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Boundary Heat Flux Estimation In Quasi-Static Thermoelastic Systems

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***Abstract*— The paper treats a heat flux estimation problem for a system that models the temperature evolution in a thermoelastic rod, which is allowed to come into contact with a rigid obstacle. The dynamics of the process are described by a nonlinear nonlocal PDE of parabolic type. Sensor configurations, which can provide measurements guaranteeing the problem to be uniquely solvable, and relevant identification and adaptive estimation procedures are discussed.**

I. Introduction

This paper addresses an input estimation problem for a system described by the following nonlinear *nonlocal* parabolic partial differential equation [2], [3]:

$$
\begin{aligned}
&(1+a^2)\theta_t - \theta_{xx} = a\tfrac{d}{dt}\max\left\{a\int_0^1 \theta(\xi,t)d\xi - g,\ 0\right\} \\
&\qquad\qquad\qquad\qquad +\Phi(x,t),\ \text{in}\ \Omega_T, \\
&\theta_x(0,t) = \varphi(t),\ \text{in}\ \ (0,T), \\
&\theta_x(1,t) = \mathbf{u}(t),\ \text{in}\ (0,T), \\
&\theta(x,0) = \theta_0(x),\ \text{in}\ \Omega
\end{aligned}
\tag{1}
$$

where $\Omega = (0,1)$ and $\Omega_T = \Omega \times (0,T)$. In the applications, which we will describe shortly, the boundary input $u(t)$ has the following form

$$
\mathbf{u}(t) = k(g - a\int_0^1 \theta(\xi,t)d\xi)\,\theta(1,t),\ \text{in}\ (0,T), \tag{2}
$$

where $k(s)$ is a nonnegative function of real argument $s \in R$.

In the past, these type of systems have been treated for existence, uniqueness, and continuous dependence of the solution on the problem data. Various techniques were used in the literature to investigate existence and uniqueness properties of weak and strong solutions [2], [3], [6], [1]. In [17], the observability problem for system (1), (2) was studied, and in [16] these results were employed to prove the distributed controllability.

To explain the interest in studying this system, consider a long thin, homogeneous isotropic elastic rod which undergoes a motion with longitudinal, or axial, displacement. Then the displacement $u(x,t)$ and the temperature $\theta(x,t)$, where $t \geq 0$ and $0 \leq x \leq 1$, satisfy the equations of dynamic linear thermoelasticity [11]

$$
\begin{aligned}
\kappa\theta_{xx} &= \varrho c\theta_t + (3\lambda+2\mu)\alpha\theta^* u_{xt}, \\
(\lambda+2\mu)u_{xx} &= (3\lambda+2\mu)\alpha\theta_x + \varrho u_{tt},
\end{aligned}
$$

where θ^* is a constant reference temperature and the other parameters that describe thermal and elasticity properties of the material are also constant. The change of variables $x \to \frac{x}{l}$, $t \to \frac{\kappa t}{\varrho c l^2}$, $\theta \to \frac{\theta-\theta^*}{\theta^*}$, $u \to \frac{u}{l}\sqrt{\frac{\lambda+2\mu}{\varrho c\theta^*}}$, leads to the non-dimensional equations

$$
\begin{aligned}
\theta_{xx} &= \theta_t + a u_{xt}, \\
b u_{tt} - u_{xx} &= -a\theta_x,
\end{aligned}
\tag{3}
$$

where

$$
a^2 = \frac{\theta^*\alpha^2(3\lambda+2\mu)^2}{\varrho c(\lambda+2\mu)},\ b = \frac{\kappa^2}{\varrho c^2(\lambda+2\mu)l^2}.
$$

We impose the initial conditions

$$
\theta(x,0) = \theta_0(x),\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x).
$$

To establish the boundary conditions, we consider the case when the rod is situated between two walls (see Figure 1). The left end of the rod is permanently attached to a wall, while the right end is free to expand or contract and eventually may come into contact with a rigid obstacle. We choose the Signorini boundary conditions to model the displacement of the free end [8]:

$$
u(1,t) \leq g,\ \sigma \leq 0,\ (u(1,t)-g)\sigma = 0, \tag{4}
$$

where $\sigma = u_x(1,t) - a\theta(1,t)$ is the stress and the constant g is the nominal gap size between the wall and the free end in the reference configuration.

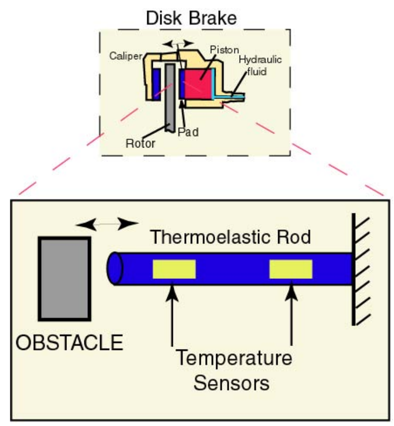

Fig. 1. The thermoelastic rod with two temperature sensors that measure average temperature over parts of the rod. The thermoelastic rod is a prototype problem for a number of applications including disk brakes.

