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Stability and stabilizability of discrete-time switched linear systems with state delay

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Abstract— **The problems of stability and stabilizability of discrete-time switched linear systems whose subsystems are subject to state delays are investigated in this paper using linear matrix inequalities. The conditions, based on Lyapunov-Krasovskii functionals, assure the stability of the system for arbitrary switching functions, irrespective of the value of the time-delay. The switching function is assumed not known *a priori* but available in real time, the state as well as the delayed state vectors are supposed available for feedback, and the time-delay is considered unknown and unbounded. The proposed conditions can reduce the conservatism of the analysis and, thanks to extra variables, can be extended for synthesis purposes, providing switched gains for the state feedback control which encompass the fixed gains obtained from the standard quadratic stabilizability approach. Numerical examples illustrate the results.**

I. INTRODUCTION

Switched systems are a very important class of dynamic systems, defined by several modes of operation (subsystems) and a rule that determines which mode is active at each instant of time [1], [2]. For instance, switched dynamics can be found in electrical circuits with electronic switches (power converters) [3] and in systems subject to switching control laws [4], [5], [6], [7]. Conditions to assess the stability and the stabilizability of this class of systems have been investigated in the last few years [8], [9], [10], [11]. Linear matrix inequality (LMI) conditions based on switched Lyapunov functions have been given to test the stability and the stabilizability of discrete-time switched linear systems subject to arbitrary switching functions in [12]. These conditions can be seen as an extension of the results for time-varying uncertain system in [13] and have been recently applied to control design with pole location and structural constraints [14]. It is important to recall that [12], [14] deal with stability, stabilizability and performance in the case of delay-free switched linear systems and thus their conditions cannot be directly applied when the system model includes state delays. To the knowledge of the authors, there are no convex conditions to assess the stability and the stabilizability of discrete-time switched linear systems with arbitrary switching functions and with unknown and unbounded time-delays on the state variables. This paper aims on the solution of these problems using the LMI framework based on Lyapunov-Krasovskii functionals.

Time-delays rise quite frequently in control systems, as for instance in systems with linear models (see [15], [16], [17] and references therein). One approach to deal with known or unknown but bounded time-delays on the state variables is to use augmented state vectors [18]. This approach, however, cannot be applied to the study of delay independent stability since the dimension of the augmented state vectors would be infinite. Robust delay-independent stability conditions have been proposed in several papers, based on the concept of quadratic stability [19]. Norm-bounded uncertainties are considered in [20], where non-convex necessary and sufficient conditions for quadratic stability and quadratic stabilization are given for fixed and known delays. Results based on LMIs and scaling parameters have been given in [21], still resulting in nonconvex conditions for control design. Sufficient conditions for the design of an $\mathcal{H}_\infty$ state feedback control are given in terms of LMIs and a scaling parameter, but only norm-bounded uncertainties are considered and the delay is supposed to be fixed. Other results include [22] (defining an augmented descriptor system), [23], [24], [25] where some nonconvex strategies are proposed for control design. All of these results are based on a fixed Lyapunov matrix that can provide, in some cases, a robust control gain by means of nonconvex conditions.

In this paper, LMI conditions for the stability of discrete-time switched linear systems with state delays are given. All the system matrices are assumed to be switched, under an arbitrary switching rule. The stability is assured, independently of the size of the time-delays (which can be unknown), by means of a Lyapunov-Krasovskii functional with switched matrices, encompassing the results based on fixed matrices (quadratic stability). Thanks to some extra matrix variables, the stability conditions can be extended to cope with the design of a stabilizing state feedback control law. The closed-loop stability is assured by means of switched Lyapunov-Krasovskii matrices which are not directly used to compute the switched control gains, allowing to extend the results to deal with decentralized control without constraining the matrices used to assess stability. By imposing fixed Lyapunov-Krasovskii matrices, the stabilizability conditions can be used to determine fixed stabilizing gains for the system as well. When the delayed state is available for feedback, a term based on past values

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of the state can be included in the control law, providing stabilization in situations where a memoryless stabilizing control gain may not exist.

