2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Thermal Convection Loop Control by Continuous Backstepping and Singular Perturbations

Rafael Vazquez and Miroslav Krstic

Abstract— **A state feedback boundary control law that stabilizes fluid flow in a 2D thermal convection loop is presented. The fluid is enclosed between two cylinders, heated from above and cooled from below, which makes its motion unstable for a large enough Rayleigh number. The actuation is at the boundary through rotation (direct velocity actuation) and heat flux (heating or cooling) of the outer boundary. The design is a new approach for this kind of a coupled PDE problem, based on a combination of singular perturbation theory and the backstepping method for infinite dimensional linear systems. Stability is proved by Lyapunov method. Though only a linearized version of the plant is considered in the design, an extensive closed loop simulation study of the nonlinear PDE model shows that the result holds for reasonably large initial conditions. A highly accurate approximation to the control law is found in closed form.**

I. Introduction

A feedback boundary control law is designed for a thermal fluid confined in a closed convection loop, which is created by heating the lower half of the loop and cooling the upper half. The temperature gradient induces density differences, creating a motion that is opposed by viscosity and thermal diffusivity. For a large enough Rayleigh number, which is a function of physical constants, geometry and temperature difference between the top and the bottom, the plant develops an instability that the control law is able to stop.

Other controllers have been designed for this problem, including an LQG controller by Burns et al [2], and a nonlinear backstepping design for a discretized version of the plant [1]. The present design is *simpler* than the former, not needing a solution of Ricatti equations, only a linear hyperbolic equation; and more *rigorous* than the latter, which does not hold in the limit when the discrete grid approaches the continuous domain.

Our controller is designed for the linearized plant using a combination of singular perturbation theory and the backstepping method for infinite dimensional linear systems. Singular perturbation theory is a mature area [4] with a wealth of control applications, while backstepping for infinite dimensional linear systems has just been recently developed [6]. Combining both methods it is possible to design an stabilizing boundary state feedback control law; this is proved for a large enough Prandtl number, which is the ratio between kinematic viscosity and thermal difussivity, whose inverse plays the role of the singular perturbation parameter.

We start the paper stating the mathematical model of the convection loop (Section II) and transforming it into a suitable form for application of singular perturbation methods. In Section III we introduce the main assumption of this paper which allows for the application of singular perturbation theory. The quasi-steady state and the reduced model are then found, and the state feedback controller for velocity is set. Section IV is divided in several subsections, and deals with the reduced system using backstepping to stabilize the PDE. A coordinate transformation (infinite dimensional, represented by a linear Volterra operator) is introduced to transform the original PDE into a stable linear PDE (a heat equation, to be exact). Finding the kernel of the transformation is the main design task. For this we derive a linear hyperbolic PDE which is verified by the kernel, and an equivalent integral equation. Either of them can be used to numerically or symbolically find the kernel. The temperature feedback control law is then presented in terms of this kernel and the state and a highly accurate approximation to the control law is found in closed form. In Section V we present the main result of the paper, a proof of stability based on both singular perturbation and infinite dimensional backstepping theory. The theoretical result is finally supported by a simulation study, presented in Section VI, in which closed loop simulations and control effort are shown. In these simulations the Rayleigh number is large enough for the plant to go open loop unstable, but the controller is able to overcome the instability.

II. Problem Statement

We employ the model derived in [1]. The geometry of the problem is shown in Fig. 1, and consists of fluid confined between two concentric cylinders standing

This work was supported by NSF.