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In situations where it is reasonable to expect small acceleration, the term bu_{tt} in (3) can be treated as negligible. This assumption (referred to as the *quasi-static* assumption) together with Signorini's boundary conditions allows the problem (3) to be decoupled and formulated in terms of the temperature only, thereby resulting in the partial differential equation (1) (see [3] for details). The functions $\Phi(x,t)$ and $\varphi(t)$ reflect the influences of the external heat sources. The condition (2) models the thermal interaction at the free end of the rod. This condition means that k is a function of the actual gap when there is no contact, and that k varies with the stress, $\sigma(t) = u_x(1,t) - a\theta(1,t)$, when there is contact.

In applied problems, for which the thermoelastic rod serves as a prototype, it may be necessary to (i) determine if there is contact with the obstacle, and if there is contact then (ii) characterize the conductivity processes at the contacting end. Since the function $k(\cdot)$ in (2) is usually poorly known[1] our objective is to construct estimation algorithms for identification of the function $\mathbf{u}(t)$ in the system (1) from the available measurements. In this paper, we examine measurements of two types:

(I) average temperature measurements:

$$y(t) = \int_0^1 \theta(\xi,t)d\xi, \quad t \in [0,T]. \tag{5}$$

(II) weighted average temperature measurements over parts of the rod:

$$z_i(t) = \int_\Omega \omega_i(\xi)\theta(\xi,t)\,d\xi, \;\; t \in [0,T], \quad i = 1,2,...,I. \tag{6}$$

We are especially interested in studying the case when the sensors are isolated from the contacting end, i.e., $\omega_i(\xi) = 0$ for all ξ sufficiently close to 1.

In the subsequent treatment we assume that the measurements are corrupted by additive measurement noise and the real data are due to either $Y(t) = y(t) + \eta(t)$ or $Z_i(t) = z_i(t) + \eta_i(t)$ where $\eta(t), \eta_i(t)$ are the measurement noises. The measurement noises are assumed to be Lebesgue measurable functions with known bounds

$$|\eta(t)| \le \mu, \;\; |\eta_i(t)| \le \mu, \;\; t \in [0,T], i = 1,2,...,I, \tag{7}$$

where μ is the bound.

We adopt the following definition.

Definition 1.1: Let $\theta_0(x)$, $\Phi(x,t)$, and $\varphi(t)$ be given and fixed. The function $\mathbf{u}(t)$, $t \in [0,T]$, is said to be identifiable from the measurements $y(t)$, $t \in [0,T]$, (respectively, $z_i(t)$, $t \in [0,T]$), if for any two solutions θ^1, θ^2 of the system (1), from the conditions $y^1(t) = y^2(t)$, $t \in [0,T]$, (respectively, $z_i^1(t) = z_i^2(t)$, $t \in [0,T]$, $i = 1,2,\cdots,I$), it follows that $\mathbf{u}^1(t) = \mathbf{u}^2(t)$, $\;t \in [0,T]$.

We now comment further on the motivation for considering the problem treated in the paper. If an estimate of $\theta_x(1,t) = u(t)$ is available, then from (1), $\theta(x,t)$ can be reconstructed at any x. From this information, and depending on a particular problem setup, the rod deformation, stresses/loads within the rod, the contact or no contact conditions, the material property dependent function k, and the wall temperature can be reconstructed. This is useful in a number of practical problems [4] for which the thermoelastic rod serves as a prototype. Specific examples of such problems include monitoring and actively controlling the thermally induced loads internal to bearings and machine tools, in situations when their internal temperature cannot be directly measured and need to be inferred. The life of these devices can be extended if their internal loads are kept within the specified limits; this in a manufacturing environment can translate into reduced down-time due to fewer machine failures. In other applications such as friction brake pads and clutches, the knowledge of spatially distributed contact pressure and of potential "no contact" regions as well as of the temperature of the contacting surfaces can be useful for monitoring the condition of these devices [18]. More complex models and finite-element approximations [9] may be required to treat these practical problems. However, useful insights into necessary sensor configurations and adaptive estimation procedures can be gained with a mathematically rigorous treatment of a prototype problem for the thermoelastic rod.

In Section II of this paper, we formulate the existence and uniqueness theorem for system (1). Section III is devoted to the study of the system with the average temperature measurements (5). Here we present an identifiability result and an on-line algorithm for estimating the function $\mathbf{u}(t)$ from the measurements $Y(t) = y(t) + \eta(t)$, and we illustrate our results with a simulation example. Some remarks on the application of the results to estimating the function k will be given. Section IV presents our results on identifiability of the function $\mathbf{u}(t)$ from the measurements (6). Finally, concluding remarks are made in Section V.

