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Finite Horizon Optimal Control of Switched Distributed Parameter Systems with Moving Actuators

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***Abstract*— The objective of this work is to provide a theoretical formulation for optimal switching of a moving (or scanning) actuator for distributed parameter systems. The proposed hybrid controller switches both the location and control signal of the plant at the beginning of a time interval and remains unchanged over the duration of the time interval. This is repeated for different time intervals. The method employed is based on LQR optimal control. First, a set of admissible locations for which the moving actuator can reside at throughout the duration of a given time interval is considered. Guiding the moving actuator at these a priori selected positions at different time intervals is made possible by solving a double optimization problem. Further, an optimal set of admissible locations at which the moving actuator can reside at throughout the duration of a given time interval is chosen. A numerical example with simulation results is presented.**

KEYWORDS: Moving actuator, actuator scheduling, distributed parameter systems (DPS), optimal control, hybrid systems, switched systems.

I. INTRODUCTION

The motivation of this work stems from recent work dealing with issues pertaining to the optimal (in certain sense) placement of actuators and sensors for DPS. For example, a commonly used method in flexible structures, employs modal controllability and observability indices/grammians to judiciously locate sensors and actuators for flexible space structures (FSS). A more detailed treatment of this along with a rather exhaustive and very descriptive survey on actuator and sensor placement for FSS can be found in the book by Gawronski [12], the book by Moheimani [15] and also in the survey of van de Wal and de Jager in [18]. For highly dissipative linear parabolic systems, methods based on numerical considerations were presented by Burns and King [2], and Burns and Rubio [3]. An analog to a portion of the current manuscript which is in the same spirit, namely using LQR/LQG control measures to find the optimal actuator and sensor locations was presented by Geromel, [13], [6] for finite dimensional systems.

The rationale for this note is based on the tenet that as the ultimate goal of the control task is the design and implementation of the best (with respect to an appropriate measure) controller (robust, cheap), then the choice of the actuator/sensor location should naturally be based on a control point of view. Specifically, using optimal control law based on Linear Quadratic Regulator (LQR) or, in the absence of full state measurements, Linear Quadratic Gaussian (LQG), the idea is to embed the actuator/sensor location problem into the control design problem. To be precise, for arbitrary but fixed actuator and sensor location, one can obtain the optimal value associated with the $\mathcal{H}_2$/LQG, or alternatively $\mathcal{H}_\infty$, performance index. Finding the minimum of this optimal value (of the performance index) would then give the "best" location over all possible combinations of sensor and actuator locations. If the problem at hand is to minimize the performance index by finding a control policy, then to find the minimum over all optimal values appears to be the *truly optimal way of choosing the sensor and actuator locations*. This was treated in the book by Omatu and Seinfeld [16] and references therein where both the optimal sensor and actuator locations for a class of distributed parameter systems were found using LQR-type measures. This was also treated in [7] for systems governed by parabolic PDEs.

In the case of moving actuators, which is the main focus of this manuscript, fewer results can be found, see for example [8], [9]. Whether one has many actuators, activates only one during a given time interval and switches to a new actuator in a subsequent time interval, *or* has a single actuator that is mobile and resides at a specific location for the duration of a certain time interval, it has little effect on the proposed algorithm as the two cases are in fact equivalent. The treatment here is somewhat different from earlier efforts of a moving actuator in FSS, in the sense that it is based on minimizing the optimal value of a performance index over a time interval in order to find the optimal actuator position for that interval, as opposed to considering observability/controllability and energy measures. While earlier works [8], [9] concentrated on the switching of force or moment actuators on flexible structures using LQR/LQG measures, they only provided a suboptimal algorithm for the actuator scheduling based on the solution of an associate Algebraic Riccati Equation (ARE). This approach sacrificed actuator switching optimality in favor of computational efficiency. A similar scheme was presented in [11], [17], [10] for thermal manufacturing systems which are governed by parabolic DPS and which similarly used a suboptimal actuator guidance policy in order to attain ease of real-time implementation and achieve savings in computational load.

The system under consideration is assumed to have multiple *but* finite actuator candidate positions with only a single location used by the actuating device over a certain time interval. The position of the activating device

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changes at the beginning of a new time interval and remains unchanged throughout the duration of the interval. If power restrictions/considerations only allow a smaller set (or even just one) of the available actuators to be active in a given time interval, then a performance-based actuator allocation would exhibit enhanced performance over a stationary actuator allocation would. By assuming a fixed dwell time, necessary for stability under switching [14], the choice of the actuator location will be changing at the beginning of each new time interval using a performance criterion. The novelty here is that *both* the actuator position and its associated stabilizing control signal are switched at the beginning of every time interval using a performance criterion.