II. PROBLEM FORMULATION

Consider the system

$$x(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d) + B_{\sigma(k)}u(\sigma(k), x(k)) + B_{d\sigma(k)}u_d(\sigma(k), x(k-d)) \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state vector ($x(k) = \mathbf{0}$ for $k < 0$), $x(k-d) \in \mathbb{R}^n$ is the state vector subject to the time-delay $d \in \mathbb{N}$, $u(\sigma(k), x(k)) \in \mathbb{R}^{m_1}$ and $u_d(\sigma(k), x(k-d)) \in \mathbb{R}^{m_2}$ are the control inputs. Matrices $A_{\sigma(k)}$, $A_{d\sigma(k)}$, $B_{\sigma(k)}$ and $B_{d\sigma(k)}$ are switched matrices and

$$\sigma(k) : \mathbb{N} \to \mathcal{I} \quad , \quad \mathcal{I} = \{1, \ldots, N\} \quad (2)$$

is the switching function that arbitrarily selects the active subsystem in the set $(A, A_d, B, B_d)_i$, $i = 1, \ldots, N$.

The discrete-time switched linear system with state delay (1) is subject to the assumptions: A1) the switching function (2) is not known *a priori* but is available in real time; A2) all the system matrices switch simultaneously ruled by the arbitrary switching function $\sigma(k)$; A3) the time-delay d can be unknown and unbounded; A4) the state vector $x(k)$ and the delayed state vector $x(k-d)$ are available for feedback.

The problems to be solved are:

Problem 1: Determine if the system (1) with $u(\sigma(k), x(k)) = u_d(\sigma(k), x(k-d)) = \mathbf{0}$ (i.e. autonomous system) is stable for arbitrary switching functions and irrespective of the value of the time-delay.

Problem 2: Find, if possible, switched gains $K_i \in \mathbb{R}^{m_1 \times n}$ and $K_{di} \in \mathbb{R}^{m_2 \times n}$, $i = 1, \ldots, N$ yielding the linear state feedback control laws

$$u(\sigma(k), x(k)) = K_{\sigma(k)}x(k) \ , \quad u_d(\sigma(k), x(k-d)) = K_{d\sigma(k)}x(k-d) \quad (3)$$

such that the closed-loop system

$$x(k+1) = \tilde{A}_{\sigma(k)}x(k) + \tilde{A}_{d\sigma(k)}x(k-d) \quad (4)$$

with

$$\tilde{A}_{\sigma(k)} = A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma(k)} \ , \quad \tilde{A}_{d\sigma(k)} = A_{d\sigma(k)} + B_{d\sigma(k)}K_{d\sigma(k)} \quad (5)$$

is stable for arbitrary switching functions $\sigma(k)$, irrespective of the value d of the time-delay.

In the next sections, an approach based on Lyapunov-Krasovskii functionals is used to derive LMI conditions to solve Problems 1 and 2.

III. STABILITY ANALYSIS

System (1) can be rewritten as

$$x(k+1) = A(\alpha(k))x(k) + A_d(\alpha(k))x(k-d) + B(\alpha(k))u(\alpha(k), x(k)) + B_d(\alpha(k))u_d(\alpha(k), x(k-d)) \quad (6)$$

where

$$(A, A_d, B, B_d)(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)(A, A_d, B, B_d)_i \quad (7)$$

with

$$\alpha_i(k) = \begin{cases} 1, & \text{when the system operates at the } i\text{-th mode} \\ 0, & \text{otherwise} \end{cases}$$

Consider the Lyapunov-Krasovskii functional

$$v(x(k), \alpha(k)) = x(k)'P(\alpha(k))x(k) + \sum_{\ell=1}^{d} x(k-\ell)'Sx(k-\ell) \quad (8)$$

with

$$P(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)P_i \ , \ P_i = P_i' > \mathbf{0} \ , \ S = S' > \mathbf{0} \, , \quad i = 1, \ldots, N \quad (9)$$

Thus

$$v(x(k+1), \alpha(k+1)) = x(k+1)'P(\alpha(k+1))x(k+1) + \sum_{\ell=1}^{d} x(k+1-\ell)'Sx(k+1-\ell) \quad (10)$$

where

$$P(\alpha(k+1)) = \sum_{i=1}^{N} \alpha_i(k+1)P_i = \sum_{j=1}^{N} \beta_j(k)P_j \quad (11)$$

The difference function $\Delta(v(x))$ is defined as

$$\Delta(v(x)) = v(x(k+1), \alpha(k+1)) - v(x(k), \alpha(k)) \quad (12)$$

Next theorem presents four equivalent sufficient conditions to solve Problem 1.

