

Robust Parametric Predictive Control Design for Polytopically Uncertain Systems

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Abstract—In this paper we present an algorithm for the design of robust model based predictive controller for polytopically uncertain systems via parametric programming. A min-max approach is adopted to design robust parametric predictive controllers, which guarantees feasible plant operation for maximum violation of polytopic uncertainty in system matrices and bounded input disturbances. The resulting piecewise affine optimal control law is a function of states and can be implemented on-line as a sequence of simple function evaluations. An example is presented to illustrate the details of the proposed robust parametric predictive controller design.

I. INTRODUCTION

Influence of uncertainty to process systems engineering applications is inevitable due to parameter variations and exogenous disturbances, which may cause infeasibility—severely affecting the economics and systems operational performance. Plant models represented by linear time-varying (LTV) systems cover parameter dependent uncertainty and external additive disturbances that can be represented by multi-model or polytopic systems. In order to absorb the disturbances and minimize the impact of uncertainty on desired outputs, a key control objective is to achieve robust stability and robust performance while guaranteeing economics, performance and operational safety.

Kothare *et al.*, [1], ensure robustness to worst-case uncertain systems by implementing a robust model-based predictive control (MPC) policy using Linear Matrix Inequalities (LMIs), [2]. However, their approach requires rigorous on-line computations for which the off-line design was later reported in Wan and Kothare, [3]. Wang and Rawlings, [4], model the time-varying model combination trajectory of the uncertain models, similar to the state tree trajectory of Scokaert and Mayne, [5]. While by using dynamic programming Diehl and Björnberg, [6], have proposed a min-max MPC. Some of the other approaches based on invariant set computations, [7], [8], [9], are [10], [11], [12] and [13].

Recently, by amending to the explicit control policy introduced by Bemporad *et al.*, [14], Sakizlis *et al.*, [15], have proposed an off-line robust MPC strategy for constrained linear dynamical system via parametric programming; however polytopic uncertainty is not explicitly addressed. Bemporad *et al.*, [16], have considered both additive and polytopic uncertainty and proposed parametric min-max control formulation based on linear performance criterion.

Although the linear performance criteria is computationally desirable, it is unsuitable for process control applications due to disadvantage that it may yield either dead-beat or idle control performance, [17].

The present work achieves robustness for the polytopically uncertain system by adhering to the quadratic performance criteria enforced by the popular constrained MPC. We propose a min-max formulation to derive the explicit control policy to safeguard against the worst-case uncertainty scenario, thus guaranteeing process feasibility as well as process stability for efficient plant operation.

The paper is organized as follows. Section II presents the problem formulation. Section III presents theoretical developments of the proposed controller design. The feasibility of the system is guaranteed by using the flexibility analysis theory of Pistikopoulos and Grossmann, [18], while the stability is achieved via LMI principles. Design examples are studied in section IV.

II. PROBLEM STATEMENT

In this paper we intend to solve the following constrained MPC problem for the uncertain linear time-varying (LTV) system:

$$\Gamma(x(0)) = \min_{u(\cdot)} \left\{ \begin{array}{l} \sum_{k=0}^{N-1} x(k)^T Q x(k) + u(k)^T R u(k) \\ + x(N)^T P x(N) \end{array} \right\} \quad (1)$$

$$s.t. \quad x(k+1) = A(\theta)x(k) + B(\theta)u(k) + Gw(k)$$

$$x(k) \in \mathcal{X} \subseteq \mathbb{R}^n$$

$$u(k) \in \mathcal{U} \subseteq \mathbb{R}^m$$

$$w(k) \in \mathcal{W} \subseteq \mathbb{R}^p$$

$$\theta \in \Theta \subseteq \mathbb{R}^q$$

$$x(N) \in \mathcal{O}_\infty \subseteq \mathcal{X}; \quad \forall k = 0, \dots, N-1$$

where $x(k)$ are states with $x(0) = x_0$ being the current state, $u(k)$ are control input, $w(k)$ are external disturbances with $\mathcal{W} \equiv \{w_{lb} \leq w(k) \leq w_{ub}; \forall w(k) \in \mathbb{R}^p\}$, and θ are the uncertain system parameters with $\Theta \equiv \{\theta_{lb} \leq \theta(k) \leq \theta_{ub}; \forall \theta \in \mathbb{R}^q\}$. System matrices $[A(\theta), B(\theta)] \in \Omega$, where Ω is a prespecified polytope.

