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New Results on Practical Stabilization and Practical Reachability of Switched Systems

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***Abstract*— The present paper has two main contributions. First, we report new sufficient conditions for practical stabilizability of switched systems. Such conditions are geometrically more appealing and easier to check than conditions proposed in our previous papers. The conditions are applicable to practical stabilizability problems over infinite or finite time intervals. Second, we study the practical reachability problems in the light of our practical stabilization results. Sufficient conditions for practical reachability are presented.**

I. INTRODUCTION

Recently, it has been observed that, under appropriate switching laws, switched systems whose subsystems have no common equilibrium may still exhibit interesting behavior similar to that of a conventional stable system near an equilibrium. In [11], [12], [13], we define such behavior as practical stability. Such practical stability notions are extensions of the traditional concepts on practical and finite time stability (see, e.g., [4], [5], [8], [9]), which are concerned with bringing the system trajectory to be within given bounds. They are quite different from the notions of Lyapunov stability for hybrid systems (see, e.g., [2], [3], [6], [7]) and can lead to interesting results in stability analysis, tracking, and reachability problems.

In [12], we formally introduce the notions of ϵ-practical stability and practical stabilizability. Sufficient conditions are proven in [12] for the practical stabilizability of switched systems consisting of time varying subsystems. Moreover, in the proof, a valid switching law under which the system is ϵ-practically stable is constructed. In this paper, we report some new results on practical stabilizability and practical reachability problems. There are two main contributions. The first contribution is that we report new sufficient conditions for practical stabilizability of switched systems. Such conditions are geometrically more appealing and easier to check than conditions in [12]. Moreover, the conditions are applicable to practical stabilizability problems over infinite or finite time intervals. Our second contribution is that we formally propose the notion of practical reachability, which concerns the problem of driving the system state from a neighborhood of one point to a neighborhood of another point. We show that our practical stabilization results can be applied to study such reachability problems. In particular, we propose conditions for practical reachability along some nominal guiding paths.

II. PRELIMINARIES

A. Switched Systems and Switching Laws

We consider switched systems consisting of subsystems

$$\dot{x} = f_i(x,t), \quad i \in I \triangleq \{1, 2, \cdots, M\}. \tag{1}$$

In (1), every $f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous vector field. The active subsystem at each instant is orchestrated by a switching law, which will be formally defined below. Given any initial time t_0 and state $x(t_0)$, the law generates a switching sequence $\sigma = \big((t_0, i_0), (t_1, i_1), \cdots, (t_k, i_k), \cdots\big)$ ($t_0 \le t_1 \le \cdots \le t_k \le \cdots$, $i_k \in I$) which indicates that subsystem i_k is active in $[t_k, t_{k+1})$. For a switched system to be well-behaved, we only consider **nonZeno** sequences which switch at most a finite number of times in any finite time interval.

In studying the behavior of switched system (1), we usually specify some time interval $\mathcal{T}$ in which a state trajectory is generated. In the sequel, when we refer to $\mathcal{T}$, we mean some infinite time interval $\mathcal{T} = [t_0, \infty)$ or finite time interval $\mathcal{T} = [t_0, t_f]$. The following definition specifies more clearly what we mean by a switching law.

Definition 1 (Switching Law over $\mathcal{T}$): Given a time interval $\mathcal{T}$ ($\mathcal{T} = [t_0, \infty)$ or $\mathcal{T} = [t_0, t_f]$), a switching law $\mathcal{S}$ over $\mathcal{T}$ is defined to be a mapping $\mathcal{S} : \mathbb{R}^n \to \Sigma_{\mathcal{T}}$ which specifies a nonZeno switching sequence $\sigma \in \Sigma_{\mathcal{T}}$ for any initial state $x(t_0)$. Here $\Sigma_{\mathcal{T}} \triangleq \{\text{switching sequence } \sigma \text{ over } \mathcal{T}\}$. □

Remark 1: $\mathcal{S}$ over a given $\mathcal{T}$ is often determined by some rules or algorithms, which describe how to generate a switching sequence for a given $x(t_0)$, rather than mathematical formulae. In this paper, we specify switching laws using such descriptions. □

B. Review of Some Practical Stabilization Results

Now we review some notions and results from [12]. Due to our interest in reachability problems which often concern finite time intervals, we will slightly modify these notions and results so as to include finite interval cases. In fact, finite time stability problems for dynamical systems (see, e.g., [10]) and hybrid systems (see, e.g., [14], [15]) deserve attention not only in their own right but also in their close connection to reachability problems. In the following, the vector (and matrix) norm $\|\cdot\|$ denotes the 2-norm; and $B[x, r]$ denotes the closed ball $\{y \in \mathbb{R}^n : \|y - x\| \le r\}$.