II. The Existence and Uniqueness

To formulate the existence and uniqueness result for system (1), we follow [1] and transform the variable $\theta(x,t)$ to $\tilde{\theta}(x,t)$, where

$$\tilde{\theta}(x,t) = (1+a^2)\theta(x,t) - a\max\{a\int_0^1 \theta(\xi,t)\,d\xi - g, 0\}.$$

The last integral equation can be uniquely solved for $\theta(x,t)$ and the solution is

$$\theta(x,t) = \frac{\tilde{\theta}(x,t)}{1+a^2} + \frac{a}{1+a^2}\max\{a\int_0^1 \tilde{\theta}(\xi,t)\,d\xi - a^2 g - g, 0\}. \tag{8}$$

The transformed equations are

$$\begin{aligned} &\tilde{\theta}_t = \tfrac{1}{1+a^2}\tilde{\theta}_{xx} + \Phi(x,t), && \text{in } \Omega_T, \\ &\tilde{\theta}_x(0,t) = (1+a^2)\varphi(t), && \text{in } (0,T), \\ &\tilde{\theta}_x(1,t) = (1+a^2)\mathbf{u}(t), && \text{in } (0,T), \\ &\tilde{\theta}(x,0) = \tilde{\theta}_0(x), && \text{in } \Omega \end{aligned} \tag{9}$$

[1] In fact, in the applications the problem may actually be to identify $k(\cdot)$.

$$\tilde{\theta}_0(x) \equiv (1+a^2)\theta_0(x) - a\max\left\{a\int\limits_0^1 \theta_0(\xi)d\xi - g,\ 0\right\}.$$

Referring to the classical theory of linear parabolic equations [14, Theorem 6.1], we can formulate the following result for system (9): if $\theta_0 \in H^1(\Omega)$, $\mathbf{u}, \varphi \in H^{1/4}(0,T)$, $\Phi \in L^2(\Omega_T)$, then the problem (9) has a unique solution in $H^{2,1}(\Omega_T)$. Since for $\tilde{\theta} \in H^{2,1}(\Omega_T)$, the function $\theta(x,t)$ in (8) also belongs to $H^{2,1}(\Omega_T)$, we can formulate the following result for the original system (1).

Theorem 2.1: Let $\theta_0 \in H^1(\Omega)$, $\mathbf{u}, \varphi \in H^{1/4}(0,T)$, $\Phi \in L^2(\Omega_T)$. Then the problem (1) has a unique solution in $H^{2,1}(\Omega_T)$.

III. The average temperature measurements

A. Identifiability theorem

Theorem 3.1: The function $\mathbf{u}(t)$, $t \in [0,T]$, is identifiable from the measurements $y(t)$, $t \in [0,T]$, defined by (5).

Proof of Theorem 3.1: Integrating the first equation in system (1) over the spatial interval $\Omega = [0,1]$, we obtain

$$\begin{array}{l}(1+a^2)\dot{y}(t) = \mathbf{u}(t) - \varphi(t) \\ +a\frac{d}{dt}\max\{ay(t)-g,\ 0\} + \int_0^1 \Phi(x,t)\,dx.\end{array} \tag{10}$$

This equation has a unique solution $\mathbf{u}(t)$ provided the function $y(t) = \int\limits_0^1 \theta(\xi,t)d\xi,\ t \in [0,T]$, is known. Hence, the function $\mathbf{u}(t)$,$t \in [0,T]$, is identifiable from the measurement data, $y(t)$, $t \in [0,T]$, defined by (5), and the proof is complete.

Remark 3.1: Let us assume the representation (2) for the input in system (1). From the theory of linear parabolic equations [14], we know that, given appropriately smooth θ_0, $\Phi(x,t)$, $\varphi(t)$, and known y from $H^1(0,T)$, then there is only one solution $\theta(x,t)$ to the system

$$\left\{\begin{array}{l}(1+a^2)\theta_t - \theta_{xx} = a\frac{d}{dt}\max\{ay(t)-g,\ 0\} \\ \qquad +\Phi(x,t), \quad \text{in } \Omega_T, \\ \theta_x(0,t) = \varphi(t), \quad \text{in } (0,T), \\ \theta_x(1,t) = \mathbf{u}(t), \quad \text{in } (0,T), \\ \theta(x,0) = \theta_0(x), \quad \text{in } \Omega.\end{array}\right. \tag{11}$$

and, hence, the trace $\theta(1,t)$ can be identified uniquely. Then, for all t where $\theta(1,t) \neq 0$, the value of function $k(g - a\int\limits_0^1 \theta(\xi,t)d\xi)$ is determined uniquely from (2).