R. Vazquez is with the Department of Mechanical and Aerospace Engineering, University of California at San Diego, 9500 Gilman Drive, CA 92093, USA `rvazquez@ucsd.edu`

M. Krstic is with Faculty of Mechanical and Aerospace Engineering, University of California at San Diego, 9500 Gilman Drive, CA 92093, USA `krstic@ucsd.edu`

0-7803-9098-9/05/$25.00 ©2005 AACC

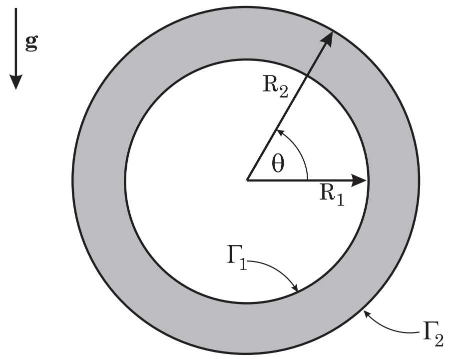

Fig. 1. Convection Loop

in a vertical plane. The main assumption is a small gap between the cylinders compared to their size, i.e. $R_2 - R_1 \ll R1 < R2$. Then, introducing the Boussinesq approximation, other standard assumptions for the velocity in this 2D configuration, and integrating the momentum equation along circles of fixed radius r, the following plant equations are derived

$$\begin{aligned} v_t &= \frac{\gamma}{2\pi}\int_0^{2\pi} T(t,s,\phi)\cos\phi d\phi \\ &\quad +\nu\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right), \end{aligned} \tag{1}$$

$$\begin{aligned} T_t &= -\frac{v}{r}T_\theta \\ &\quad +\chi\left(\frac{T_{\theta\theta}}{r^2}+\frac{T_r}{r}+T_{rr}\right), \end{aligned} \tag{2}$$

where v stands for velocity, which only depends on the radius r, T for the temperature, which depends on both r and the angle θ, ν is the kinematic viscosity, χ the thermal diffusivity, and $\gamma = g\beta$, with g the acceleration due to gravity and β the coefficient of thermal expansion. The boundary conditions are Dirichlet for velocity, with actuation by rotating the outer boundary, while the temperature has Neumann boundary conditions, namely $T_r(t, R_1, \theta) = T_r(t, R_2, \theta) = K \sin\theta$, with K a constant parameter representing the imposed heating and cooling in the boundaries. We actuate the heat flux, which is more realistic than direct temperature actuation.

Defining $\tau = T - Kr\sin\theta$ we shift the equilibrium to the origin. Then, we introduce nondimensional coordinates and variables, $r' = r/d$, $t' = t\chi/d^2$, $v' = vd/\chi$, $\tau' = \tau/\Delta T$, $R_a = (1/C)\gamma\Delta d^3/2\nu\chi$, $P = \nu/\chi$, where $d = R2 - R1$, $\Delta T = -(4/\pi)K(R_1 + R_2/2)$, and R_a and P are respectively the Rayleigh and Prandtl numbers. C is set so the system is stable for Rayleigh numbers less than unity and unstable otherwise [1]. The nondimensional plant equations are, dropping primes, as follows:

$$\begin{aligned} v_t &= \frac{1}{\pi}PR_aC\int_0^{2\pi} \tau(t,s,\phi)\cos\phi d\phi \\ &\quad +P\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right), \end{aligned} \tag{3}$$

$$\begin{aligned} \tau_t &= \frac{d\pi}{2(R_1+R_2)}v\cos\theta - \frac{v}{r}\tau_\theta + \frac{\tau_{\theta\theta}}{r^2} \\ &\quad +\frac{\tau_r}{r}+\tau_{rr}. \end{aligned} \tag{4}$$

The boundary conditions are:

$$v(t,R_1) = 0, \tag{5}$$
$$v(t,R_2) = V(t), \tag{6}$$
$$\tau_r(t,R_1,\theta) = 0, \tag{7}$$
$$\tau_r(t,R_2,\theta) = U(t,\theta) \tag{8}$$

where V and U are, respectively, the nondimensional velocity and temperature control, and periodic boundary conditions in angle for τ.

Defining $\epsilon = P^{-1}$, $A_1 = R_aC/\pi$, $A_2 = d\pi/2(R_1 + R_2)$, dropping time dependence, and neglecting the nonlinear term, the linearized plant equations are:

$$\begin{aligned} \epsilon v_t &= A_1\int_0^{2\pi} \tau(r,\phi)\cos\phi d\phi - \frac{v}{r^2} \\ &\quad +\frac{v_r}{r}+v_{rr}, \end{aligned} \tag{9}$$

$$\tau_t = A_2 v\cos\theta + \frac{\tau_{\theta\theta}}{r^2}+\frac{\tau_r}{r}+\tau_{rr}, \tag{10}$$

with the same boundary conditions (5)-(8).