Without loss of generality, we only discuss the practical stability of the system around the origin.

Definition 2 (ϵ-Practical Stability over $\mathcal{T}$): Assume that a time interval $\mathcal{T}$ ($\mathcal{T} = [t_0, \infty)$ or $\mathcal{T} = [t_0, t_f]$) is given. Also assume that a switching law $\mathcal{S}$ over $\mathcal{T}$ is given for switched system (1). Given an $\epsilon > 0$, the system is said to be *ϵ-practically stable over $\mathcal{T}$ under $\mathcal{S}$* if $\exists\delta > 0$ (δ depends on ϵ and the interval $\mathcal{T}$) such that $x(t) \in B[0, \epsilon]$, $\forall t \in \mathcal{T}$, whenever $x(t_0) \in B[0, \delta]$. □

Remark 2: Unlike the classical stability concept, we do not assume $f_i(0, t) = 0, \forall i \in I$, i.e., the origin doesn't have to be a common equilibrium for system (1). Moreover, unlike the classical definition which is based on the existence of δ for any ϵ, here the ϵ is given and hence does not vary (this is why we term it ϵ-practical stability). □

Definition 3 (ϵ-Practical Stabilizability over $\mathcal{T}$): Assume that a time interval $\mathcal{T}$ ($\mathcal{T} = [t_0, \infty)$ or $\mathcal{T} = [t_0, t_f]$) is given. Given an $\epsilon > 0$, system (1) is said to be *ϵ-practically stabilizable over $\mathcal{T}$* if there exists a switching law $\mathcal{S}$ over $\mathcal{T}$ ($\mathcal{S}$ depends on ϵ) such that the system is ϵ-practically stable over $\mathcal{T}$ under $\mathcal{S}$. □

Definition 4 (Practical Stabilizability over $\mathcal{T}$): Assume a time interval $\mathcal{T}$ ($\mathcal{T} = [t_0, \infty)$ or $\mathcal{T} = [t_0, t_f]$) is given. System (1) is said to be *practically stabilizable over $\mathcal{T}$* if it is ϵ-practically stabilizable over $\mathcal{T}$ for any $\epsilon > 0$. □

Remark 3: In Definition 4, ϵ can be chosen to be any positive value as opposed to the fixed ϵ in Definition 2. Hence a practically stabilizable system over $\mathcal{T}$ has the property that, for any given $B[0, \epsilon]$, a valid switching law can be found that keeps the system trajectories, whose initial states are in $B[0, \delta]$, in $B[0, \epsilon]$. □

The following lemma from [12] provides us with some sufficient conditions for practical stabilizability.

Lemma 1 ([12]): Given a time interval $\mathcal{T}$ ($\mathcal{T} = [t_0, \infty)$ or $\mathcal{T} = [t_0, t_f]$), switched system (1) is practically stabilizable over $\mathcal{T}$ if it satisfies the following conditions:

(a). There exists a $G_1 > 0$ such that for any $t \in \mathcal{T}$, every $x \in B[0, 1]$ can be expressed as

$$x = \sum_{i=1}^{M} \gamma_i(t) f_i(0, t) \tag{2}$$

where the $\gamma_i(t)$'s satisfy $\gamma_i(t) \leq 0$, $i \in I$ and

$$\sum_{i=1}^{M} |\gamma_i(t)| \leq G_1. \tag{3}$$

(b). There exists a $G_2 > 0$ such that $\|f_i(0, t)\| \leq G_2$, $\forall t \in \mathcal{T}, \forall i \in I$.

(c). $f_i(x, t)$'s satisfy the Lipschitz condition in x around the origin, i.e., $\exists L_1 > 0$ such that

$$\|f_i(x_1, t) - f_i(x_2, t)\| \leq L_1 \|x_1 - x_2\| \tag{4}$$

$\forall x_1, x_2$ in some ball $B[0, r]$ and $\forall t \in \mathcal{T}, \forall i \in I$.