$Q \succeq 0$ and $R \succ 0$ are the weighting matrices for state and control while positive definite P is the terminal cost for the prediction horizon N , which satisfies a certain Lyapunov criterion and is used to guarantee asymptotic stability of the MPC. Additionally, we assume that both \mathcal{X} and \mathcal{U} are closed polyhedral sets containing the origin in their interiors. Furthermore after the N^{th} time step we enforce the states to lie in the positive invariant set, [8], [9], containing the origin in its interior by defining the positive invariant set $\mathcal{O}_\infty \subseteq \mathcal{X}$:

$$\mathcal{O}_\infty \equiv \left\{ \begin{array}{l} x(k) \in \mathfrak{X}^n, u(k) \in \mathfrak{U}^m \mid Kx(k) \in \mathcal{U}, \\ (A_i + B_i K)x(k) + Gw(k) \in \mathcal{X}; \forall w \in \mathcal{W} \end{array} \right\} \quad (2)$$

where K is the optimal feedback gain.

Using the current state information, $x(0)$, problem (1) is solved to derive the predicted control policy, $u(\cdot)$ while only the 1^{th} input sequence is implemented to derive the new state predictions, $x(\cdot)$. Now, using the new current states the problem (1) is solved repetitively and the procedure is continued at each time step in receding horizon manner to obtain the next forecast of control actions.

A. Model Reformulation

The system model is reformulated into a multi-model polytopic system by defining the $A(\theta)$ and $B(\theta)$ as a convex hull of the polytopic system given by,

$$\Omega \equiv \left\{ \begin{array}{l} A(\theta) = \sum_{i=1}^s A_i \lambda_i; B(\theta) = \sum_{i=1}^s B_i \lambda_i \\ \sum_{i=1}^s \lambda_i = 1; \forall \lambda_i \geq 0; i = 1, \dots, s \end{array} \right\}. \quad (3)$$

Now, if we satisfy the feasibility and stability for each of the i^{th} polytopic system in Ω , then we guarantee the feasibility and stability for the overall system $x(k+1) = \sum_{i=1}^s A_i \lambda_i x(k) + \sum_{i=1}^s B_i \lambda_i u(k) + Gw(k)$. Next, by substituting for $x(k) = A_i^k x(0) + \sum_{j=0}^{k-1} \{A_i^j B_i u(k-1-j) + A_i^j Gw(k-1-j)\}$ for every i^{th} system into constraint set \mathcal{X} and \mathcal{U} , we reformulate problem (1) into a multiparametric quadratic program as

$$\Gamma(x(0)) = \min_U \underset{W \in \mathcal{W}^N}{E} [\Phi(x(0), U, W)] \quad (4)$$

s.t. $g_j(x(0), U, W) \leq 0$

where the column vector $U \triangleq [u(0)^T, \dots, u(N-1)^T]^T$ is the optimization vector, the column vector $W \triangleq [w(0)^T, \dots, w(N-1)^T]^T$ is the expected disturbance vector. After substituting $x(k) = \sum_{i=1}^s \{A_i^k x(0) + \sum_{j=0}^{k-1} \{A_i^j B_i u(k-1-j) + A_i^j Gw(k-1-j)\}\}$ into the problem (1) gives $\Phi(x(0), U, W)$ as the quadratic objective function and $g_j(x(0), U, W)$ as all the exponentially many combinations of the system constraints written in explicit form with $j = 1, \dots, J$ where $J = nNs^N$.

III. THEORETICAL DEVELOPMENTS

A. Feasibility

Definition 1.1: The robust polytopic parametric predictive controller steers the plant into the feasible operating region for a specific range of uncertain variations, $\forall k \geq 0$.

