs of all those variables are shown on the entire simulation interval from $T = 0$ to $T = 80$ (Fig. 1), and around the reference time points: $T = 40$ (Fig. 2), $T = 60$ (Fig. 3), and $T = 80$ (Fig. 4). It can also be noted that the error variance $P(t)$ converges to zero, since the optimal estimate (21) converges to the real state (19).

The following values of the reference state variable $x(t)$ and the estimates $m(t)$ and $m_K(t)$ are obtained at the reference time points: for $T = 40$, $x(40) = 12.55$, $m(40) = 12.62$, $m_K(40) = 12.75$; for $T = 60$, $x(60) = 51.56$, $m(60) = 51.50$, $m_K(60) = 52.12$; for $T = 80$, $x(80) = 211.92$, $m(80) = 211.96$, $m_K(80) = 214.08$.

Thus, it can be concluded that the obtained optimal filter for a linear systems with state delay and over linear observations with delay (21)–(23) yield better estimates than the conventional Kalman-Bucy filter. Subsequent discussion of the obtained simulation results can be found in Conclusions.

V. Conclusions

The simulation results show that the values of the estimate calculated by using the obtained optimal filter for a linear state with delay over linear observations with delay are noticeably closer to the real values of the reference variable than the values of the Kalman-Bucy estimates. Moreover, it can be seen that the estimate produced by the optimal filter for a linear state with delay over linear observations asymptotically converges to the real values of the reference variable as time tends to infinity, although the reference system (19) itself is unstable. On the contrary, the conventionally designed (non-optimal) Kalman-Bucy estimates do not converge to the real values. This significant improvement in the estimate behavior is obtained due to the more careful selection of the filter gain matrix using the multi-equational system (21)–(23), which compensates for unstable dynamics of the reference system, as it should be in the optimal filter. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

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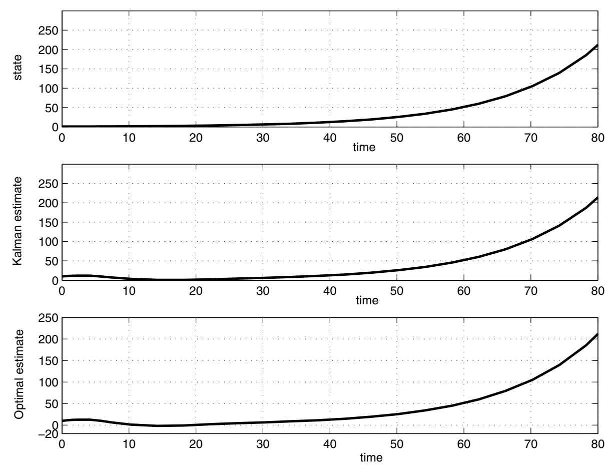

Fig. 1. Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ on the entire simulation interval $[0,80]$.

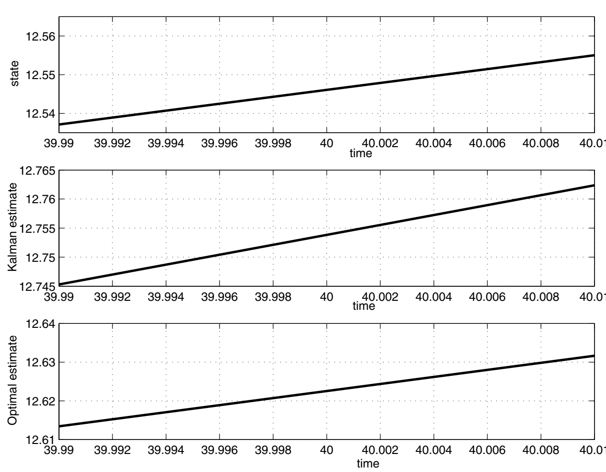

Fig. 2. Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ around the reference time point $T=40$.

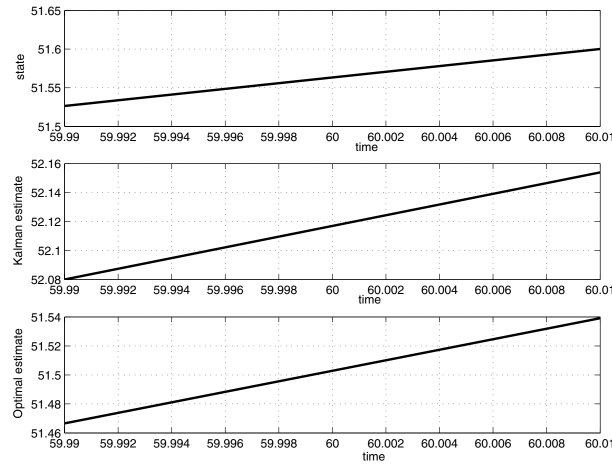

Fig. 3. Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ around the reference time point $T=60$.

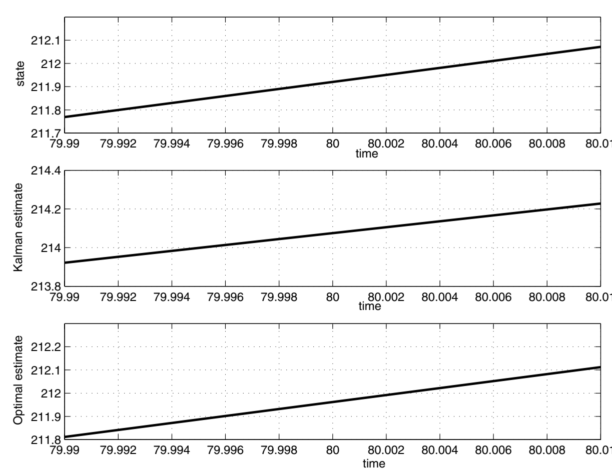

Fig. 4. Graphs of the reference state variable $x(t)$ and the estimates $m_K(t)$ and $m(t)$ around the reference time point $T=80$.