The problem under study is formulated in the next section. Prior results on the finite horizon optimal control are summarized in Section 3. The algorithm for the controller and actuator switching for a family of switched distributed parameter systems is given in Section 4 along with a proof of the optimality of the algorithm. An example of the heat equation system described by a parabolic partial differential equation along with its numerical results are presented in Section 5. Conclusions with directions for future research are also included in Section 6.

II. OPTIMAL CONTROL PROBLEMS ON A FINITE-TIME INTERVAL

Let $(S_p)_{p\in\mathcal{P}}$, for some index set $\mathcal{P}$, be the family of linear continuous-time systems which, for each fixed $p\in\mathcal{P}$, is given by a state linear system (A,B_p,C_p) of the form

$$(S_p)\begin{cases} \dfrac{d}{dt}z(t)=Az(t)+B_pu(t),\\ y(t)=C_pz(t)\end{cases} \tag{1}$$

where the operator A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Hilbert space Z, and B_p and C_p are bounded linear operators from a Hilbert space U to Z, and from Z to a Hilbert space Y, respectively. We consider the finite-time interval $[t_0,t_f]$. To $(S_p)_{p\in\mathcal{P}}$, the set of all possible switches between the given systems

$$\Sigma=\{\sigma\mid\sigma:[t_0,t_f]\to\mathcal{P}\text{ piecewise constant}\}$$

is associated. The *family of switched systems* $((S_p)_{p\in\mathcal{P}},\Sigma)$ taken under consideration in this paper are the hybrid dynamical systems consisting of the family of continuous-time systems $(S_p)_{p\in\mathcal{P}}$ together with all switching rules $\sigma\in\Sigma$, all initial states $z(0)=z_0\in Z$, and all inputs $u\in L_2([t_0,t_f];U)$. Denote by $t_0<t_1<t_2<...<t_n=t_f$, where $n\in\mathbb{N}\setminus\{0\}$, the finite set of switching time instants, namely the discontinuities of the switching functions σ.

Suppose that a switching function σ, an initial condition z_0 and an input u are given. Then, on each interval $\{[t_i,t_{i+1})\}_{i=0}^{n-1}$, the state $z_\sigma(t)$ of the switched system $((S_p)_{p\in\mathcal{P}},\sigma)$ is the mild solution of the Cauchy problem (1) (see [5, Chapter 3]), i.e. for $t_i\le t\le t_{i+1}$

$$z_\sigma(t)=T(t)z_\sigma(t_i)+\int_{t_i}^{t}T(t-s)B_{\sigma(t_i)}u(s)ds. \tag{2}$$

Then, the output trajectory on each interval $\{[t_i,t_{i+1})\}_{i=0}^{n-1}$ is

$$y_\sigma(t)=C_{\sigma(t_i)}z_\sigma(t). \tag{3}$$

The initial conditions at the beginning of each interval are $\{z_\sigma(t_i)\}_{i=0}^{n-1}$, and they are considered to be the end values of the solution on the previous time-interval. Consequently, the mild solution of any of the switched system $((S_p)_{p\in\mathcal{P}},\sigma)$ is continuous. With the trajectories (2),(3) we associate the following cost functional

$$\begin{aligned}J_\sigma(z_0;u,t_0,t_f)&=\langle z_\sigma(t_f),Mz_\sigma(t_f)\rangle\\&\quad+\int_{t_0}^{t_f}\Big(\langle y_\sigma(s),y_\sigma(s)\rangle+\langle u(s),Ru(s)\rangle\Big)ds,\end{aligned} \tag{4}$$

where $z_0\in Z$ is the initial condition, $u\in L_2([t_0,t_f];U)$ is the input trajectory, M is a self-adjoint, nonnegative, bounded operator on Z and R is a coercive, bounded operator on U.

The aim is to minimize the cost (4) over all possible trajectories (2),(3) of the switched systems from the given family $((S_p)_{p\in\mathcal{P}},\Sigma)$. One can formulate the following double optimization problem.