Theorem 1: System (1) is stable for any arbitrary switching function $\sigma(k)$, irrespective of the value of the time-delay d, if there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, N$ and $S \in \mathbb{R}^{n \times n}$ such that any of the following equivalent conditions holds ($\star$ represents symmetric blocks in the LMIs):

a) The Lyapunov-Krasovskii functional (8) is positive definite and the difference function $\Delta(v(x))$ given by (12) is negative definite for all $(x(k), x(k-d)) \neq 0$.

b)

$$\Theta \triangleq \begin{bmatrix} P_i - A_i'(S + P_j)A_i & -A_i'(S + P_j)A_{di} \\ \star & S - A_{di}'(S + P_j)A_{di} \end{bmatrix} > \mathbf{0}, \quad (i, j) \in \mathcal{I} \times \mathcal{I} \quad (13)$$

c)

$$\Upsilon \triangleq \begin{bmatrix} S+P_j & -(S+P_j)A_i & -(S+P_j)A_{di} \\ \star & P_i & \mathbf{0} \\ \star & \star & S \end{bmatrix} > \mathbf{0}, \quad (i,j) \in \mathcal{I} \times \mathcal{I} \qquad (14)$$

d) There exist matrices $F_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n\times n}$ and $H_i \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{M} \triangleq \begin{bmatrix} -(F_i+F_i'+P_j+S) & F_iA_i - G_i' & F_iA_{di} - H_i' \\ \star & P_i + G_iA_i + A_i'G_i' & G_iA_{di} + A_i'H_i' \\ \star & \star & S + H_iA_{di} + A_{di}'H_i' \end{bmatrix} > \mathbf{0}, \quad (i,j) \in \mathcal{I} \times \mathcal{I} \qquad (15)$$

Proof: Taking (8) and (10) into account, the difference function $\Delta(v(x))$ given by (12) is equivalent to

$$\Delta(v(x)) = -\begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}' Q \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix} \qquad (16)$$

with

$$Q \triangleq \begin{bmatrix} P(\alpha(k)) - A(\alpha(k))'(S+P(\alpha(k+1)))A(\alpha(k)) & -A(\alpha(k))'(S+P(\alpha(k+1)))A_d(\alpha(k)) \\ \star & S - A_d(\alpha(k))'(S+P(\alpha(k+1)))A_d(\alpha(k)) \end{bmatrix} \qquad (17)$$

A necessary and sufficient condition for $Q > \mathbf{0}$ in (17) (i.e. $\Delta(v(x)) < 0$) for matrices defined by (7), (9), (11) is given by condition *b)*. Clearly, for $P_i > \mathbf{0}$, and $S > \mathbf{0}$ *a)* is verified if and only if *b)* holds. The equivalence between *b)* and *c)* is established straightforwardly by Schur complement. Notice that if *c)* holds, then (15) is verified for the special choice $F_i = F_i' = -(S+P_j)$, $G_i = H_i = \mathbf{0}$ and, conversely, if *d)* holds, then $\Theta = T'\mathcal{M}T > \mathbf{0}$ with T' given by

$$T' = \begin{bmatrix} A_i' & \mathbf{I} & \mathbf{0} \\ A_{di}' & \mathbf{0} & \mathbf{I} \end{bmatrix} \qquad (18)$$

■

A first remark on Theorem 1 is that it provides LMI conditions to assess the stability of system (1) for arbitrary switching functions and irrespective of the time-delays, which can be even unbounded. Although the conditions *a)-d)* are equivalent for the stability analysis, the extra matrix variables F_i, G_i, H_i in *d)* can be used to reduce the conservatism in the control synthesis problem. A second remark is that $P(\alpha(k))$ is a switched matrix in the Lyapunov-Krasovskii functional, thus encompassing the results of the functional with fixed matrices P and S. Indeed, the quadratic stability condition can be recovered from the special choice of fixed matrices $P_i = P$, $F_i = F_i' = -(S+P)$, $G_i = H_i = \mathbf{0}$ in *d)*, as stated in the next corollary.