We will stabilize this linearized plant around its equilibrium at zero, therefore stabilizing —at least locally— the full nonlinear plant.

III. Reduced Model

We assume that the parameter ϵ is small enough so we can use singular perturbation theory.

For obtaining the quasi-steady-state, we set $\epsilon = 0$ and solve (9), subject to (5)–(6). The solution to this two-point boundary value problem is [5]:

$$\begin{aligned} v &= \frac{R_2}{r}\frac{r^2-R_1^2}{R_2^2-R_1^2}\left(V(t)+\frac{A_1}{2}\right. \\ &\quad \left.\times\int_{R_1}^{R_2}\int_0^{2\pi}\frac{R_2^2-s^2}{R_2}\cos\phi\tau(s,\phi)dsd\phi\right) \\ &\quad -\frac{A_1}{2}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2-s^2}{r}\cos\phi\tau(s,\phi)dsd\phi. \end{aligned} \tag{11}$$

The quasi-steady-state, substituted into (10), gives the reduced system, which will be stabilized via the back-stepping method. For this procedure to be applicable we need the quasy-steady-state to have a strict integral feedback form, i.e., $v(t,r)$ should not depend on any value of τ after r. Based on this consideration we set the velocity actuation:

$$V = -\frac{A_1}{2}\int_{R_1}^{R_2}\int_0^{2\pi}\frac{R_2^2-s^2}{R_2}\cos\phi\,\tau(s,\phi)ds d\phi, \quad (12)$$

and then the expression for the quasi-steady-state is

$$v = -\frac{A_1}{2}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2-s^2}{r}\cos\phi\,\tau(t,s,\phi)ds d\phi, \quad (13)$$

which plugged into equation (10) renders the following reduced system:

$$\begin{aligned}\tau_t &= -A_{12}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2-s^2}{r}\cos\phi\cos\theta\,\tau(s,\phi)ds d\phi \\ &\quad +\frac{\tau_{\theta\theta}}{r^2}+\frac{\tau_r}{r}+\tau_{rr}, \quad (14)\end{aligned}$$

where $A_{12} = A_1 A_2/2$, and boundary conditions (7)-(8).

IV. Backstepping Controller for Temperature

For stabilization of the reduced system we apply the backstepping technique for parabolic PDEs [6], which allows for compensation of strict-feedback integral terms. This methods consists on finding a transformation of the original system (14) into an exponentially stable *target* system; the control law is then found from the transformation.

A. Target system

The target system is going to be:

$$w_t = \frac{w_{\theta\theta}}{r^2}+\frac{w_r}{r}+w_{rr}, \quad (15)$$

with boundary conditions $w_r(R_1) = 0$ and $w_r(R_2) = qw(R_2)$, where q is a negative number, used for tweaking. Note that this system is exponentially stable, which follows from a standard Lyapunov argument.

B. Backstepping transformation

For transforming (14) into (15) we are going to use the following change of variables:

$$w = \tau(r,\theta) - \int_{R_1}^{r}\int_0^{2\pi} k(r,\theta,s,\phi)\tau(s,\phi)ds d\phi. \quad (16)$$

For calculating the kernel, we introduce (16) into (15) and then we apply integration by parts to arrive at an ultra-hyperbolic PDE,

$$\begin{aligned}k_{rr} &= -\frac{k_{\theta\theta}}{r^2}-\frac{k_r}{r}+\frac{k_{\phi\phi}}{s^2}-\frac{k_s}{s}+k_{ss}+\frac{k}{s^2} \\ &\quad +A_{12}\left(\int_s^r\int_0^{2\pi}k(r,\theta,\rho,\psi)\frac{\rho^2-s^2}{\rho}\right. \\ &\quad \left.\times\cos\psi d\rho d\psi - \frac{r^2-s^2}{r}\cos\theta\right)\cos\phi, (17)\end{aligned}$$

with periodic boundary conditions in both ϕ and ψ, and

$$\begin{aligned}k_s(r,\theta,R_1,\phi) &= \frac{k(r,\theta,R_1,\phi)}{R_1}, \quad (18)\\ k(r,\theta,r,\phi) &= 0. \quad (19)\end{aligned}$$