(d). $f_i(0, t)$'s as functions of t satisfy the Lipschitz condition in t, i.e., $\exists L_2 > 0$ such that

$$\|f_i(0, t_1) - f_i(0, t_2)\| \leq L_2 |t_1 - t_2| \tag{5}$$

$\forall t_1, t_2 \in \mathcal{T}$ and $\forall i \in I$. □

In the proof of Lemma 1 in [12], we obtain the δ as in Definition 2 for any given ϵ as follows. We choose $\delta = \frac{\epsilon_1}{2G}$ where $G = 1 + G_1 G_2$ and ϵ_1 satisfies $0 < \epsilon_1 \leq \min\{\epsilon, \frac{1}{(L_1 + L_2 \frac{G_1}{2G}) G_1}, r\}$. Moreover, an ϵ-practically stabilizing switching law over $\mathcal{T}$ is constructed as:

Switching Law (for system (1) with $x(t_0) \in B[0, \delta]$)

(1). Assume that the system trajectory starts from $x(t_0) \in B[0, \delta]$. Set $k = 0$, $T_k = t_0$ and the current state $x(T_k) = x(t_0)$.

(2). Express $\frac{x(T_k)}{\delta} = \sum_{i=1}^{M} \gamma_i(T_k) f_i(0, T_k)$ (in this case, $\frac{x(T_k)}{\delta} \in B[0, 1]$, hence condition (a) of Lemma 1 applies). So the current state $x(T_k) = \sum_{i=1}^{M} \delta\gamma_i(T_k) f_i(0, T_k)$. First switch to subsystem 1 and stay for time $\delta|\gamma_1(T_k)|$, then switch to subsystem 2 and stay for time $\delta|\gamma_2(T_k)|$ and so on, i.e., we obtain a switching sequence $\big((T_k, 1), (T_k + \delta|\gamma_1(T_k)|, 2), (T_k + \delta|\gamma_1(T_k)| + \delta|\gamma_2(T_k)|, 3), \cdots, (T_k + \delta|\gamma_1(T_k)| + \cdots + \delta|\gamma_{M-1}(T_k)|, M)\big)$ from time T_k to $\tilde{T}_k \triangleq T_k + \sum_{i=1}^{M} \delta|\gamma_i(T_k)|$.

(3). From time $\tilde{T}_k$ on, let subsystem M be active until the state trajectory intersects the δ-sphere.

(4). When the state intersects the δ-sphere, set $k = k + 1$ and denote T_k to be the time instant of intersection (hence, if intersection takes place, then $x(T_k)$ is the intersecting point). Go back to step (2). □

III. New Sufficient Conditions for Practical Stabilizability

Conditions (b), (c), and (d) in Lemma 1 are straightforward to check. However, checking condition (a) presents some difficulty. In this section, we will present new conditions that are equivalent to condition (a) and are geometrically more appealing. In the following, we use conv(A) to denote the convex hull of a finite subset $A = \{a_1, \cdots, a_M\}$ of $\mathbb{R}^n$, i.e., $\text{conv}(A) = \{\sum_{i=1}^{M} \lambda_i a_i : \lambda_1 \geq 0, \cdots, \lambda_M \geq 0, \sum_{i=1}^{M} \lambda_i = 1\}$. We denote the interior of a set C as Int(C) and the boundary of C as ∂C.

The following theorem presents a condition which is equivalent to condition (a) in Lemma 1.

Theorem 1: Condition (a) in Lemma 1 is satisfied if and only if $0 \in \text{Int}\big(\bigcap_{t \in \mathcal{T}} C(t)\big)$ where $C(t)$ is the convex hull $C(t) = \text{conv}\big(\{f_i(0, t) : i \in I\}\big)$.