Corollary 1: If there exist symmetric positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} P+S & -(P+S)\tilde{A}_i & -(P+S)\tilde{A}_{di} \\ \star & P & \mathbf{0} \\ \star & \star & S \end{bmatrix} > \mathbf{0}, \quad i = 1,\dots,N \qquad (19)$$

then (8)-(12) hold with $P(\alpha) = P(\alpha(k+1)) = P$, implying that system (1), with $u(\sigma(k), x(k)) = u(k-d) = \mathbf{0}$, is stable for arbitrary switching functions, irrespective of the value d of the time delay.

In the next section, condition *d)* of Theorem 1 is extended to cope with Problem 2, providing LMI conditions for the synthesis of switched stabilizing gains for system (1).

IV. STABILIZABILITY

Theorem 2: If there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$, and matrices $F_i \in \mathbb{R}^{n\times n}$, $Z_i \in \mathbb{R}^{m_1\times n}$ and $Z_{di} \in \mathbb{R}^{m_2\times n}$, $i = 1,\dots,N$, such that

$$\begin{bmatrix} -(F_i+F_i'+P_j+S) & F_iA_i' + Z_i'B_i' & F_iA_{di}' + Z_{di}'B_{di}' \\ \star & P_i & \mathbf{0} \\ \star & \star & S \end{bmatrix} > \mathbf{0}, \quad (i,j) \in \mathcal{I} \times \mathcal{I} \qquad (20)$$

then the switched state feedback gains K_i and K_{di} given by

$$K_i = Z_i(F_i')^{-1}, \quad K_{di} = Z_{di}(F_i')^{-1}, \quad i = 1,\dots,N \qquad (21)$$

are such that the closed-loop system (4) is stable for any arbitrary switching function $\sigma(k)$, irrespective of the value d of the time delay.

Proof: Notice that, since $P_j > \mathbf{0}$, $S > \mathbf{0}$, one has $-(F_i + F_i') > \mathbf{0}$ thus implying that F_i are nonsingular matrices. Condition (20) comes from (15), for the special choice $G_i = H_i = \mathbf{0}$, $i = 1,\dots,N$ and replacing A_i by $(A_i + B_iK_i)'$ and A_{di} by $(A_{di} + B_{di}K_{di})'$ (dual system). ■

Observe that Theorem 2 gives a solution to Problem 2 based on the feasibility of a set of LMIs. The closed-loop stability is assured by means of the Lyapunov-Krasovskii functional given by (8). Moreover, with the conditions of Theorem 2 it is possible to determine switched gains (21) for the state feedback control law (3). Notice that the value of the delay d does not appear in the conditions of Theorem 2, but the delayed state vector $x(k-d)$ can be used in the control action. A special choice of matrices in (20) allows to determine fixed (robust) stabilizing gains, as shown in the next corollary.

Corollary 2: If there exist symmetric positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ and matrices $F \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{m_1\times n}$ and $Z_d \in \mathbb{R}^{m_2\times n}$ such that

$$\left[\begin{array}{cc} -(F+F'+P+S) & FA_i'+Z'B_i' \\ \star & P \\ \star & \star \end{array}\right.$$
$$\left.\begin{array}{c} FA_{di}'+Z_d'B_{di}' \\ \mathbf{0} \\ S \end{array}\right] > \mathbf{0}, \quad i=1,\ldots,N \qquad (22)$$

then the robust state feedback gains K and K_d given by

$$K = Z(F')^{-1} \quad , \quad K_d = Z_d(F')^{-1} \qquad (23)$$

are such that the the closed-loop system

$$x(k+1) = (A_{\sigma(k)} + B_{\sigma(k)}K)x(k)) + (A_{d\sigma(k)} + B_{d\sigma(k)}K_d)x(k-d) \qquad (24)$$

is quadratically stable for any arbitrary switching function $\sigma(k)$, irrespective of the value of the d of the time-delay.

Proof: Straightforward, choosing $P_j = P_i = P$ and $F_i = F_i' = F$ in (20). ■

Note that Theorem 2 encompasses Corollary 2, leading to less conservative results for the synthesis at the price of implementing a switched control strategy.