B. An on-line estimation algorithm

In this Section, we present an estimation procedure for $\mathbf{u}(t)$ given the measurement data $Y(t) = y(t) + \eta(t)$, $t \in [0,T]$ where the measurement noises $\eta(t)$ satisfy (7).

There is an extensive literature on adaptive estimation in infinite dimensional systems (see, e.g., [7], [15] and references therein). In this paper, we adopt an input observer of the form proposed in [5] for finite-dimensional applications. We also note that we will need some regularization conditions to guarantee convergence because directly solving equation (10) with respect to $\mathbf{u}(t)$, $t \geq 0$, is an ill-posed problem, and would involve the differentiation of the measurement signal. We will treat the observer gain as a regularization parameter for an ill-posed problem.

Consider the system

$$\begin{array}{rcl}\hat{\mathbf{u}}(t) &=& \epsilon(t) + \alpha_0(1+a^2)Y(t) - \beta(t) \\ &-& \alpha_0\, a\max\{aY(t)-g, 0\} + \Psi(t)\end{array} \tag{12}$$

where

$$\begin{array}{l}\dot{\epsilon}(t) = -\alpha_0\epsilon(t) - \alpha_0^2 Y(t), \\ \dot{\beta}(t) = -\alpha_0\beta(t) + \alpha_0^2 a\max\{aY(t)-g, 0\} \\ \dot{\Psi}(t) = -\alpha_0\Psi(t) + \alpha_0(\varphi(t) - \int_0^1 \Phi(x,t)dx), \\ \epsilon(0) = 0,\ \beta(0) = 0,\ \Psi(0) = 0.\end{array} \tag{13}$$

Here $\alpha_0 > 0$ is the observer gain while the constant a is fixed and the same as in (1). System (12) is linear with constant coefficients and with the non-homogeneity of the square integrable type. Hence, there is a unique solution of the Caratheodory type $(\epsilon(t), \beta(t), \Psi(t))$ which is bounded on any finite interval $0 \leq t \leq T$.

Theorem 3.2: Suppose the conditions of Theorem 2.1 are satisfied. Let $\mathbf{u}(t)$ be continuous, then for any $t \in (0,T]$, we have

$$\hat{\mathbf{u}}(t) \to \mathbf{u}(t) \text{ as } \alpha_0 \to \infty,\ \mu \to 0,\ \alpha_0\mu \to 0. \tag{14}$$

Proof of Theorem 3.2: Let

$$\begin{array}{l}\bar{\hat{\mathbf{u}}}(t) = \epsilon(t) + \alpha_0(1+a^2)y(t) \\ -\beta(t) - \alpha_0\, a\max\{ay(t)-g, 0\} + \Psi(t)\end{array},$$

$$\mathbf{u}_\delta(t) = \frac{1}{2\delta}\int\limits_{t-\delta}^{t+\delta} \mathbf{u}(\tau)\,d\tau,$$

where $\delta > 0$ will be chosen later and $\mathbf{u}(t)$ is assumed to be extended beyond the original interval with the values $\mathbf{u}(0)$ for $t < 0$ and $\mathbf{u}(T)$ for $f > T$. Then,

$$(\hat{\mathbf{u}} - \mathbf{u})^2 \quad \leq \quad 3(\hat{\mathbf{u}} - \bar{\hat{\mathbf{u}}})^2 + 3(\bar{\hat{\mathbf{u}}} - \mathbf{u}_\delta)^2 + 3(\mathbf{u}_\delta - \mathbf{u})^2 \tag{15}$$

It is well known (see, e.g. [12, p. 85]) that $\|\mathbf{u}_\delta - \mathbf{u}\|_{L^2(0,T)} \to 0$ as $\delta \to 0$. Next, $\|\dot{\mathbf{u}}_\delta\|_{L^2(0,T)} \leq \delta^{-1}\|\mathbf{u}\|_{L^2(0,T)}$, and, finally, for any $t \in [0,T]$, we have $\mathbf{u}_\delta(t) \to \mathbf{u}(t)$ as $\delta \to 0$. Then, taking into account that

$$\begin{array}{rcl}|\hat{\mathbf{u}}(t) - \bar{\hat{\mathbf{u}}}(t)| &\leq& \alpha_0|\eta(t)| + \alpha_0 a|\max\{aY(t)-g, 0\} \\ &-& \max\{ay(t)-g, 0\}| \leq (1+a^2)\alpha_0\mu,\end{array}$$

we conclude that the first and the third terms on the right hand side of inequality (15) tend to zero for all $t \in [0,T]$ under the conditions stated in Theorem 3.2 and the additional condition that $\delta \to 0$. To prove that $\bar{\hat{\mathbf{u}}}(t) \to \mathbf{u}_\delta(t)$ for $t \in (0,T]$, we consider a Lyapunov-like function

$$V(t) = \frac{1}{2}(\bar{\hat{\mathbf{u}}}(t) - \mathbf{u}_\delta(t))^2.$$

Then after straightforward algebraic manipulations, we obtain

$$\begin{array}{l}\dot{V} \leq -\alpha_0 V(t) + \frac{1}{2\alpha_0}(\alpha_0|\mathbf{u}_\delta - \mathbf{u}| \\ +(1+2a^2)\alpha_0^2\mu + |\dot{\mathbf{u}}_\delta|)^2.\end{array}$$