Problem 1: Given a family of switched systems $((S_p)_{p\in\mathcal{P}},\Sigma)$, and an initial condition $z_0\in Z$, find an optimal control $u_{opt}\in L_2([t_0,t_f];U)$ and an optimal switching function $\sigma_{opt}\in\Sigma$ that minimize the cost functional (4) over all possible trajectories (2),(3). In other words, find

$$J_{opt}=\min_{\sigma\in\Sigma,\,u\in L_2([t_0,t_f],U)}J_\sigma(z_0;u). \tag{5}$$

Additional assumptions, motivated from engineering applications and specifically thermal manufacturing [10], are introduced. In these manufacturing applications, the actuator device (heat source) can most likely be mounted on a robotic arm, and thus cannot transverse large distances in infinitesimal time intervals. In addition, due to computational restrictions, on-line optimization at every time instance is not feasible. Instead, it is assumed that the moving actuator can reside at a given location for a time interval of small duration (dwell time) whose length is dictated by hardware limitations and bandwidth, and closed loop system stability bounds. Thus, we will look at the optimal problem at each of the subintervals. For simplicity, in this exposition we consider time intervals of constant length.

Assumption 1 (Finite candidate locations): There are only a finite number $m\ge 2$ of admissible locations for the moving actuator. Denote by $\mathcal{P}^m=\{p_1,p_2,\dots,p_m\}\subset\mathcal{P}$ the corresponding index set, and by Σ^m the set of switching functions σ.

Assumption 2 (Zero transverse time): The time required by the actuating device to transverse from location $p_1\in\mathcal{P}$ at the end of the time interval $[t_i,t_i+\Delta t]$ to location $p_2\in\mathcal{P}$ for the subsequent interval $[t_i+\Delta t,t_i+2\Delta t]$ is negligible and may be assumed to be zero, i.e. one has inertialess moving source. Alternatively, one may assume that there are many actuators available, and only one is to be activated and remain active throughout a given time interval. Thus the actuator at location $p_1 \in \mathcal{P}$ will be active during the time interval $[t_i, t_i + \Delta t]$ and the actuator at location $p_2 \in \mathcal{P}$ will be activated for the subsequent interval $[t_i + \Delta t, t_i + 2\Delta t]$ with no additional time required for activating and disengaging p_2 and p_1, respectively.

Assumption 3 (Minimum residence time): The choice of the residence time Δt is chosen to be larger than the minimum dwell time τ_d [14], allowable for stability under switching.

Under the above assumptions, one may now formulate the following two optimal problems.

Problem 2: Given a family of switched systems $((S_p)_{p\in\mathcal{P}^m}, \Sigma^m)$ which satisfies Assumptions 2 and 3, and an initial condition $z_0 \in Z$, find an optimal control $u_{opt} \in L_2([t_0, t_f]; U)$ and an optimal switching function $\sigma_{opt} \in \Sigma$ that minimize the cost functional $J_\sigma(z_0; u, t_0, t_f)$ over all possible trajectories (2),(3), i.e. solve

$$J^m_{opt}(z_0; t_0, t_f) = \min_{\sigma\in\Sigma^m,\, u\in L_2} J^m_\sigma(z_0; u, t_0, t_f). \tag{6}$$

Problem 3: Given a family of switched systems $((S_p)_{p\in\mathcal{P}}, \Sigma)$ which satisfies Assumptions 2 and 3, and an initial condition $z_0 \in Z$, find an optimal $\mathcal{P}^m_{opt} \subset \mathcal{P}$ that minimizes

$$J_{opt} = \min_{\mathcal{P}^m_{opt}\subset\mathcal{P}} J^m_{opt}. \tag{7}$$

Problem 2 and Problem 3 have two, respectively three, degrees of freedom. We seek an optimal control from a class of available controls. Problem 2 and Problem 3 can be found in the general classification provided in [4, Figure 1.4, page 45].

An algorithm for solving the optimal control problem on a finite-time interval for switched distributed parameter systems will be proposed after recalling some standard results from optimal control for distributed parameter systems.