Proof: "If" part: if $0 \in \text{Int}\big(\bigcap_{t \in \mathcal{T}} C(t)\big)$, then $\exists r_1 > 0$ s.t. $B[0, r_1] \subseteq \bigcap_{t \in \mathcal{T}} C(t)$. So for any $t \in \mathcal{T}$, $B[0, r_1] \subseteq C(t)$. Note that any $x \in B[0, 1]$ can always be represented as $x = \frac{y}{r_1}$ where $y \in B[0, r_1]$. For this y, we have $-y \in B[0, r_1] \subseteq C(t)$ and hence $-y$ can be represented as a convex combination $-y = \sum_{i=1}^{M} \lambda_i(t) f_i(0, t)$ with $\lambda_i(t) \geq 0, i \in I$ and $\sum_{i=1}^{M} \lambda_i(t) = 1$ at any $t \in \mathcal{T}$. This leads to $x = \frac{y}{r_1} = \sum_{i=1}^{M} \big(-\frac{\lambda_i(t)}{r_1}\big) f_i(0, t)$, $\forall t \in \mathcal{T}$. Such an expression of x satisfies condition (a) in Lemma 1 since we can let

$\gamma_i(t) = -\frac{\lambda_i(t)}{r_1}$. Also note that since $\sum_{i=1}^{M} |\gamma_i(t)| = \frac{1}{r_1}$, $\forall t \in \mathcal{T}$, we can choose $G_1 = \frac{1}{r_1}$ in condition (a).

"Only if" part: we only need to note that condition (a) in Lemma 1 implies that $B[0,1] \subseteq \{\sum_{i=1}^{M} \lambda_i(t) f_i(0,t) : \lambda_i(t) \geq 0, \forall i \in I$ and $\sum_{i=1}^{M} \lambda_i(t) = G_1\}$. Hence $B[0, \frac{1}{G_1}] \subseteq C(t), \forall t \in \mathcal{T}$. □

Unlike condition (a) in Lemma 1 which is algebraic in nature, the condition proposed in Theorem 1 has direct geometric relevance. In particular, in the case of finite interval, the above condition can be further reduced to conditions more amenable to be checked.

Theorem 2: For a finite time interval $\mathcal{T} = [t_0, t_f]$, $0 \in \text{Int}\big(\bigcap_{t\in T} C(t)\big)$ if and only if $0 \in \text{Int}\big(C(t)\big)$ for any $t \in \mathcal{T}$.

Proof: "Only if" part: it is clear that $0 \in \text{Int}\big(\bigcap_{t\in T} C(t)\big)$ directly leads to $0 \in \text{Int}\big(C(t)\big)$, $\forall t \in \mathcal{T}$.

"If" part: for any convex hull $C(t)$, we define the distance between the origin and the boundary of the set $C(t)$ to be $d(t) = \min_{y \in \partial C(t)} \|y\|$. Due to our assumption that every $f_i(x,t)$ is continuous and hence $f_i(0,t)$ is continuous in t, we conclude the boundary $\partial C(t)$ also evolves in a continuous fashion. Therefore $d(t)$ is a continuous function of t over $\mathcal{T}$. Since $\mathcal{T} = [t_0, t_f]$ is compact, $d(t)$ reaches its minimum at some point $t^* \in \mathcal{T}$, which in turn is a positive minimum $d(t^*) > 0$ by assumption. Consequently we have $d(t) \geq d(t^*)$, $\forall t \in \mathcal{T}$, which leads to $B[0, d(t^*)] \subseteq C(t), \forall t \in \mathcal{T}$. □

Theorem 2 provides us with an easy way of verifying condition (a) in Lemma 1 if a finite time interval is considered. In particular, it avoids the burden of calculating $\bigcap_{t\in T} C(t)$. In this case, we only need to check if 0 is in the interior of every $C(t)$. Such a task can usually be accomplished by geometric observations or appropriate computational methods.

Example 1: Consider a switched system (1) in $\mathbb{R}^2$ which consists of 4 subsystems with $f_1(x,t) = [x_1 + t,\ x_2 + 2t]^T$, $f_2(x,t) = [-x_1 - 2t,\ x_2 + t]^T$, $f_3(x,t) = [x_1 - 2t,\ -x_2 - 3t]^T$, $f_4(x,t) = [2x_1 + t,\ x_2 - 3t]^T$. Given any finite time interval $\mathcal{T} = [t_0, t_f]$ in which $0 < t_0 < t_f$, we can show that the system is practically stabilizable over $\mathcal{T}$ as follows.

First consider $f_1(0,t) = [t,\ 2t]^T$, $f_2(0,t) = [-2t,\ t]^T$, $f_3(0,t) = [-2t,\ -3t]^T$, $f_4(0,t) = [t,\ -3t]^T$. For any $t \in \mathcal{T}$ (hence $t > 0$) we observe that $f_i(0,t)$ is inside the i-th quadrant ($i = 1,2,3,4$). Therefore $C(t)$ includes 0 as an interior point. Consequently, the condition of Theorem 2 is satisfied. This then establishes condition (a) in Lemma 1.