By inspection of (17) we set:

$$k(r,\theta,s,\phi) = \cos\theta\cos\phi\sqrt{\frac{s}{r}}\hat{k}(r,s), \quad (20)$$

which verifies the periodic boundary conditions, and substituted in (17) renders:

$$\begin{aligned}\hat{k}_{rr}-\hat{k}_{ss} &= \frac{3}{4}\left(\frac{1}{r^2}-\frac{1}{s^2}\right)\hat{k} - A_{12}\left(\frac{r^2-s^2}{\sqrt{rs}}\right. \\ &\quad \left. -\pi\int_s^r \hat{k}(r,\rho)\frac{\rho^2-s^2}{\sqrt{\rho s}}d\rho\right), \quad (21)\end{aligned}$$

a hyperbolic partial integro-differential equation, in the region $\mathcal{T} = \{(r,s) : R_1 \le r \le R_2, R_1 \le s \le r\}$ with boundary conditions:

$$\begin{aligned}\hat{k}_s(r,R_1) &= \frac{\hat{k}(r,R_1)}{2R_1}, \quad (22)\\ \hat{k}(r,r) &= 0. \quad (23)\end{aligned}$$

The kernel $\hat{k}$ can be calculated numerically, or rewritten into an integral equation [6]. For this we introduce new coordinates $\xi = r+s$, $\eta = r-s$, and denote

$$G(\xi,\eta) = \hat{k}(r,s) = \hat{k}\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right). \quad (24)$$

G verifies the following integral equation:

$$\begin{aligned}G &= G_0(\xi,\eta) \\ &\quad +\int_{2R_1+\eta}^{\xi}\int_0^{\eta}\left(3\left(\frac{\gamma\sigma}{(\sigma^2-\gamma^2)^2}\right)G(\sigma,\gamma)\right. \\ &\quad +\int_0^{2\gamma}A_{12}\pi G(\sigma+\frac{\rho}{2},\gamma-\frac{\rho}{2}) \\ &\quad \left.\times\frac{(\rho+\sigma-\gamma)^2-(\sigma-\gamma)^2}{2\sqrt{(\rho+\sigma-\gamma)(\sigma-\gamma)}}d\rho\right)d\gamma d\sigma \\ &\quad +\int_0^{\eta}\int_0^{\sigma}e^{\frac{\eta-\sigma}{R_1}}\left(6G(\sigma,\gamma)\frac{(2R_1+\sigma)\gamma}{((2R_1+\sigma)^2-\gamma^2)^2}\right. \\ &\quad +\int_0^{2\gamma}\frac{(\rho+2R_1+\sigma-\gamma)^2-(2R_1+\sigma-\gamma)^2}{\sqrt{(\rho+2R_1+\sigma-\gamma)(2R_1+\sigma-\gamma)}} \\ &\quad \left.\times A_{12}\pi G(2R_1+\sigma+\frac{\rho}{2},\gamma-\frac{\rho}{2})d\rho\right)d\gamma d\sigma. \quad (25)\end{aligned}$$

G_0 represents the initial term in a successive approximation series for G and is defined as

$$\begin{aligned} G_0(\xi,\eta) \;=\; & -A_{12}\left[\frac{1}{6}(\xi^3-\eta^3-(\xi^2-\eta^2)^{3/2})\right. \\ & +\frac{5}{2}\sqrt{\pi}R_1^3 e^{1+\eta/R_1} \\ & \times\left(\mathrm{erf}(1)-\mathrm{erf}(\sqrt{1+\eta/R_1})\right) \\ & +R_1^3\left(6e^{\eta/R_1}-34/3\right)-8R_1^2\eta-2R_1\eta^2 \\ & \left.+\frac{5}{3}\sqrt{R_1^2+R_1\eta}\left(5R_1^2+2R_1\eta\right)\right]. \end{aligned} \tag{26}$$

Using (25) and the same argument as in [6] the following result holds:

Theorem 1: The equation (21) with boundary conditions (22)-(23) has a unique $\mathcal{C}^2(\mathcal{T})$ solution.