III. STANDARD OPTIMAL CONTROL ON A FINITE-TIME INTERVAL

In this section we summarize well-known results from [5, Chapter 6]. For each fixed actuator location $p \in \mathcal{P}^m$, let M_p be a self-adjoint, nonnegative, bounded operator on Z and R as in (4). The optimal control signal that minimizes the finite horizon cost on the finite time interval $[t_a, t_b]$

$$\begin{aligned} J_p(z_0; t_a, t_b, u) &= \langle z_p(t_b), M_p z_p(t_b)\rangle \\ &+ \int_{t_a}^{t_b} \Big(\langle y_p(s), y_p(s)\rangle + \langle u(s), Ru(s)\rangle\Big)\, ds, \end{aligned} \tag{8}$$

over all trajectories of the system S_p ($\sigma(t) = p$ for all $t \geq 0$), $z_0 = z(t_a)$, is given by

$$\begin{aligned} u^{opt}_p(t; z_0, t_a, t_b) &= -R^{-1}B^*_p \Pi_p(t) z^{opt}_p(t; z_0, t_a, t_b) \\ &= K_p(t) z^{opt}_p(t; z_0, t_a, t_b), \end{aligned} \tag{9}$$

with $K_p(t) \triangleq -R^{-1}B^*_p\Pi_p(t)$. The optimal state, $z^{opt}_p(t; z_0, t_a, t_b)$, is the mild solution of the abstract evolution equation

$$\begin{aligned} \frac{d}{dt}z(t) &= (A - B_p K_p(t))z(t) := A_{cl,p}(t)z(t) \\ z(t_a) &= z_0. \end{aligned} \tag{10}$$

Then $A_{cl,p}(t)$ is the generator of the mild evolution operator $U_{cl,p}(t,s)$ on the set $\{(t,s); t_a \leq s \leq t \leq t_b\}$ and $z^{opt}_p(t; z_0, s, t_b) = U_{cl,p}(t,s)z_0$.

The self-adjoint nonnegative operator $\Pi_p(t) \in \mathcal{L}(Z)$ for all $t \in [t_a, t_b]$, and satisfies the Operator Differential Riccati Equation (ODRE)

$$\begin{aligned} &\frac{d}{dt}\langle\phi, \Pi_p(t)\psi\rangle = -\langle\phi, \Pi_p(t)A\psi\rangle \\ &\quad -\langle A\phi, \Pi_p(t)\psi\rangle - \langle C_p\phi, C_p\psi\rangle \\ &\quad +\langle \Pi_p(t)B_pR^{-1}B^*_p\Pi_p(t)\phi, \psi\rangle, \;\; t \in [t_a, t_b], \\ &\Pi_p(t_b) = M_p, \end{aligned} \tag{11}$$

for $\phi, \psi \in \mathcal{D}(A)$. Moreover, it is the unique solution of this differential Riccati equation in the class of strongly continuous, self-adjoint operators in $\mathcal{L}(Z)$ such that $\langle\phi, \Pi_p(t)\psi\rangle$ is differentiable for $\phi, \psi \in \mathcal{D}(A)$ and $t \in [t_a, t_b]$.

The following relationship between the minimum of $J_p(z_0; t_a, t_b, u)$, defined by (9), and $\Pi_p(t_a)$ holds

$$\min_{u\in L_2([t_a,t_b],U)} J_p(z_0; t_a, t_b, u) = \langle z_0, \Pi_p(t_a)z_0\rangle. \tag{12}$$

IV. ALGORITHMS FOR SOLVING PROBLEM 2 AND PROBLEM 3

In this section we assume that the family $((S_p)_{p\in\mathcal{P}^m}, \Sigma^m)$ satisfies Assumptions 2 and 3 and an initial condition $z_0 \in Z$ is given. Then solutions to the formulated optimal control problems on a finite-time interval $[t_0, t_f]$, Problem 2 and Problem 3, are provided. We must first consider a set of m fixed values (locations) $\mathcal{P}^m = \{p_1, p_2, ..., p_m\}$ in which the switching function may take values (the moving actuator may reside at during certain time intervals). The expression (12) may be used to obtain $\mathcal{P}^m$ from a larger set of admissible actuator locations.

The following algorithm provides a solution for Problem 2.

Algorithm 1: Consider a family of switched systems $((S_p)_{p\in\mathcal{P}^m}, \Sigma^m)$.

Part A: Solve m ODREs in each subinterval backwards in time with the terminal condition for an interval being the initial condition from the next subinterval.

Step 1: Divide the interval $[t_0, t_f]$ into $k = [\frac{t_f - t_0}{\Delta t}]$ subintervals of length Δt. ($[\cdot]$ denotes the integer part of a real number).