Since $\mathcal{T}$ is bounded, $f_i(0,t)$'s are bounded for any $t \in \mathcal{T}$ and any $i \in \{1,2,3,4\}$. Hence condition (b) in Lemma 1 is satisfied. Also note that since every f_i is linear in x and linear in t, conditions (c) and (d) in Lemma 1 are also satisfied. □

Remark 4: Given a $t \in \mathcal{T}$, mathematical programming methods can also be used to determine whether the origin is outside or on the boundary of or inside $C(t)$. To do this, we can solve the following mathematical programming problem in $a \in \mathbb{R}$ and $d = [d_1, \cdots, d_n]^T \in \mathbb{R}^n$

$$\min\ a \tag{6}$$

$$\text{s.t.} \begin{cases} d^T f_i(0,t) \leq a,\ i = 1,2,\cdots,M \\ a \leq 0, \\ |d_1| + \cdots + |d_n| = 1 \end{cases} \tag{7}$$

If $a_{\min} < 0$, then $\exists d$ s.t. the angles between d and all $f_i(0,t)$'s are greater than $\frac{\pi}{2}$. So there exists a hyperplane strictly separating $C(t)$ and the origin. Similar argument leads to the conclusion that if $a_{\min} = 0$, then 0 is on $\partial C(t)$; if there is no feasible solution, then $0 \in \text{Int}\big(C(t)\big)$. The above optimization problem can be easily decomposed into several linear programming problems (only need to rewrite $|d_1| + \cdots + |d_n| = 1$ into different possible linear equalities).

To check whether $0 \in \text{Int}\big(C(t)\big)$, $\forall t \in \mathcal{T} = [t_0, t_f]$, we can partition $\mathcal{T}$ into small intervals and only check whether $0 \in \text{Int}\big(C(t)\big)$ at the grid points. This lets us solve only a finite number of problems (6)-(7). Based on continuity, when the intervals are sufficiently small, such a method can determine whether $0 \in \text{Int}\big(C(t)\big)$, $\forall t \in \mathcal{T}$. □

IV. Practical Reachability

In this section, we apply our practical stabilization results to practical reachability problems, which are concerned with driving the state from a neighborhood of a given initial point x_0 to a neighborhood of a given target point x_f.

Definition 5 (ϵ-Practical Reachability): Consider switched system (1). Given t_0 and x_0, a target point $x_f \in \mathbb{R}^n$ is said to be *ϵ-practically reachable from x_0 at t_0* if a switching law $\mathcal{S}$ over some finite interval $\mathcal{T} = [t_0, t_f]$, and a $\delta > 0$ (δ depends on ϵ, x_0, x_f, and t_0) exist, such that $x(t_f) \in B[x_f, \epsilon]$ whenever $x(t_0) \in B[x_0, \delta]$. □

Remark 5: The ϵ-practical reachability concerns the problem of driving the state from the δ-neighborhood of x_0 to the ϵ-neighborhood of x_f within finite time. Such neighborhood to neighborhood reachability is often desired in practice (see [10]), e.g., in sending a rocket from point A to point B over some nominal trajectory. □

Definition 6 (Practical Reachability): Consider switched system (1). Given t_0 and x_0, a target point $x_f \in \mathbb{R}^n$ is said to be *practically reachable from x_0 at t_0* if it is ϵ-practically reachable from x_0 at t_0 for any $\epsilon > 0$. □

Remark 6: In Definition 6, ϵ can be chosen to be any positive value as opposed to the fixed ϵ in Definition 5. So if x_f is practically reachable from x_0, we can find appropriate switching laws to achieve any desired accuracy specified by ϵ. However, for different accuracy requirements, the time intervals for reachability need not be the same. □

A. Practical Reachability along Guiding Paths

If a nominal guiding path from x_0 to x_f is specified, then we can study whether x_f is practically reachable from x_0 along it. Such guided reachability is in the sense that a switching law can be designed to keep the state within a neighborhood of the guiding path and eventually lead the state into a neighborhood of x_f.