Therefore a smooth solution exists for equation (17) with boundary conditions (18)-(19).

C. Control law

Substituting the transformation into the target system outer boundary condition, we find the control law:

$$\begin{aligned} U(t,\theta) \;=\; & q\tau(R_2,\theta)-\cos\theta\int_{R_1}^{R_2}\int_0^{2\pi}\frac{\sqrt{s}\cos\phi}{\sqrt{R_2}} \\ & \times\left((q+\frac{1}{2R_2})\hat{k}(R_2,s)\right. \\ & \left.-\hat{k}_r(R_2,s)\right)\tau(t,s,\phi)dsd\phi. \end{aligned} \tag{27}$$

As we shall see in Section VI, $\hat{k}(R_2,s)$ is very close to $G_0(R_2+s,R_2-s)$ and $\hat{k}_r(R_2,s)$ is very close to $\frac{\partial G_0}{\partial\xi}(R_2+s,R_2-s)+\frac{\partial G_0}{\partial\eta}(R_2+s,R_2-s)$. This means that, introducing these approximations, we get explicit control laws.

D. Inverse transformation

The backstepping change of variables is invertible, and its inverse, which will be needed later, is defined as

$$\tau = w(r,\theta)-\int_{R_1}^{r}\int_0^{2\pi} l(r,\theta,s,\phi)w(s,\phi)dsd\phi. \tag{28}$$

Like for the direct transformation we set $l(r,\theta,s,\phi)=\cos\theta\cos\phi\sqrt{\frac{s}{r}}\hat{l}(r,s)$. Then, $\hat{l}$ verifies a similar equation to (21) and the same result as Theorem 1 holds for the inverse kernel, see [6].

V. Singular Pertubation Analysis for the Entire System

Now that we have derived a control law for the reduced system, we can drop the assumption that $\epsilon=0$ and instead consider it a small but nonzero parameter. The quasi-steady-state will no longer be the exact solution of the v PDE. Let us call it v_{ss},

$$v_{ss} = -\frac{A_1}{2}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2-s^2}{r}\cos\phi\,\tau(s,\phi)dsd\phi, \tag{29}$$

and introduce the error variable z:

$$z(t,r)=v(t,r)-v_{ss}(t,r), \tag{30}$$

which is found to verify the PDE

$$\begin{aligned} \epsilon z_t \;=\; & -\frac{z}{r^2}+\frac{z_r}{r}+z_{rr}+\epsilon\frac{A_1}{2}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2-s^2}{r} \\ & \times\cos\phi\,\tau_t(s,\phi)dsd\phi. \end{aligned} \tag{31}$$

This PDE without the last term is usually referred to as the Boundary Layer model; note that it is exponentially stable. Using the backstepping transformation to express (10) and all τ dependence in terms of w coordinate, we can write the overall plant in terms of (z,w) variables

$$\begin{aligned} \epsilon z_t \;=\; & -\frac{z}{r^2}+\frac{z_r}{r}+z_{rr}+\epsilon\left(\int_{R_1}^{r}Q_{zz}(r,s)z(s)ds\right. \\ & +\int_{R_1}^{r}\int_0^{2\pi}Q_{zw}^1(r,s,\phi)w(s,\phi)dsd\phi \\ & +\int_0^{2\pi}A_1\cos\phi\, w(r,\phi)d\phi \\ & \left.+\int_0^{2\pi}Q_{zw_0}(r,\phi)w(R_1,\phi)d\phi\right), \end{aligned} \tag{32}$$

$$\begin{aligned} w_t \;=\; & \frac{w_{\theta\theta}}{r^2}+\frac{w_r}{r}+w_{rr}+A_2\cos\theta z(r) \\ & +\int_{R_1}^{r}Q_{wz}^1(r,s,\theta)z(s)ds, \end{aligned} \tag{33}$$