This, after integration, yields

$$\begin{aligned} V(t) &\leq e^{-\alpha_0 t}V(0) + \tfrac{3}{2}\max_{0\leq t\leq T}|\mathbf{u}_\delta(t)-\mathbf{u}(t)|^2 \\ &+ \tfrac{3}{2}(1+2a^2)^2\alpha_0^2\mu^2 + \tfrac{3}{2\alpha_0}\int_0^t |\dot{\mathbf{u}}_\delta|^2\,d\tau. \end{aligned}$$

So, we conclude that $V(t)\to 0$ for $t>0$ as $\alpha_0\to\infty,\ \mu\to 0,\ \alpha_0\mu\to 0$ and if we take $\delta=\alpha_0^{-1/4}$. The proof of Theorem 3.2 is complete.

Remark 3.2: Theorem 3.2 was proved under the assumption that $\mathbf{u}(t)$ is continuous. This assumption is satisfied if in (2) $k\in C^1(R)$ and is nonnegative, and if $\theta_0\in H^1(\Omega)$, $\phi\in H^1(0,T)$, $\theta_{0_x}(0)=\phi(0)$, and $\Phi\in H^{1,1}(\Omega_T)$. Under these assumptions there is a unique solution to system (1), (2) in the space $W^{2,1}(\Omega_T)$, and, hence, the function (2) is continuous.

Remark 3.3: Consider an observer $\hat{\theta}(x,t)$ for $\theta(x,t)$ which is due to system (1) with $\mathbf{u}(t)$ replaced by $\hat{\mathbf{u}}(t)$. Then under the conditions of Theorem 3.2 which ensure that $\hat{\mathbf{u}}\to\mathbf{u}$ in $L^2(0,T)$, it is straightforward to prove that $\hat{\theta}\to\theta$ in $H^{\frac{3}{2},\frac{3}{4}}(\Omega_T)$.

Applying the trace theorem (see, e.g., [14, P.9]), we conclude that there exists a trace $\hat{\theta}(1,\cdot)\in H^{\frac{1}{2}}(0,T)$. Due to the compactness of the inclusion $H^{\frac{1}{2}}(0,T)\subset L^2(0,T)$, under the conditions of Theorem 2.1, we have $\hat{\theta}(1,\cdot)\to\theta(1,\cdot)$ in $L^2(0,T)$.

Remark 3.4: The input observer (12), (13) for $\mathbf{u}(t)$ based on (10) was developed under the assumption that $\mathbf{u}(t)$ is only continuous. This result is stronger then those proved in the prior literature including [5].

C. Simulation example

To illustrate the functionality and performance of the input observer, we consider a simple simulation example. The function $\theta(x,t)$ shown in Figure 2 is of the form

$$\theta(x,t)=u(t)\sigma(t)\cos(\rho(t)x),\ x\in[0,1],\ t\in[0,6]$$

where $\sigma(t)$ and $\rho(t)$ are appropriately defined functions from $C^1[0,T]$. The measurement data $Y(t)$ are presented in Figure 3 (left). The measurement noise bound here is $\mu=0.005$. The parameter α_0 is set to 10. Figure 3 (right) confirms good performance of the input observer. The function $\mathbf{u}(t)$ is shown in Figure 3 (right) by the solid line and its estimate, $\hat{\mathbf{u}}(t)$, is shown by the dashed line.

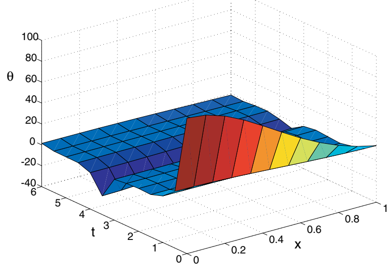

Fig. 2. The temperature distribution through the rod (deviation from nominal).

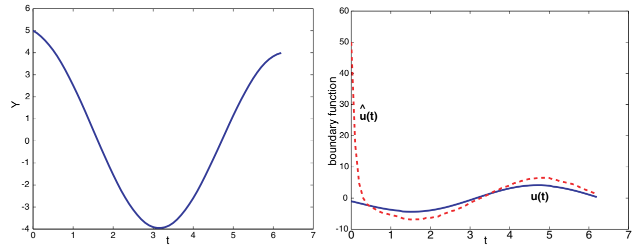

Fig. 3. Left: The measurement data. Right: The true boundary function $\mathbf{u}(t)$ (solid line) and the estimate from the input observer $\hat{\mathbf{u}}(t)$ (dashed line).