Step 2: If $t_f > k\Delta t$ solve ODRE (11) with the initial condition $\Pi_{\sigma(k\Delta t)}(t_f) = M$, on the interval $[k\Delta t, t_f]$, for every possible value of $\sigma(k\Delta t)$. Set

$$M_{\sigma(k\Delta t)} = \Pi_{\sigma(k\Delta t)}(k\Delta t).$$

The number of ODREs to be solved is m. Else, for each $\sigma(k\Delta t) = p \in \mathcal{P}^m$, set $M_{\sigma(k\Delta t)} = M$ and proceed with *Step 3*.

Step 3: Solve, on the interval $[t_{k-1}, t_k]$, the ODRE (11) for each of the m-initial conditions $\Pi_{\sigma(t_{k-1})}(t_k) = M_{\sigma(t_k)}$ and each value of $\sigma(t_{k-1})$. The number of ODREs to be solved is m^2 if $t_f > t_k$ and m if $t_f = t_k$.

Step 4: Continue the procedure until the last interval $[t_0, t_1]$. The number of ODREs to be solved is m^{l+1} if $t_f > t_k$, and m^l if $t_f = t_k$.

Part B: Find the optimal switching σ^{opt} and $u^{opt}_{\sigma^{opt}}$.

Step 5: Find the minimal cost of (6) as

$$\begin{aligned} J^m_{opt}(t_0, t_f) &= \min_{\sigma \in \Sigma^m} J^m_\sigma(z_0; t_0, t_f, u^{opt}_\sigma) \\ &= \min_{\sigma \in \Sigma^m} \langle z_0, \Pi_\sigma(t_0) z_0 \rangle. \end{aligned} \tag{13}$$

Step 6: Choose σ^{opt} as the switching function corresponding to the optimal cost $J^m_{opt}(t_0, t_f)$. Then the optimal input is $u^{opt}_{\sigma^{opt}}$.

The total number of ODREs to be solved in the above algorithm is

$$\begin{aligned} &\textstyle\sum_{l=1}^{k+1} m^l = \frac{m(m^{k+1}-1)}{m-1}, && \text{if } t_f > k\Delta t, \text{ and} \\ &\textstyle\sum_{l=1}^{k} m^l = \frac{m(m^{k}-1)}{m-1}, && \text{when } t_f = k\Delta t. \end{aligned}$$

Theorem 1: Consider the switched system $((S_p)_{p\in\mathcal{P}}, \sigma)$ with the cost functional (4). For every $z_0 \in Z$ there exists a unique input $u^{opt}_\sigma(\cdot; z_0, t_0, t_f) \in L_2([t_0, t_f], U)$ such that

$$J\Big(z_0; t_0, t_f, u^{opt}_\sigma(\cdot; z_0, t_0, t_f)\Big) \le J(z_0; t_0, t_f, u)$$

for all $u \in L_2([t_0, t_f], U)$. Moreover, the Algorithm 1 is optimal.

Proof: Perform Step 1 of the Algorithm. Since the switching function σ is fixed, applying [5, Theorem 6.1.4] on each of the intervals $[t_j, t_{j+1}]$, $j = 0, \ldots, k$, $t_{k+1} := t_f$, one obtains that there exists a unique input $u^{opt}_\sigma(\cdot; z_\sigma(t_j), t_j, t_{j+1}) \in L_2([t_j, t_{j+1}], U)$ such that

$$J\left(z_\sigma(t_j); t_j, t_{j+1}, u^{opt}_\sigma(\cdot; z_\sigma(t_j); t_j, t_{j+1})\right) \le J(z_\sigma(t_j); t_j, t_{j+1}, u)$$

for all $u \in L_2([t_j, t_{j+1}], U)$. Combine all optimal inputs on subintervals in one input function $u^{opt}_\sigma \in L_2([t_0, t_f], U)$. Recall that $M_{\sigma(t_j)} = \Pi_{\sigma(t_j)}(t_j)$, where $t_j := t_0 + j\Delta t$, for $j = 1, \ldots, k$. Denote

$$\alpha_{\sigma(t_j)}(s) = \langle y_{\sigma(t_j)}(s), y_{\sigma(t_j)}(s) \rangle + \langle u^{opt}_{\sigma(t_j)}(s), R u^{opt}_{\sigma(t_j)}(s) \rangle.$$