Both results of Theorem 2 and Corollary 2 can be used to determine decentralized control gains, by imposing block diagonal structures to the matrices, as for instance, by imposing to the matrices of Corollary 2 the constraints

$$F = F_D = \text{block-diag}\{F^1, \ \ldots, \ F^M\} \qquad (25)$$

$$Z = Z_D = \text{block-diag}\{Z^1, \ \ldots, \ Z^M\} \qquad (26)$$

$$Z_d = Z_{dD} = \text{block-diag}\{Z_d^1, \ \ldots, \ Z_d^M\} \qquad (27)$$

with M being the number of subsystems, yielding the block-diagonal stabilizing feedback gains

$$K_D = Z_D(F_D')^{-1} \quad , \quad K_{dD} = Z_{dD}(F_D')^{-1} \qquad (28)$$

Note that in this case no structural constraint is imposed to matrices P_i, $i=1,\ldots,N$, and S used in the Lyapunov-Krasovskii functional. Moreover, it is also possible to design state feedback control laws that use only $x(k)$ or $x(k-d)$ for feedback, decentralized or not, by simply fixing $Z = \mathbf{0}$ or $Z_d = \mathbf{0}$ in the LMIs.

Considering expressions (13)-(15), one has that the number of scalar variables are: $\mathcal{K}_{14} = n(N+1)(n+1)/2$, $\mathcal{K}_{15} = \mathcal{K}_{14}$, $\mathcal{K}_{16} = \mathcal{K}_{14} + 3n^2N$, and the number of LMI rows are: $\mathcal{L}_{14} = 2nN^2$, $\mathcal{L}_{15} = 3nN^2$, $\mathcal{L}_{16} = \mathcal{L}_{15}$. Efficient algorithms can solve these problems in polynomial time nowadays.

V. EXAMPLES

Example 1: consider the autonomous system (1) (i.e. $u(\sigma(k), x(k)) = u(k-d) = \mathbf{0}$) with four randomly generated subsystems given by

$$A_1 = \begin{bmatrix} 0.166 & 0.209 \\ -0.273 & -0.001 \end{bmatrix}, A_{d1} = \begin{bmatrix} -0.148 & -0.150 \\ -0.351 & 0.098 \end{bmatrix} \qquad (29)$$

$$A_2 = \begin{bmatrix} -0.123 & -0.206 \\ 0.269 & -0.013 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.210 & -0.123 \\ -0.209 & 0.055 \end{bmatrix} \qquad (30)$$

$$A_3 = \begin{bmatrix} 0.430 & -0.465 \\ 0.120 & 0.003 \end{bmatrix}, \ A_{d3} = \begin{bmatrix} -0.537 & -0.461 \\ 0.324 & 0.308 \end{bmatrix} \qquad (31)$$

$$A_4 = \begin{bmatrix} 0.210 & -0.359 \\ 0.107 & 0.088 \end{bmatrix}, \ A_{d4} = \begin{bmatrix} 0.215 & -0.235 \\ -0.131 & -0.157 \end{bmatrix} \qquad (32)$$

This system is not quadratically stable, that is, Corollary 1, based on the quadratic stability condition, is unfeasible. On the other hand, Theorem 1 provides a feasible solution assuring that the switched system with matrices (29)-(32) is stable for any arbitrary switching function $\sigma(k)$ and irrespective of the value d of the time-delay. This illustrates how the use of a switched Lyapunov-Krasovskii functional can reduce the conservatism of the stability analysis.

Example 2: the second example is borrowed from [20] with the matrices

$$A_1 = \begin{bmatrix} -0.545 & -0.43 \\ 0.185 & -0.61 \end{bmatrix}, \ A_{d1} = \begin{bmatrix} 0.24 & 0.07 \\ -0.12 & 0.09 \end{bmatrix} \qquad (33)$$

$$A_2 = \begin{bmatrix} -0.455 & -0.37 \\ 0.215 & -0.59 \end{bmatrix}, \ A_{d2} = \begin{bmatrix} 0.36 & 0.13 \\ -0.08 & 0.11 \end{bmatrix} \qquad (34)$$

representing here two subsystems of a discrete-time switched system with delay.

Using the conditions of Corollary 1 one gets

$$P = \begin{bmatrix} 0.568 & -0.041 \\ -0.041 & 0.971 \end{bmatrix} \ S = \begin{bmatrix} 0.373 & -0.020 \\ -0.020 & 0.402 \end{bmatrix}$$

implying that the uncertain system is quadratically stable irrespective of the value d of the time-delay (not only for $d=2$ as in [20]) and for any arbitrary switching function.