D. More general case of a single sensor

We have shown above that the heat flux can be estimated from the average temperature measurements across the whole rod. A more general identifiability result can be obtained by considered a single weighted average temperature measurement, i.e.,

$$y(t)=\int_\Omega \omega(\xi)\theta(\xi,t)\,d\xi,\ t\in[0,T],\quad \Omega=(0,1).$$

We denote

$$\omega^{(n)}=\int_\Omega \omega(x)\cos(\pi n x)dx,\ n=0,1,\ldots,$$

$$P(t)=\omega^{(0)}+2\sum_{n=1}^{\infty}(-1)^n\omega^{(n)}\,e^{-\frac{\pi^2n^2t}{1+a^2}},\ t\geq 0.$$

Theorem 3.3: Let

$$\begin{array}{ll} \text{(i)} & \omega\in C^2(\overline{\Omega}),\quad \omega'(0)=\omega'(1)=0,\\ \text{(ii)} & P(0)\neq 0,\\ \text{(iii)} & \frac{a^2}{1+a^2}\left|1-\frac{\omega^{(0)})}{P(0)}\right|<1,\\ \text{(iv)} & \mathbf{u}(\cdot),\varphi(\cdot)\in C[0,T],\quad \Phi\in L^2(\Omega_T). \end{array} \tag{16}$$

Then the function $\mathbf{u}(t)$, $t\in[0,T]$, is identifiable from $y(t)$, $t\in[0,T]$.

The proof of Theorem 3.3, based on an application of a fixed point theorem, is omitted due to lack of space.

Remark 3.5: Theorem 3.3 gives sufficient conditions for $\omega(x)$ to guarantee that the function $\mathbf{u}(t)$ is identifiable. The condition (i) provides some regularity properties for the Fourier series of $\omega(x)$ with respect to the system $\{\cos(\pi nx)\}_{n=0}^\infty$, $x\in[0,1]$. If (i) is satisfied, the condition (ii) means that $\omega(1)=P(0)\neq 0$. As in the average temperature case, the measurement for the weighted average temperature case must contain information about the temperature from the contacting end of the rod. The condition (iii), roughly speaking, holds when either a is small enough or $\omega(1)\approx\int_0^1\omega(\xi)\,d\xi$.

We next investigate if $u(t),\ t\in[0,T]$, can be identified from multiple temperature sensors *isolated* from the contacting end.

IV. The average observations in parts of the rod

Now suppose that average observations over parts of the rod, (6), contaminated by the measurement noise (7) are available for system (1). The questions are, first, determine whether the function $\mathbf{u}(t)$, $t \in [0,T]$, is identifiable from the noise-free measurements, $z_i(t)$, $t \in [0,T]$, and, second, given the measurements $Z_i(t)$, $t \in [0,T]$, to obtain an estimate for $\mathbf{u}(t)$, $t \in [0,T]$.

A. Identifability result. I

In this Subsection, we treat in detail the special case when $I = 2$ and the measurements $z_i(t)$, $i = 1,2$ are available for all $t \geq 0$. We show that if functions ω_1 and ω_2 are $C^2[0,1]$ and not identical, while $\int_\Omega \omega_i(\xi)\, d\xi \neq 0$, $i = 1,2$, (without loss of generality, we assume that $\int_\Omega \omega_i(\xi)\, d\xi = 1$), then the function $\mathbf{u}(t)$ is identifiable from the measurements $z_1(t)$, $z_2(t)$, $t \geq 0$ defined by (6). We define,

$$\omega_i^{(n)} = \int_\Omega \omega_i(x)\cos(\pi n x)\, dx, \quad \bar{\omega}^{(n)} = \omega_2^{(n)} - \omega_1^{(n)}, \quad i = 1,2, \quad n = 1,2,...,$$

$$P(t) = 2\sum_{n=1}^{\infty} (-1)^n \bar{\omega}^{(n)}\, e^{-\frac{\pi^2 n^2 t}{1+a^2}}. \tag{17}$$

If $t = 0$ the series (17) is the Fourier Cosine expansion of $\omega_2(x) - \omega_1(x)$ evaluated at $x = 1$. Since $\omega_1(x)$ and $\omega_2(x)$ are zero outside of $\chi_i \subset [0,1)$, it can be demonstrated that for $t = 0$ the series converges and $P(0) = 0$. For $t > 0$, the series in (17) converges due to the exponential decay of its elements with n.