For any fixed switching function σ, using equality (12) on intervals $[t_j, t_{j+1}]$, one can write

$$\begin{aligned} & J^m_\sigma(z_0; u^{opt}_\sigma, t_0, t_f) \\ = & \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \alpha_{\sigma(t_j)}(s) ds \\ + & \int_{t_k}^{t_f} \alpha_{\sigma(t_k)}(s) ds + \langle z(t_f), M z(t_f) \rangle \\ = & \sum_{j=0}^{k-2} \int_{t_j}^{t_{j+1}} \alpha_{\sigma(t_j)}(s) ds + \int_{t_{k-1}}^{t_k} \alpha_{\sigma(t_{k-1})}(s) ds \\ + & \langle z_{\sigma(t_k)}(t_k), M_{\sigma(t_k)} z_{\sigma(t_k)}(t_k) \rangle \\ = & \sum_{j=0}^{k-2} \int_{t_j}^{t_{j+1}} \alpha_{\sigma(t_j)}(s) ds \\ + & \langle z_{\sigma(t_{k-1})}(t_{k-1}), M_{\sigma(t_{k-1})} z_{\sigma(t_{k-1})}(t_{k-1}) \rangle \\ = & \ldots\ldots = \langle z_0, M_\sigma(t_0) z_0 \rangle. \end{aligned}$$

The above was written for $t_f > k\Delta t$, but a similar argument works for $t_f = k\Delta t$. Then

$$J(z_0; t_0, t_f, u^{opt}_\sigma(\cdot; z_0, t_0, t_f)) \le J(z_0; t_0, t_f, u)$$

for all $u \in L_2([t_0, t_f], U)$.

The optimality of the above algorithm follows from the first part of this theorem and the fact that all possible switches are considered when $J^m_{opt}(t_0, t_f)$ of (13) is computed. ■

Consider a variable initial time t_λ with $t_0 \le t_\lambda \le t_f$. In Theorem 1 no condition on the initial time t_0 was necessary. Consequently, for a fixed switching function σ, $z(t_\lambda) = z_{0,\lambda}$ and the cost functional (8) with $t_a = t_\lambda$ and $t_b = t_f$, an optimal input signal $u^{opt}_\sigma(\cdot; z_{0,\lambda}, t_\lambda, t_f)$ is obtained. Further, $z^{opt}_\sigma(\cdot; z_0, t_0, t_f)$ and $z^{opt}_\sigma(\cdot; z_{0,\lambda}, t_\lambda, t_f)$ will denote the optimal state trajectories corresponding to each of the two intervals.

The following principle of optimality follows easily from the uniqueness of the optimal trajectory for a fixed switching function, proved in Theorem 1.

Lemma 1: Consider a fixed switching function $\sigma \in \Sigma^m$. Then the equality

$$z^{opt}_\sigma(s; z_0, t_0, t_f) = z^{opt}_\sigma(\cdot; z^{opt}_\sigma(t_\lambda; z_0, t_0, t_f), t_\lambda, t_f).$$

holds for all $s \in [t_\lambda, t_f]$. The above statement is not true when the switching function is not fixed, i.e., one allows to choose the switching function during computation of the optimal input.

Some simple properties that can be associated to the above analysis are given as follows.

Lemma 2: Consider a fixed switching function $\sigma \in \Sigma^m$. Then the following statements hold:

1) The optimal trajectory does not depend on the initial choice of the time t_0.
2) If $t_0 \le t_1 \le t_2 \le t_k$, then $\Pi_\sigma(t_2) \le \Pi_\sigma(t_1)$.
3) $\Pi_\sigma(\cdot)$ is strongly continuous from the right in $[t_0, t_f]$.
4) If $t_0 \le t_1 \le t_2$, then

$$J^m_\sigma(z_0; u^{opt}_\sigma, t_0, t_1) \le J^m_\sigma(z_0; u^{opt}_\sigma, t_0, t_2).$$

Solving the optimal control problem on a finite-time interval $[t_0, t_f]$ for the class of switched systems

$((S_p)_{p\in\mathcal{P}^m},\Sigma^m)$ did not require restrictive assumptions on the component systems. As a consequence, one can apply the above algorithm in a very general setting.

Further, a solution to Problem 3 is proposed for the case when the set $\mathcal{P}$ is finite with its cardinality n (the number of available positions for placing actuators) smaller than the cardinality of the set of available actuators (chosen already to be m). The solution is very simple and it is provided by the following algorithm.