Consider switched system (1). Given t_0, x_0, and x_f, we want to study whether x_f is practically reachable from x_0 at t_0 along some specified guiding path $y(t)$ that connects x_0 and x_f. We assume that $y(t)$ is continuous and has piecewise continuous derivative $g(t)$, i.e.,

$$\dot{y} = g(t) \tag{8}$$

over a finite time interval $\mathcal{T} = [t_0, t_f]$ (note t_f will be clearly specified later; at present t_f may simply be regarded as a prespecified constant) and $y(t_0) = x_0$, $y(t_f) = x_f$.

The practical reachability problem along the given guiding path requires us to find an appropriate switching law over some finite time interval $\mathcal{T} = [t_0, t_f]$ so that $x(t)$ is guided by $y(t)$ in the sense that $\|x(t) - y(t)\| \leq \epsilon$, $\forall t \in \mathcal{T}$ (here ϵ is a given tolerance level). Such a problem can be converted to a practical stabilization problem by subtracting (8) from (1) and defining $z(t) \stackrel{\triangle}{=} x(t) - y(t)$. By doing so, we obtain the following switched system whose state variable is $z \in \mathbb{R}^n$.

$$\dot{z} = f_i(x, t) - g(t) = f_i\big(z + y(t), t\big) - g(t),\ i \in I \tag{9}$$

In (9), we can regard $y(t)$ as a function of t (since it is the prespecified guiding path) and then define

$$h_i(z, t) \stackrel{\triangle}{=} f_i\big(z + y(t), t\big) - g(t). \tag{10}$$

Using (10), we can rewrite (9) as

$$\dot{z} = h_i(z, t),\ i \in I. \tag{11}$$

It can now be seen that the practical reachability problem along the guiding path for switched system (1) is equivalent to the practical stabilizability problem for switched system (11) over $\mathcal{T}$. Hence, the results in Sections II and III for practical stabilizability can be applied to system (11).

If every f_i in (1) is time invariant, i.e.,

$$\dot{x} = f_i(x, t) = f_i(x),\ i \in I \tag{12}$$

then $h_i(z, t) = f_i\big(z + y(t)\big) - g\big(y(t)\big)$. In such a case, t_0 is no longer important, we can simply choose it as 0. Below we consider such time invariant subsystems and consider a particularly important type of guiding path connecting x_0 and x_f — the line segment connecting them. Such a line segment can be generated by

$$\dot{y} = g(t) = a \tag{13}$$

where $a \in \mathbb{R}^n$ is a nonzero constant vector in the direction of $x_f - x_0$ (or, equivalently, $a = \frac{1}{t_f}(x_f - x_0)$ for some $t_f > 0$). By choosing an appropriate t_f, we have $\mathcal{T} = [0, t_f]$, and $y(t) = x_0 + at$ for $t \in \mathcal{T}$, and $z(t)$ satisfies

$$\dot{z} = h_i(z, t) = f_i\big(z + y(t)\big) - a,\ i \in I. \tag{14}$$

In particular, we have

$$h_i(0, t) = f_i\big(y(t)\big) - a,\ i \in I. \tag{15}$$

If such an a exists so that the switched system (14) is practically stabilizable over $\mathcal{T}$, then for switched system (12), x_f is practically reachable from x_0 along the line segment connecting them. The following theorem provides us with sufficient conditions for this to be true.

Theorem 3: Consider switched system (12). Given x_0 and x_f, x_f is practically reachable from x_0 along the line segment connecting them if the following conditions hold

(a). All $f_i(x)$'s are locally Lipschitz continuous.

(b). $0 \in \text{Int}\big(C(y)\big)$ for any y on the line segment connecting x_0 and x_f. Here $C(y) = \text{conv}\big(\{f_i(y) : i \in I\}\big)$.

Proof: We only need to prove that an $a \in \mathbb{R}^n$ exists such that the switched system (14) is practically stabilizable over $\mathcal{T} = [0, t_f]$ where $t_f = \frac{\|x_f - x_0\|}{\|a\|}$. So we only need to show that an a exists so that all the conditions of Lemma 1 can be satisfied for system (14).