Suppose now that the subsystems (33)-(34) are perturbed by a fixed positive value $\rho \geq 1$, yielding a description $(\rho A_1, \rho A_2, \rho A_{d1}, \rho A_{d2})$. Moreover, consider a control problem, with input matrices

$$B_1 = B_2 = B_{d1} = B_{d2} = \begin{bmatrix} 0 & 1 \end{bmatrix}' \qquad (35)$$

Table I summarizes the comparison of the bounds of stabilizability, measured by the maximum value of the perturbation ρ in this case, for six control strategies: $T2(K_i, K_{di})$ for the system controlled by the switched gains K_i, K_{di} from Theorem 2, $T2(K_i)$ for the system controlled by the switched gains K_i, from Theorem 2, $T2(K_{di})$ for the system controlled by the switched gains K_{di}, from Theorem 2, $C2(K, K_d)$ for the system controlled by the fixed gains K, K_d from Corollary 2, $C2(K)$ for the system controlled by the fixed gains K, from Corollary 2, and $C2(K_d)$ for the system controlled by the fixed gains K_d, from Corollary 2.

The switching control strategy provides larger bounds of stabilizability, as expected. Of course, when both state vectors are available for feedback, Theorem 2 and Corollary 2 provide their respective largest bounds of stabilizability.

TABLE I

BOUNDS OF STABILIZABILITY FOR THE SYSTEM OF EXAMPLE 2.

Control strategy	$\max(\rho)$
$T2(K_i, K_{di})$	4.36
$T2(K_i)$	3.07
$T2(K_{di})$	1.24
$C2(K, K_d)$	3.93
$C2(K)$	2.84
$C2(K_d)$	1.22

Example 3: consider the system matrices

$$A_1 = \begin{bmatrix} 0.998 & -0.603 \\ 0.398 & 0.497 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.498 & 0.199 \\ 0.598 & 0.399 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.002 & -0.597 \\ 0.402 & 0.503 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} 0.502 & 0.201 \\ 0.602 & 0.401 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.10 & 0.05 \\ -0.20 & -0.05 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.30 & 0.35 \\ 0.20 & 0.25 \end{bmatrix}$$

and $B_{d1} = B_{d2} = \mathbf{0}$, representing here two subsystems of the discrete-time switched system given by (1). This system was found quadratically stabilizable through a memoryless feedback gain, irrespective of the value of the delay d in [26]. The use of switched matrices can provide improved results for the stabilizability.

For instance, consider that matrix B_1 is multiplied by a parameter $0 < \rho \leq 1$, such that ρB_1 models possible failures in the actuators when the system operates at subsystem 1. When ρ tends to zero, subsystem 1 tends to a condition of total failure in the actuators and when $\rho = 1$, no failure occurs.

Corollary 2, based on fixed matrices P and S, is feasible for $0.16 \leq \rho \leq 1$, while Theorem 2 is feasible for $0.04 \leq \rho \leq 1$, which represents a significant improvement in the bounds of stabilizability provided by the use of switched matrices in the Lyapunov-Krasovskii functional.

Example 4: consider the system with randomly generated subsystems

$$A_1 = \begin{bmatrix} 0.1441 & 0.3466 \\ 0.4120 & 0.0997 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.3427 & 0.5191 \\ 0.9600 & 0.6097 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.7860 & 0.2990 \\ 0.0326 & 0.7344 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} 0.1843 & 0.3964 \\ 0.9692 & 0.4705 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B_{d1} = \begin{bmatrix} 0.7206 \\ 0.5428 \end{bmatrix}, \; B_{d2} = \begin{bmatrix} 0.1027 \\ 0.3854 \end{bmatrix}$$

Note that, by assumption, only the delayed control input $u(k-d)$ is used for feedback. This system is not stabilizable by fixed gains from Corollary 2 but the switched gains

$$K_{d1} = \begin{bmatrix} -134.1310 & -39.2750 \end{bmatrix}, \quad K_{d2} = \begin{bmatrix} 238.9429 & 113.0072 \end{bmatrix}$$

from Theorem 2 can stabilize the system for arbitrary switching functions, based only on the information of the delayed state.