Theorem 4.1: Let $I = 2$ and suppose that the functions ω_i, $i = 1,2$, satisfy the assumptions made in Section I and, in addition, $P(t)$ is not identically zero. Suppose also that $\mathbf{u} \in L^1(0,+\infty)$. Then the function $\mathbf{u}(t)$, $t \geq 0$, is identifiable from the measurements $z_1(t), z_2(t)$, $t \geq 0$ obtained from (6).

Proof of Theorem 4.1: Let $\theta(x,t)$ solve the initial-boundary value problem (1). Using the notations from the previous section, we will denote $y(t) = \int_0^1 \theta(\xi,t)\, d\xi$, but in contrast to the functions $z_1(t), z_2(t)$, defined by (6), this function is now assumed unknown for $t > 0$.

Using Green's function for a parabolic equation in a one-dimensional domain with Neumann boundary conditions

$$G(x,\xi,t) = 1 + 2\sum_{n=1}^{\infty} \cos(\pi n x)\cos(\pi n \xi) e^{-\frac{\pi^2 n^2 t}{1+a^2}}, \quad x,\xi \in \Omega,\ t > 0, \tag{18}$$

and taking into account that $\int_\Omega \omega_i(x)\, dx = 1$, the measurement equation (6) can be written as follows:

$$\begin{aligned} z_i(t) = {} & \tfrac{1}{1+a^2} \int_\Omega \int_\Omega \omega_i(x) G(x,\xi,t)\theta_0(\xi)\, d\xi dx \\ & + \tfrac{a}{1+a^2}(\max\{ay(t) - g,\ 0\} - \max\{ay(0) - g,\ 0\}) \\ & + \tfrac{a}{1+a^2} \int_\Omega \int_0^t \int_\Omega G(x,\xi,t-\tau)\Phi(\xi,\tau)\omega_i(x)\, d\xi d\tau dx \\ & - \tfrac{1}{1+a^2} \int_\Omega \int_0^t G(x,0,t-\tau)\varphi(\tau)\omega_i(x)\, d\tau dx \\ & + \tfrac{1}{1+a^2} \int_\Omega \int_0^t G(x,1,t-\tau)\mathbf{u}(\tau)\omega_i(x)\, d\tau dx. \end{aligned} \tag{19}$$

Then, recalling the notation (17), we can write the difference $z_2(t) - z_1(t)$ formally in the form:

$$z_2(t) - z_1(t) = \int_0^t P(t-\tau)\mathbf{u}(\tau)\, d\tau + F(t),\ t \geq 0 \tag{20}$$

where $F(t)$ is a function that depends only on the known data $\varphi(t)$, $\Phi(x,t)$, $\theta_0(x)$, $\omega_1(x)$, and $\omega_2(x)$, and where $P(t)$ is defined by (17). We will prove that (20) has a unique solution.

Suppose the contrary, i.e. let (20) have two solutions. Then the equation

$$\int_0^t P(t-\tau)\mathbf{u}(\tau)\, d\tau = 0 \tag{21}$$

considered for $t \geq 0$, has a nontrivial solution $\bar{\mathbf{u}}(t)$. Since $\omega_i \in C^2[0,1]$, the function $P(t)$ is in $L^1(0,\infty)$. Then, applying the Laplace transform to each side of (21), we get

$$\bar{\mathbf{U}}(s) \cdot \sum_{n=1}^{\infty} \frac{(-1)^n \bar{\omega}^{(n)}}{s + \frac{\pi^2 n^2}{1+a^2}} = 0,\ s = \sigma + j\, p. \tag{22}$$

The second factor in this product is an analytic function in the domain $\sigma > 0$. Thus the product can be equal to zero only if $\bar{\mathbf{U}}(s) = 0$, and we conclude, that $\bar{\mathbf{u}}(t) = 0$, $t \geq 0$.

B. A numerical solution

To numerically solve the integral equation (20), the following off-line procedure can be used. Consider a function $u_\epsilon(t)$ yielding the following Volterra integral equation of the second kind:

$$\epsilon\mathbf{u}(t) + \int_0^t K(t,\tau)\mathbf{u}(\tau)\, d\tau = \mathcal{F}(t),\ t \in [0,T], \tag{23}$$

where $K(t,\tau) = \int_{\max\{t,\tau\}}^T P(s-\tau)P(s-t)\, ds$, $\mathcal{F}(t) = \int_t^T P(\tau - t)[Z_2(\tau) - Z_1(\tau) - F(\tau)]\, d\tau$. Assuming that $\epsilon > 0$, there exists a unique solution to (23) $\mathbf{u}_\epsilon \in L^2(0,T)$. As shown in [10, Chapter 3, §9], $\mathbf{u}_\epsilon \to \mathbf{u}$ strongly in $L^2(0,T)$ as $\alpha \to 0$ and $\mu = o(\sqrt{\epsilon})$. So, to approximate u in (1), we may use any numerical scheme available for Volterra integral equations of the second kind (see e.g., [10]).