Using the flexibility analysis theory of Pistikopoulos and Grossmann, [18], the maximum violation of the disturbances define the feasible operating region. This feasible region is enclosed by the feasibility constraints, $\psi(x(0), U) \leq 0$ given by,

$$\psi(x(0), U) \leq 0 \Leftrightarrow \max_{W, j} \left\{ \begin{array}{l} g_j(x(0), U, W); \forall j = 1, \dots, J \\ U \in \mathcal{U}^N, x(0) \in \mathcal{X}, W \in \mathcal{W}^N \end{array} \right\}. \quad (5)$$

Equation (5) can be solved by identifying critical uncertainty points for each maximization as,

$$\text{if } \frac{\partial g_j}{\partial w(k)} > 0 \Rightarrow w(k)^{cr} = w(k)^{ub}; \forall j \in J, \forall k \geq 0 \quad (6)$$

$$\text{if } \frac{\partial g_j}{\partial w(k)} < 0 \Rightarrow w(k)^{cr} = w(k)^{lb}; \forall j \in J, \forall k \geq 0. \quad (7)$$

Thus, by substituting the sequence of critical uncertainty, $w(k)^{cr}$ in the constraints set $g_j(x(0), U, W)$, a multiparametric linear program (mpLP) is formulated as,

$$\begin{aligned} \psi(x(0), U) &= \max_j g_j(x(0), U, W^{cr}) \quad (8) \\ &= \min_\varepsilon \left\{ \begin{array}{l} \varepsilon \geq g_j(x(0), U, W^{cr}) \\ \forall j \in J, U \in \mathcal{U}^N, \\ x(0) \in \mathcal{X}, W \in \mathcal{W}^N \end{array} \right\}. \quad (9) \end{aligned}$$

Equation (9) can then be solved using the formal comparison procedure of Acevedo and Pistikopoulos, [19].

B. Stability and Terminal Cost

Definition 2.1: Assuming pairs (A_i, B_i) are both stabilizable and detectable, system (A_i, B_i) is asymptotically stable if there exists quadratic Lyapunov function given by $V(x(k)) = x(k)^T P x(k) > 0$.

The following theorem finds the positive definite P that achieves stability for all the pairs of A_i, B_i , in turn for system $(A(\theta), B(\theta))$.

Theorem 1 (Lyapunov stability): According to Lyapunov stability theorem, an open-loop system is stable if and only if

$$\{\exists P = P^T \succ 0 \mid A_i^T P A_i - P < 0; \forall i \in s\} \quad (10)$$

and closed-loop system pairs (A_i, B_i) are stable if and only if

$$\left\{ \begin{array}{l} \exists P = P^T \succ 0 \mid \\ (A_i + B_i K)^T P (A_i + B_i K) - P < 0; \forall i \in s \end{array} \right\}. \quad (11)$$

With $X = P^{-1}$ and $Y = KX$ equation (11) is converted into following set of LMIs, (for proof see, [2]).

$$\left[\begin{array}{cc} X & (A_i X + B_i Y)^T \\ (A_i X + B_i Y) & X \end{array} \right] \succ 0. \quad (12)$$

Remark 3.1: Equations (10) and (12) are both LMIs. After N^{th} , time step the control law $u(k) = Kx(k)$, with control gain $K = YX^{-1}$ is implemented.

C. Open-Loop Robust Parametric Predictive Controller

The feasibility constraints from section III-A are incorporated in problem (4) to obtain the following open-loop robust predictive control problem,

$$\Gamma_{ol}(x(0)) = \min_{U, W \in \mathcal{W}^N} E [\Phi(x(0), U, W)] \quad (13)$$

$$s.t. \quad g_j(x(0), U, W) \leq 0$$

$$\max_{W, j} \left\{ \begin{array}{l} g_j(x(0), U, W); \quad \forall j = 1, \dots, J \\ U \in \mathcal{U}^N, x(0) \in \mathcal{X}, W \in \mathcal{W}^N \end{array} \right\} \leq 0.$$

This open-loop robust predictive control problem is a bilevel optimization problem. Note that the inner minimization problem is equivalent to equation (9), which can be solved separately resulting into a set of linear feasibility constraints, $\psi(x(0), U) \leq 0$. Substituting it into equation (13) results in the following single-level optimization problem,

$$\begin{aligned} \Gamma_{ol}(x(0)) &= \min_{U, W \in \mathcal{W}^N} E [\Phi(x(0), U, W)] \quad (14) \\ s.t. \quad &g_j(x(0), U, W) \leq 0 \\ &\psi(x(0), U) \leq 0. \end{aligned}$$

D. Closed-Loop Robust Parametric Controller

As discussed in Sakizlis *et al.*, [15], the drawback of open-loop is that it does not take into account the future measurements which contain the information of past uncertainty. The closed-loop problem which preserves feasibility for all uncertainty realizations is achieved by finding feasibility constraints at every time step is given as follows.