together with boundary conditions $z(R_1)=z(R_2)=0$, $w_r(R_1,\theta)=0$, $w_r(R_2,\theta)=qw(R_2,\theta)$. For simplicity, we have defined the following kernels:

$$Q_{zz} = A_{12}\pi\frac{r^2-s^2}{r}, \tag{34}$$

$$\begin{aligned} Q_{zw}^1 \;=\; & -\frac{A_1}{2}\cos\phi\left(2\pi\sqrt{\frac{s}{r}}\hat{l}(r,s)\right. \\ & \left.+A_{12}\cos\phi\frac{r^4-s^4-4r^2s^2\ln\frac{r}{s}}{4r}\right), \end{aligned} \tag{35}$$

$$Q_{wz}^1 = -A_2\pi\cos\theta\sqrt{\frac{s}{r}}\hat{k}(r,s), \tag{36}$$

$$Q_{zw_0} = \frac{A_1}{2}\frac{r^2+R_1^2}{rR_1}\cos\phi. \tag{37}$$

Now, selecting the following Lyapunov function,

$$\begin{aligned} E(t) \;=\; & \frac{1}{2}\int_0^{2\pi}\int_{R_1}^{R_2}w^2(t,s,\phi)sdsd\phi \\ & +\frac{1}{2}\int_{R_1}^{R_2}z^2(t,s)sds. \end{aligned} \tag{38}$$

we find its time derivative to be:

$$
\begin{aligned}
\frac{dE(t)}{dt} \quad < \quad & \left(\frac{R_2}{2}(q+\frac{R_2}{4(R_2-R_1)})\right) \\
& \times \int_0^{2\pi} w(t,R_2,\phi)^2 d\phi \\
& -\frac{1}{8(R_2-R_1)^2}\int_0^{2\pi}\int_{R_1}^{R_2} w^2 sdsd\phi \\
& +(\beta_1+\beta_2)\left(\int_{R_1}^{R_2} z^2(s)sds\right)^{\frac{1}{2}} \\
& \times \left(\int_0^{2\pi}\int_{R_1}^{R_2} w^2(s,\phi)sdsd\phi\right)^{\frac{1}{2}} \\
& -\left(\frac{1}{\epsilon R_2^2}-\gamma\right)\int_{R_1}^{R_2} z^2 sds. \qquad (39)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0 &= \sqrt{(R_2^2-R_1^2)\ln\frac{R_2}{R_1}} && (40)\\
\beta_1 &= \sqrt{2\pi}\left(A_2+\beta_0\|Q_{wz}^1\|_\infty\right). && (41)\\
\beta_2 &= \sqrt{2\pi}\left(A_1+\beta_0\|Q_{zw}^1\|_\infty\right), && (42)\\
\beta_3 &= -\frac{R_2}{2}(q+\frac{R_2}{4(R_2-R_1)}), && (43)\\
\gamma &= \beta_0\|Q_{zz}\|_\infty+2\pi^2(R_2-R_1)^2\|Q_{zw_0}\|_\infty^2 R_2 \\
& \quad +\frac{\pi^2}{2\beta_3}(R_2-R_1)\|Q_{zw_0}\|_\infty^2 && (44)
\end{aligned}
$$

In the calculations repeated use of Cauchy-Schwartz's, Young's, and Poincare's inequality has been made.

We need to verify $dE/dt<0$. Choosing first q as

$$q=-1-\frac{R_2}{4(R_2-R_1)}, \qquad (45)$$

we need to find the values for ϵ that make the right hand side of (39) negative. Now, we identify the quadratic form in (39) and call its matrix A:

$$A=-\begin{pmatrix} \frac{1}{8(R_2-R_1)^2} & -\frac{\beta_1+\beta_2}{2} \\ -\frac{\beta_1+\beta_2}{2} & \frac{1}{\epsilon R_2^2}-\gamma \end{pmatrix}. \qquad (46)$$

From Sylvester's criterion, $A<0$ when $\epsilon<\epsilon^*$, where

$$
\begin{aligned}
\frac{1}{\epsilon *} \quad = \quad & 2R_2^2(R_2-R_1)^2 \; (\beta_1+\beta_2)^2 \\
& +R_2^2\left(\gamma_1+2\gamma_2+2\frac{\gamma_3}{R_2}\right). \qquad (47)
\end{aligned}
$$

Note that ϵ^* depends only on geometry and physical parameters.