Algorithm 2: Consider all (m combination of n possible choices) families of switched systems $((S_p)_{p\in\mathcal{P}^m},\Sigma^m)$ which satisfies Assumptions 2 and 3, with the initial condition $z(t_0)=z_0\in Z$.
Step 1: For each resulting family of switched systems find, using Algorithm 1, the optimal cost $J^m_{opt}(t_0,t_f)$.
Step 2: Minimize the criterion (7) from Problem 3 to find the optimal set of locations $\mathcal{P}^m_{opt}$. Then the optimal switching function and optimal input signal are σ^{opt}_m and $u^{opt}_{\sigma^{opt}_m}$.

V. GUIDANCE OF MOVING ACTUATOR OF THE HEAT EQUATION

The above formulation of the optimization problem for switched systems is more general than one needs for providing methodologies for optimal and suboptimal efficient switching of a moving actuator for distributed parameter systems. The generality comes from the fact that, for the set of systems (1), we have not imposed the restriction that the bounded operators $(B_p)_{p\in\mathcal{P}}$ and $(C_p)_{p\in\mathcal{P}}$ model point actuators and sensors. Usually, a point actuator and/or sensor is modelled as a delta distribution in the point where the actuator is applied, which does not "always" represent a bounded operator. This is one way to view the problem and provide an optimal trajectory for the actuator position, as treated in [4]. The particular example here will consider the operators $(B_p)_{p\in\mathcal{P}}$ and which approximately model a point actuator. For example when heating a metal rod [5], one can consider $U=Y=\mathbb{C}$, $Z=L_2(\Omega)$, and for each $p\in\mathcal{P}$

$$B_p u \quad = \quad \frac{1}{2\varepsilon}\mathbf{1}_{[\xi_{0,p}-\varepsilon,\xi_{0,p}+\varepsilon]}(\xi)u, \tag{14}$$

where

$$\mathbf{1}_{[\alpha,\beta]}(\xi)=\begin{cases}1 & \text{for } \alpha\le\xi\le\beta\\ 0 & \text{elswhere}\end{cases}.$$

We consider that the state is fully accessible, i.e. $C_p=I$ for every $p\in\mathcal{P}$. For the 1-D diffusion system with Dirichlet boundary conditions we take $\Omega=[0,L]$, $L=2$. Then the system is given by

$$\begin{aligned}&\frac{\partial z}{\partial t}(t,\xi)=0.01\frac{\partial^2 z}{\partial \xi^2}(t,\xi)+\frac{1}{2\varepsilon}\mathbf{1}_{[\xi_{0,p}-\varepsilon,\xi_{0,p}+\varepsilon]}(\xi)u(t,\xi)\\ &y(t,\xi)=z(t,\xi).\end{aligned} \tag{15}$$

For this system we solved Problem 2 by implementing Algorithm . The computations were carried out via codes written in Matlab® run on a dual processor DELL® workstation(Xeon 2.8GHz, 2×2GB). The closed loop system was discretized using a spline-based Galerkin approximation scheme with 20 basis elements. The resulting finite dimensional system of ordinary differential equations was integrated using the stiff ODE solver from the Matlab® ODE library, routine `ode23s` based on a fourth order Runge-Kutta scheme. All required (spatial) integrals were computed numerically via a composite two point Gauss-Legendre quadrature rule. The resulting ODREs (11) were solved using the BDF 1-step method presented in [1].

The set of candidate positions was chosen via

$$\mathcal{P}^m=\left\{p_j\in[0,L]\,:\,p_j=\frac{jL}{m+1},\quad j=1,\dots,m\right\},$$

which accounts for $m=3,4,5$, and the LQR parameters were chosen as $R=10^{-4}$, $Q=10^3 I$, $M=I$ with $\Delta t=3$ sec and the initial condition was $z(0,\xi)=10\sin(\pi\xi)$. Due to the large memory required for storage of the Riccati solutions, the interval $[0,9]$ was divided into three intervals, thus only allowing switches at $t_1=3$ sec and at $t_2=6$ sec. To add effects of disturbances, the term

$$\Big(\mathbf{1}_{[0.2L,0.3L]}(\xi)\sin(\tfrac{\pi t}{5})+\mathbf{1}_{[0.4L,0.6L]}(\xi)\sin(\tfrac{\pi t}{5}-\tfrac{\pi}{2})\\ +\mathbf{1}_{[0.8L,0.9L]}(\xi)\sin(\tfrac{\pi t}{5}+\tfrac{\pi}{2})\Big)/200$$

was added to the right hand side of (15).