This example has its appeal from a situation that can rise when discrete-time control techniques are used to control continuous-time plants. The analog variables from the plant to be used in a state feedback control strategy must be conditioned, sampled, converted to digital variables, and processed to generate the control signal. Very frequently, at the instant k, the control signal is synthesized based on the information from $k - d$, as illustrated in this example.

Example 5: this example is inspired in the combined marketing and production control problem studied in [27], where the system behavior is represented by

$$x(k+1) = Ax(k) + A_d x(k-d) + Bu(k) + B_d u(k-d) \quad (36)$$

The state vector is $x(k) = [s_1(k)\; s_2(k)\; i_1(k)\; i_2(k)]'$, where $s_1(k)$, $s_2(k)$ represent the amount of sales of the products 1 and 2, respectively, during the k-th period and $i_1(k)$, $i_2(k)$ are the amount of inventory of products 1 and 2 at the end of the k-th period. The control vector is given by $u(k) = [q_1(k)\; q_2(k)\; a_1(k)\; a_2(k)]'$, with $q_1(k) = p_1(k+1)$ and $q_2(k) = p_2(k+1)$. The variables $p_1(k)$, $p_2(k)$ are the amount of products 1 and 2 during the k-th period and $a_1(k)$, $a_2(k)$ are the advertisement costs spent for products 1 and 2 during the k-th period.

The system matrices are given by

$$A = \begin{bmatrix} 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.7 & 0 & 1.0 & 0 \\ 0 & -0.5 & 0 & 1.0 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 8.0 & 0 \\ 0 & 0 & 0 & 7.0 \\ 1.0 & 0 & -8.0 & 0 \\ 0 & 1.0 & 0 & -7.0 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0 & 0 & 0 & 2.0 \\ 0 & 0 & 3.0 & 0 \\ 0 & 0 & 0 & -2.0 \\ 0 & 0 & -3.0 & 0 \end{bmatrix}$$

This problem was studied in [27] in the context of a precisely known linear system subject to state delay, with constrained state variables and control action, and with the control law obtained through the minimization of a quadratic cost function.

Here, consider that the system (36) is subject to sudden changes on the matrices B and B_d, which can model abrupt variations in the conditions of production and in the cost of the advertisement. It is also supposed that an independent control action related with the delay can occur and that the state variables and the control action are unconstrained.

Suppose that the actuators are subject to failures modeled as $(\rho B, \rho B_d)$, with $0.5 \leq \rho \leq 1$. Both conditions of Theorem 2 and of Corollary 2 provide a feasible solution. For instance, if the aim is robust stabilization (fixed gains), Corollary 2 provides the state feedback gains

$$K = \begin{bmatrix} 0 & 0 & -1.2000 & 0 \\ 0 & 0 & 0 & -1.2000 \\ -0.1050 & 0 & 0 & 0 \\ 0 & -0.0857 & 0 & 0 \end{bmatrix}$$

$$K_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0400 & 0 & 0 & 0 \\ 0 & -0.1200 & 0 & 0 \end{bmatrix}$$

with matrices of the Lyapunov-Krasovskii functional given by

$$P = \begin{bmatrix} 22.5304 & 0 & 1.3904 & 0 \\ 0 & 23.3531 & 0 & 1.0361 \\ 1.3904 & 0 & 22.5304 & 0 \\ 0 & 1.0361 & 0 & 22.4828 \end{bmatrix}$$

$$S = \begin{bmatrix} 22.5304 & 0 & 1.3904 & 0 \\ 0 & 23.3531 & 0 & 1.0361 \\ 1.3904 & 0 & 22.5304 & 0 \\ 0 & 1.0361 & 0 & 22.4828 \end{bmatrix}$$

ensuring the closed-loop system stability.

VI. CONCLUSION

Convex delay independent LMI conditions have been given for the robust stability and robust stabilizability of discrete-time switched linear systems with arbitrary switching functions and with unknown and unbounded state delays. The use of switched matrices in the Lyapunov-Krasovskii functional and extra matrices in the conditions provide less conservative evaluations of stability domains for this class of switched systems and, additionally, allows the design of state feedback gains by convex procedures, encompassing previous results based on quadratic stability. Additional constraints can be incorporated in the feedback gains without imposing a special structure to the Lyapunov-Krasovskii matrices. Numerical examples including a marketing and production control illustrate the efficiency of the proposed techniques.

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