First we show that condition (a) in Lemma 1 can be satisfied. Assume some $a \in \mathbb{R}^n$ is chosen (we will show how to choose it below). Then for each y on the line segment, there is a corresponding $t \in \mathcal{T}$ (here $\mathcal{T} = [0, t_f]$ is finite) such that $y = y(t)$. Hence $C(y) = C(t) = \text{conv}\Big(\{f_i\big(y(t)\big) : i \in I\}\Big)$. Due to condition (b) in Theorem 3, we have $0 \in \text{Int}\big(C(t)\big) = \text{Int}\big(C(y)\big)$. From Theorem 2, we have $0 \in \text{Int}\big(\bigcap_{t \in \mathcal{T}} C(t)\big)$. Hence there exists a $r_1 > 0$ such that $B[0, r_1] \subseteq \bigcap_{t \in \mathcal{T}} C(t)$. This consequently leads to $B[0, r_1] \subseteq C(t)$, $\forall t \in \mathcal{T}$, i.e., $B[0, r_1] \subseteq C(y)$ for any y on the line segment (in fact, r_1 needs not depend on the abovementioned $\mathcal{T}$). In such a case, we can choose $a \in \mathbb{R}^n$ which is parallel to $x_f - x_0$ and satisfies $\|a\| = \frac{r_1}{2}$. Then we have $B[a, \|a\|] \subseteq B[0, r_1]$. Next consider the convex hull $C_h(t)$ generated by the $h_i(0, t)$'s. Note that $C_h(t) = \text{conv}\{h_i(0, t) : i \in I\} = \{\sum_{i=1}^{M} \lambda_i \Big(f_i\big(y(t)\big) - a\Big) : \lambda_i \geq 0, \sum_{i=1}^{M} \lambda_i = 1\} = \{\Big(\sum_{i=1}^{M} \lambda_i f_i\big(y(t)\big)\Big) - a : \lambda_i \geq 0, \sum_{i=1}^{M} \lambda_i = 1\} = C\big(y(t)\big) - a$, where we use $C\big(y\big) - a$ to denote the set $\{z : z = y - a, y \in C(y)\}$. Because $B[0, \|a\|] = B[a, \|a\|] - a \subseteq B[0, r_1] - a \subseteq C(y) - a$ for any y on the line segment, we conclude that

$$B[0, \|a\|] \subseteq C_h(t), \forall t \in \mathcal{T}. \tag{16}$$

(16) satisfies the conditions in Theorems 2 and 1. Hence with the above choice of a, condition (a) in Lemma 1 is satisfied for system (14).

Condition (b) in Lemma 1 is satisfied since f_i's are continuous and $\mathcal{T}$ is finite (hence compact), which consequently leads to the boundedness of $h_i(0, t)$, $\forall t \in \mathcal{T}$, $\forall i \in I$.

Now let us show that condition (c) in Lemma 1 is satisfied. Since a locally Lipschitz continuous function f_i is Lipschitz on any compact subset of $\mathbb{R}^n$, there exists a constant $L_1 > 0$ so that all $f_i(x)$'s are Lipschitz with constant L_1 on a compact set which includes the line segment between x_0 and x_f in its interior. We then have

$$\begin{aligned}\|h_i(z_1, t) - h_i(z_2, t)\| &= \|f_i\big(z_1 + y(t)\big) - f_i\big(z_2 + y(t)\big)\| \\ &\leq L_1 \|z_1 - z_2\|\end{aligned}$$

for any z_1, z_2 in some neighborhood $B[0, r]$ (which is inside the abovementioned compact set) and $\forall t \in \mathcal{T}$, $\forall i \in I$.

Lastly, we show that condition (d) in Lemma 1 is satisfied. To see this, we have

$$\|h_i(0,t_1) - h_i(0,t_2)\| = \|f_i\big(y(t_1)\big) - f_i\big(y(t_2)\big)\|$$

$$\leq L_1\|y(t_1) - y(t_2)\| = L_1\|a(t_1 - t_2)\| = L_1\|a\| \cdot |t_1 - t_2|$$

for any $t_1, t_2 \in \mathcal{T}$ and $\forall i \in I$. $\square$

B. Guidable Regions

Now we further explore the practical reachability problem for switched systems (12) in the light of Theorem 3. We will assume that condition (a) is satisfied for system (12). We will pay special attention to condition (b). Due to it, we only need to check if $0 \in \text{Int}\big(C(y)\big)$ for any point y on the line segment between x_0 and x_f. Taking another viewpoint, if we define a subset R of $\mathbb{R}^n$ to be the collection of all such point y satisfying $0 \in \text{Int}\big(C(y)\big)$, i.e.,

$$R \stackrel{\triangle}{=} \{y \in \mathbb{R}^n : 0 \in \text{Int}\big(C(y)\big)\}, \tag{17}$$

then condition (b) is equivalent to the following condition:

(c). The line segment between x_0 and x_f lies in R.