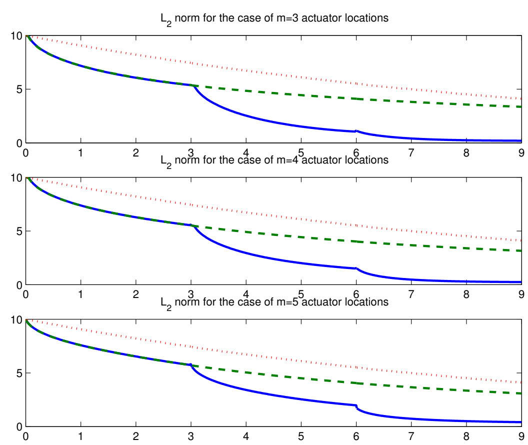

Fig. 1. $L_2(0,L)$ norm of $z(t,\xi)$ vs time for different available actuator locations.

Figure 1 depicts the evolution of the $L_2(\Omega)$ norm of $z(t,\xi)$ for the open loop (dashed), fixed actuator (dotted), and moving actuator (solid) for different choices of m. It is observed that when the actuator is allowed to move at the locations in $\mathcal{P}^m$, it improves the performance and in this case the state norm converges to zero faster than the case of a fixed actuator. Figure 2 depicts the spatial distribution versus the spatial variable ξ at $t=9$ sec. It is also observed that the convergence of the state is improved when the actuator is allowed to move. The actuator allocation for $m=3$, $m=4$ and $m=5$ is depicted in Figure 3 for both the fixed (dashed) and the switching (solid) actuator.

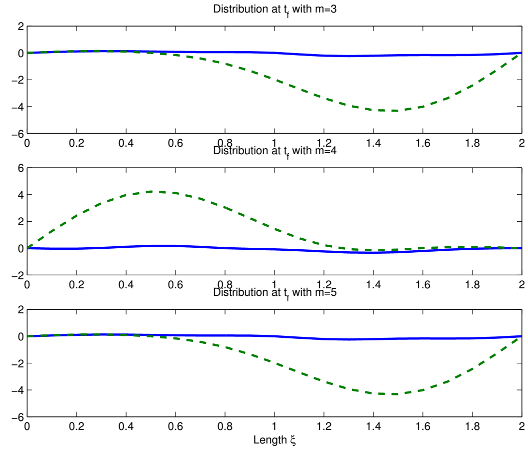

Fig. 2. Distribution at $t = t_f$ sec vs ξ for different available actuator locations.

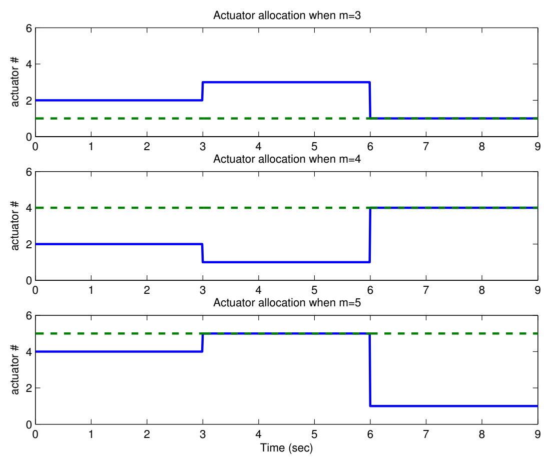

Fig. 3. Actuator allocation (switching) for $m = 3$, $m = 4$ and $m = 5$ actuator locations; switching (solid) and fixed (dashed).

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this note, we presented an algorithm and summarized the proof of the optimal switching policy of moving actuator for a class of hybrid DPS. The method employed is based on LQR optimal control on finite time interval. Unlike similar work for lumped parameter systems, here *both* the actuator and the control signal were changing at the beginning of a given time interval. The theoretical results were supported by an extensive numerical study on the heat equation with full accessible state which demonstrated the improved efficiency and enhanced performance of the moving actuator over the case of using a single stationary actuator.

B. Future Works

Due to computational issues that appear during the implementation of the optimal algorithm, we do not claim that in our theoretical approach we considered possible practical difficulties. For example we did not address in this paper the question of including disturbances or issues arising from the implementation procedure when solving a large number of ODREs. However, in the presented simulations a disturbance was added. Results providing answers to these questions appeared elsewhere or will be further investigated. Simulations for other examples will be also performed by the authors.

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