C. Identifiability result. II

Now suppose that that the measurement data $z_1(t)$, $z_2(t)$ are available on a finite time interval $0 \leq t \leq T$, $(T < \infty)$. In this case, as before, the integral equation (20) constrains the function u. By differentiating (20) with respect to time, it may appear that (20) is reducible to a well-studied Volterra integral equation of the convolution type, which would be solvable by standard numerical techniques. Unfortunately, this is not the case because $P(0) = 0$. Therefore, a different approach is required.

In the following theorem, the sufficient conditions for the signal $\mathbf{u}(t)$, $0 \leq t \leq T$, to be identifiable are given.

Theorem 4.2: Suppose $\omega_1(x)$, $\omega_2(x)$ satisfy the conditions of Theorem 4.1 and

$$\sum_{n=1}^{\infty} (-1)^n \bar{\omega}^{(n)} \frac{1}{n^2} \neq 0. \tag{24}$$

Assume also that $\mathbf{u} \in C[0,T]$ and $\mathbf{u}(t) \equiv const$, $t \in [T-\delta, T]$, for some $\delta \in (0,T]$. Then the function $\mathbf{u}(t)$, $t \in [0,T]$, is identifiable from the measurements $z_1(t)$, $z_2(t)$, $t \in [0,T]$.

Proof of Theorem 4.2: Having assumed that the function $\mathbf{u}(t)$, $t \in [0,T]$, is not identifiable, we again face equation (21) but on the finite interval $[0,T]$, and deduce that it has a nontrivial solution $\mathbf{u}(t)$, $t \in [0,T]$. According to the assumption of the theorem, this solution must be constant on some interval $[T-\delta, T]$. So, $\mathbf{u}(t) = \mathbf{u}_T$, for $t \in [T-\delta, T]$. Then from (21), we deduce

$$\begin{aligned}&\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2 t}{1+a^2}}(-1)^n \bar{\omega}^{(n)} \times \\ &\left[\int_0^{T-\delta} e^{\frac{\pi^2 n^2 \tau}{1+a^2}} \mathbf{u}(\tau)\, d\tau + e^{\frac{\pi^2 n^2 (T-\delta)}{1+a^2}} \frac{(1+a^2)\mathbf{u_T}}{\pi^2 n^2}\right] \\ &-\mathbf{u_T} \sum_{n=1}^{\infty} (-1)^n \bar{\omega}^{(n)} \frac{(1+a^2)}{\pi^2 n^2} = 0,\ t \in [T-\delta, T].\end{aligned} \tag{25}$$

The functions $\{e^{-\frac{\pi^2 n^2 t}{1+a^2}}\}_{n=0}^{\infty}$ are linearly independent in $L^2(0, T-\delta)$. Hence, from (25), we see that $\mathbf{u_T} = 0$. Then, sequentially,

$$\bar{\omega}^{(n)} \int_0^{T-\delta} e^{\frac{\pi^2 n^2 \tau}{1+a^2}} \mathbf{u}(\tau)\, d\tau = 0,\ n = 1, 2, ... \tag{26}$$

Now, let us construct the function $\bar{\mathbf{u}}(t)$, which is equal to $\mathbf{u}(t)$ for $t \in [0,T]$ and equal to zero for $t > T$. Equalities (26) will result in $\int_0^t P(t-\tau)\bar{\mathbf{u}}(\tau)\, d\tau = 0$ for all $t \geq 0$, which, as shown above, means that $\mathbf{u}(t) = 0$, $t \in [0,T]$. With this, Theorem 4.2 is proved.

Remark 4.1: It follows immediately from arguments similar to the ones we used to prove Theorem 4.2, that if, in addition to the measurements, the values of the function $\bar{\mathbf{u}}(t)$, $t \in [0,T]$, are known on a subinterval $t \in [T-\delta, T]$, then the function $\bar{\mathbf{u}}(t)$ can be identified on the whole interval $[0,T]$ provided the weights $\omega_i(x)$, $i = 1, 2$, satisfy the conditions of Theorem 4.2.

Remark 4.2: The numerical procedure described in Subsection IV-B can be used for estimating the function $\mathbf{u}(t)$, $t \in [0,T]$, if the condition of Theorem 4.2 are satisfied.

V. Conclusion

An input estimation problem for a one dimensional quasi-static thermoelastic equation has been studied where the unknown input is the Neumann-type boundary condition at the contacting end. Identifiability results and estimation procedures were described for different types of temperature measurements. It was shown that for the case of a single sensor measuring weighted average temperature of the rod, the measurement must contain information about the temperature at the contacting end of the rod. However, if two separate sensors are available, and under appropriate additional conditions, these two sensors can be isolated from the contacting end of the rod.

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