$$\Gamma_{cl}(x(0)) = \min_{U, W \in \mathcal{W}^N} E [\Phi(x(0), U, W)] \quad (15)$$

$$s.t. \quad g_j(x(0), U, W) \leq 0$$

$$\max_{w(0)} \min_{u(1)} \dots \max_{w(N-2)} \min_{u(N)} \max_{w(N-1), j} \left\{ \begin{array}{l} g_j(x(0), U, W); \quad \forall j = 1, \dots, J \\ u \in \mathcal{U}^N, x(0) \in \mathcal{X}, w \in \mathcal{W}^N \end{array} \right\} \leq 0.$$

$$\max_{w(1)} \min_{u(2)} \dots \max_{w(N-2)} \min_{u(N)} \max_{w(N-1), j} \left\{ \begin{array}{l} g_j(x(0), U, W); \quad \forall j = 1, \dots, J \\ u \in \mathcal{U}^N, x(0) \in \mathcal{X}, w \in \mathcal{W}^N \end{array} \right\} \leq 0.$$

$$\vdots$$

$$\max_{w(N-1), j} \left\{ \begin{array}{l} g_j(x(0), U, W); \quad \forall j = 1, \dots, J \\ u \in \mathcal{U}^N, x(0) \in \mathcal{X}, w \in \mathcal{W}^N \end{array} \right\} \leq 0.$$

Equation (15) is a multi-level program which can be solved separately similar to the open-loop solution procedure. Thus, for each time step the inner max problem is solved backwards in time for the entire horizon N . The resulting flexibility constraints are then incorporated into the main MPC problem (1) which is reformulated to the following mpQP similar to the open-loop case,

$$\Gamma_{cl}(x(0)) = \min_{U, W \in \mathcal{W}^N} E [\Phi(x(0), U, W)] \quad (16)$$

$$s.t. \quad g_j(x(0), U, W) \leq 0$$

$$\psi_f^{w(0)}(x(0), u(0)) \leq 0$$

$$\psi_f^{w(1)}(x(0), [u(0)^T, u(1)^T], w(0)) \leq 0$$

\vdots

$$\psi_f^{w(N-1)} \left(x(0), [u(0)^T, \dots, u(N-1)^T], [w(0)^T, \dots, w(N-2)^T] \right) \leq 0.$$

Remark 3.2: The solution obtained in sections III-C and III-D both are obtained as a piecewise affine optimal robust parametric predictive control policy as a function of states $U(x(0))$ for the critical polyhedral regions in which plant operation is stable and feasible $\forall w(k) \in \mathcal{W}$ and $\forall [A(\theta), B(\theta)] \in \Omega$.

IV. DESIGN EXAMPLE

Example 1: Mass Spring Damper System

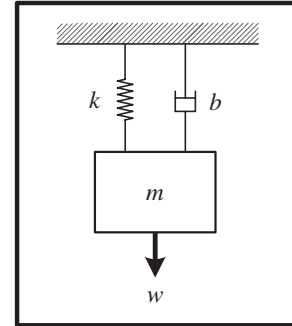


Fig. 1. Mass spring damper system

Consider the following state space model of a underdamped mass spring damper system shown in Figure 1,

$$\begin{bmatrix} \frac{dr}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k(\theta) & b(\theta) \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

where r is position, v is velocity, u is the applied force and w is the disturbance force with maximum variation between $[-0.1, 0.1]$.

The control objective is to maintain the position of the mass at steady state when disturbances perturb the

systems by applying the force with limits $[-1, 1]$ without exceeding the maximum velocity of the block between its limit of $[-0.4, 0.4]$. The maximum allowable limit for the position of the mass is between $[-0.2, 0.2]$. At two different scenarios we have $[k(\theta), b(\theta)] = Co[-(0.1, 0.3), -(1, 1)]$. These extreme vertices can be seen as the different values of spring constant and damper coefficients, i.e., conditions when the system is new and at the time it is salvaged. Thus, the goal is to achieve desired control objectives during the entire life-span of the system.

The system is discretized using sample and hold with sampling time of one second. Weights are chosen to be $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$, $R = 1$ and prediction horizon $N = 2$.