If $\epsilon\in(0,\epsilon^*)$, then $dE/dt<0$, thus establishing asymptotic stability for the plant in (z,w) coordinates. Stability for (v,τ) follows from (28) and (30). We have just proved:

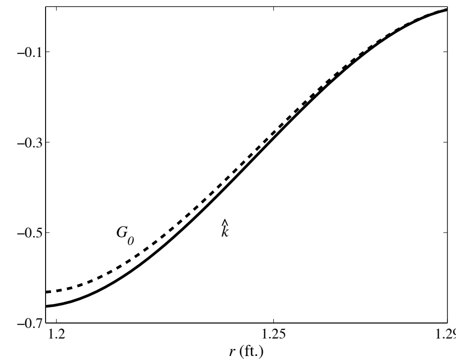

Fig. 2. Exact (solid) and approximate (dashed) control kernels at R_2.

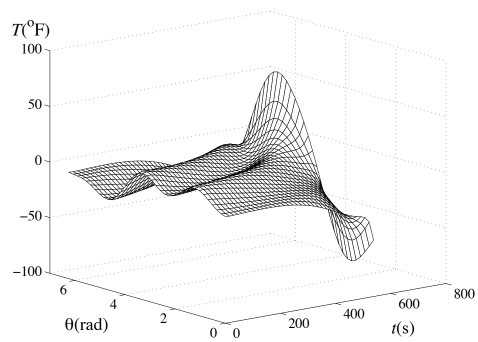

Fig. 3. Open loop evolution of temperature at radius $r=1.21$.

Theorem 2: For a sufficiently small ϵ, the system (9)-(10) with boundary conditions (5)-(8), where V and U are specified by control laws (12) and (27) respectively, has unique classical solutions and is exponentially stable at the origin in the L^2 sense, that is, there exist $M,\alpha>0$, independent of the initial conditions, such that

$$
\begin{aligned}
& \int_{R_1}^{R_2}\left(v^2(t,s)+\int_0^{2\pi}\tau^2(t,s,\phi)d\phi\right)sds \\
\leq \quad & M\mathrm{e}^{-\alpha t}\int_{R_1}^{R_2}\left(v^2(0,s)\right. \\
& \left.+\int_0^{2\pi}\tau^2(0,s,\phi)d\phi\right)sds \qquad (48)
\end{aligned}
$$

Proof of existence and uniqueness of classical solutions has been skipped, but follows from standard arguments due to linearity of (9)-(10) and the form of the boundary conditions.

VI. Simulation Study

We show a prototypical simulation case. A spectral decomposition and a Crank-Nicholson method [3] has been used, with the following numerical values:$R_1=$