From Theorem 1, stabilizing $P = \begin{bmatrix} 1.4346 & 0.2684 \\ 0.2684 & 1.0139 \end{bmatrix}$

and unconstrained control gain is $K = [-0.2241 \quad -0.9253]$. Fig. 2 depicts the nominal control policy without disturbances, illustrating infeasible operation if started from the initial condition at $[0.05 \quad 0.3]$. Thus, the resulting open-loop robust optimal control policy as a function of states is shown in Fig. 3 and the simulation results with nominal uncertainty values are depicted in Fig. 4 and 5. Fig. 6 and 7 illustrates the scenario when the random persistent disturbance with zero mean and variance of 0.1^2 enters the system.

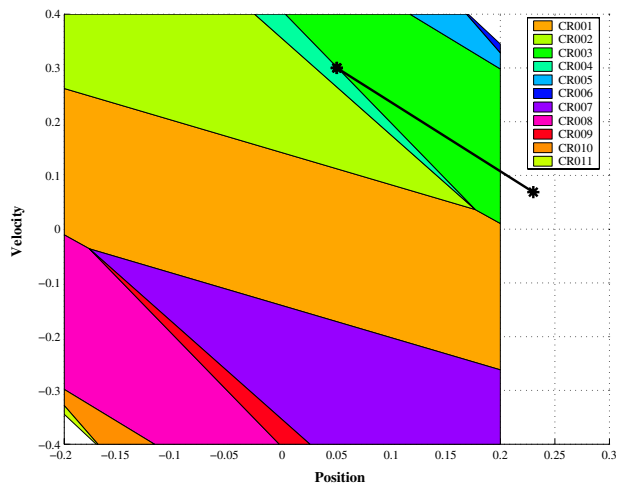


Fig. 2. State-space partition for nominal control

V. CONCLUSION

This paper presents an explicit solution to the robust MPC for polytopically uncertain linear systems via parametric programming. A min-max based feasibility analysis to deal with the worst case uncertainty is described. The open-loop and closed-loop controller performance guarantees system stability and feasible operation. The resulting controllers yield a piecewise affine control law which can be implemented on-line by simple function evaluations. The

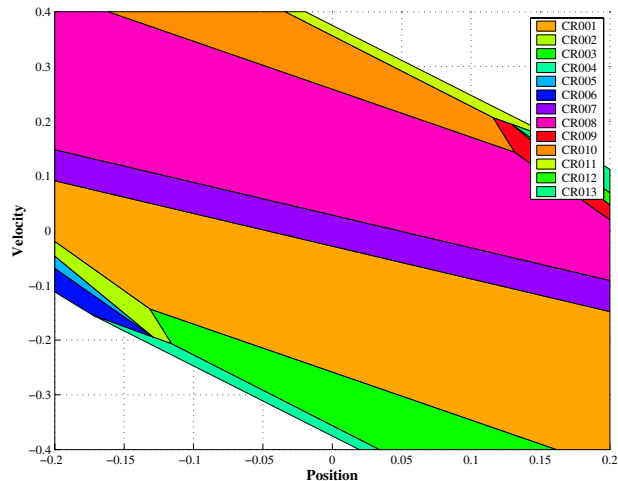


Fig. 3. State-space partition for open-loop robust control

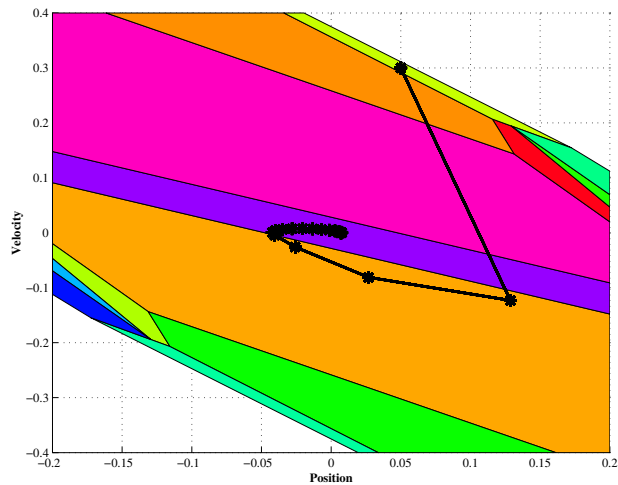


Fig. 4. Nominal state profiles

proposed framework can be extended to account for the robust control of hybrid systems, [20].

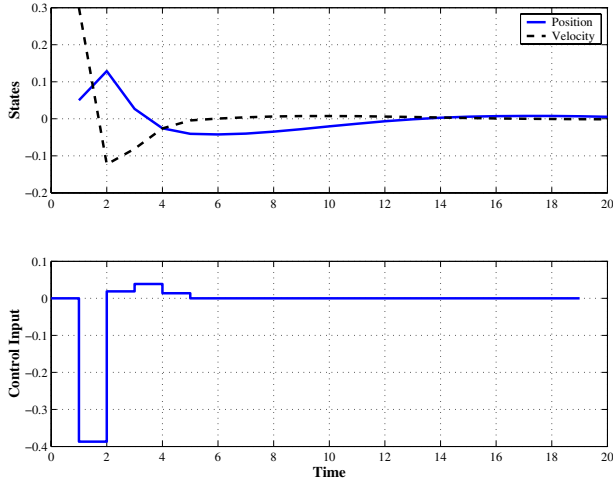


Fig. 5. Nominal state and control Trajectories

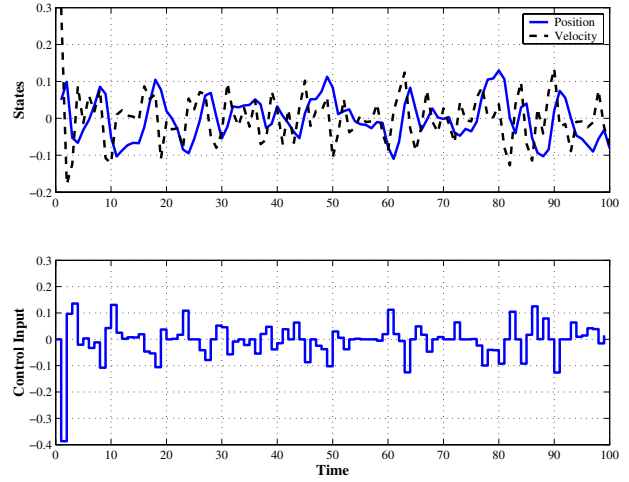


Fig. 7. State and control trajectories under persistent disturbances

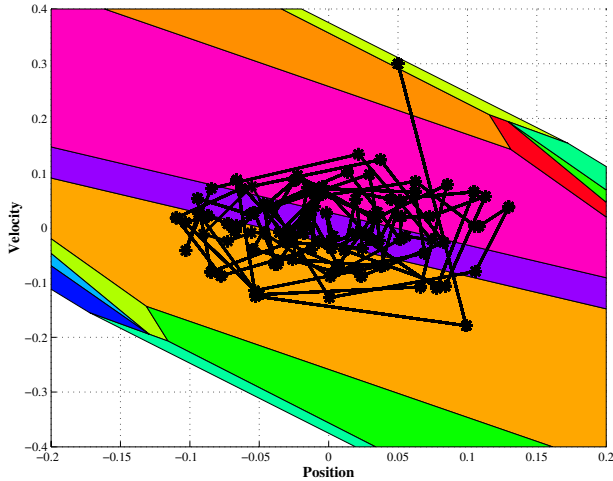


Fig. 6. State profiles under persistent disturbances

VI. NOTATIONS

Symbol	Meaning
$x(0)$	current state
θ	parametric uncertainty
$x(k), u(k), w(k)$	state, input and disturbance variables
$x_{lb}, u_{lb}, w_{lb}, \theta_{lb}$	lower bounds on x, u, w, θ
$x_{ub}, u_{ub}, w_{ub}, \theta_{ub}$	upper bounds on x, u, w, θ
A_i, B_i, G	system matrices
n, m, p, q, s	number of x, u, w, θ and system models
N	prediction horizon
$\mathcal{X}, \mathcal{U}, \mathcal{W}, \Theta$	bounded polytope for x, u, w, θ
\mathcal{O}_∞	positive invariant set
Ω	polytopic set defined by $Co(A_i, B_i)$
$Q, R,$	weighting matrices
X, Y	LMI variable
P	positive definite terminal penalty matrix
K	proportional controller gain
U, W	vector of input and disturbances
ψ	feasibility constraint
w^N	disturbance at N^{th} time step
$E_{W \in \mathcal{W}^N}$	expected value for the disturbance vector

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