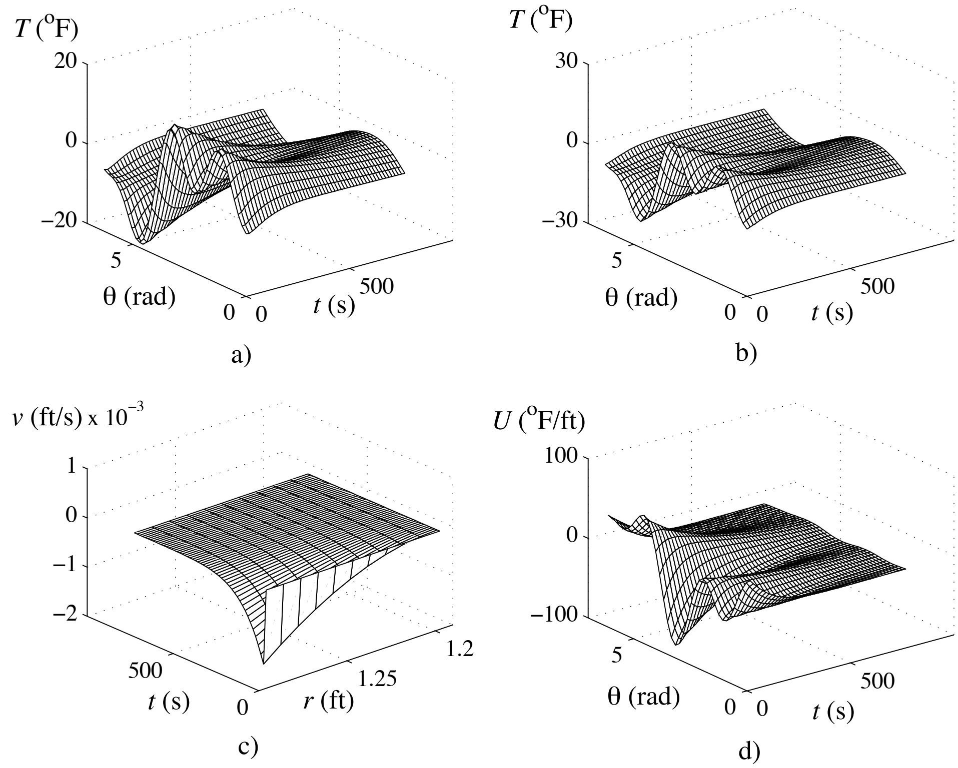

Fig. 4. Closed loop simulation. a) temperature at radius $r = 1.21$ ft. b) temperature at radius $r = 1.25$ ft. c) velocity d) temperature control effort

1.1975 ft., $R_2 = 1.2959$ ft., $P = 8.06$, $Ra = 50$, $C = 7.8962 \times 10^3$, $K = 5\ ^\circ\mathrm{F/ft}$.

Fig. 2 shows the shape of the control kernel, $\hat{k}(R_2, s)$. Information near the inner boundary is given more weight in the control law, because the boundary controller is on the opposite side and therefore has to react more agresively to compensate fluctuations of temperature in the inner side. The approximate kernel given by (26) is also shown, and it can be seen that it is an excellent approximation.

Fig. 3 is an open loop simulation of temperature, which grows very positive or very negative, depending on the angle. In Fig. 4 closed loop simulations of the plant are shown in physical variables (velocity and temperature) showing how they reach the equilibrium state quickly, staying there afterwards. The magnitude of heat flux control is also shown, while the velocity actuation can be seen by looking at the $r = R_2$ section in the velocity plot, which is the outer cylinder rotation imposed by the control law.

Acknowledgement

We thank Andrey Smyshlyaev for valuable feedback and finding the analytical expression for (26).

References

[1] D. Boskovic and M. Krstic, “Nonlinear stabilization of a thermal convection loop by state feedback,” *Automatica*, vol. 37, pp. 2033–2040, 2001.

[2] J.A. Burns, B.B. King and D. Rubio, “Feedback control of a thermal fluid using state estimation,” *International Journal of Computational Fluid Dynamics*, vol. 11, pp. 93–112, 1998.

[3] R. Haberman, *Elementary Applied Partial Differential Equations*, Englewood Cliffs, NJ: Prentice-Hall, 1987.

[4] P. Kokotovic, H.K. Khalil and J. O’Reilly, *Singular Perturbation Methods in Control*, Philadelphia, PA: SIAM Classics, 1999.

[5] G. F. Simmons, *Differential Equations With Applications and Historical Notes*, New York, NY: McGraw-Hill, 1991.

[6] A. Smyshlyaev and Miroslav Krstic, “Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations,” *IEEE Transactions on Automatic Control*, vol. 49, No. 12, pp. 2185–2202, Dec. 2004.

[7] R. Vazquez and M. Krstic, “Volterra Boundary Control Laws for a Class of Nonlinear Parabolic Partial Differential Equations,” *Proceeding of the Sixth IFAC Symposium on Nonlinear Control Systems (NOLCOS 2004)*, Stuttgart, Germany, September 2004.