It can be shown that R is open. This is because for any y satisfying $0 \in \text{Int}\big(C(y)\big)$, a neighborhood ball around y can be found such that for any point y_1 in it, we have $0 \in \text{Int}\big(C(y_1)\big)$ (due to the continuous evolvement of $\partial C(y)$ with respect to y). From mathematical analysis, we know that R can then be expressed in one and only one way as a finite or countable disjoint union of open connected subsets in $\mathbb{R}^n$ (see page 89 in [1]), i.e.,

$$R = \bigcup_i R_i,\ R_i \text{ open connected}, R_{i_1} \cap R_{i_2} = \emptyset, i_1 \neq i_2.$$

Hence, in order to satisfy condition (c), the line segment between x_0 and x_f must be within one of the open connected subsets. Such a requirement is sometimes overly restrictive. For example, it is possible that both x_0 and x_f lie in the same open connected subset R_i, but not every point on the line segment between them is in R_i. In this case, condition (c) cannot be applied. However, we can show that x_f is practically reachable from x_0 as follows.

Since an open connected subset R_i in $\mathbb{R}^n$ is also polygonally connected, i.e., every pair of points in R_i can be joined by a finite number of connected line segments (see page 89 in [1]), there exists a path in R_i from x_0 to x_f consisting of connected line segments with connection nodes in the order of $x_0, y_1, y_2, \cdots, y_K, x_f$ (such a path is also called a polygonal path). In this case, because the line segment between y_K and x_f is in R_i, we can apply Theorem 3 to conclude that x_f is practically reachable from y_K along the line segment between them. Similarly, we can conclude that y_k is practically reachable from y_{k-1} along the line segment between them for any $1 < k \leq K$, and y_1 is practically reachable from x_0 along the line segment between them. In view of these and Definition 6, it is not difficult to show that x_f is practically reachable from x_0 along this polygonal path. Fig. 1 illustrates such a case. In fact, the discussion here shows that any point in the open connected subset R_i is practically reachable from any other point in R_i along some guiding polygonal path. Because of this, R_i is also called a *guidable region*.

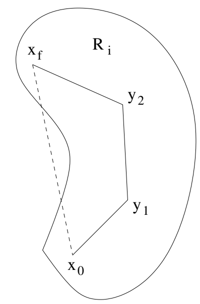

Fig. 1. x_f is not practically reachable from x_0 along the line segment between them, however, it is practically reachable from x_0 along a polygonal path.

In general, the computation of guidable regions are challenging and still under our research. However, for switched systems consisting of linear time invariant (LTI) subsystems

$$\dot{x} = A_i x,\ i \in I, \tag{18}$$

every guidable region is an open connected cone emitting from 0 and can be easily computed. This is because $0 \in \text{Int}\big(C(\alpha y)\big)$, $\forall \alpha > 0$, whenever $0 \in \text{Int}\big(C(y)\big)$. In view of this, we only need to find the portion of guidable regions on the unit sphere in order to characterize all guidable regions.

Example 2: Consider a LTI switched system (18) in $\mathbb{R}^2$ which consists of 3 subsystems:

$$\text{subsystem 1: } \dot{x} = f_1(x) = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} x,$$

$$\text{subsystem 2: } \dot{x} = f_2(x) = \begin{bmatrix} -0.2 & -0.2 \\ 0.2 & -0.2 \end{bmatrix} x,$$

$$\text{subsystem 3: } \dot{x} = f_3(x) = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.2 \end{bmatrix} x.$$

We can compute the guidable regions by observing the patterns of vector fields f_1, f_2, and f_3 on the unit circle in $\mathbb{R}^2$ (see Fig. 2). There exist 2 lines l_1 and l_2 which divide $\mathbb{R}^2$ into 4 closed conic regions $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 as shown in Fig. 2. In this case, we observe that there are two guidable regions. They are $R_1 = \text{Int}(\Omega_1)$ and $R_2 = \text{Int}(\Omega_3)$.

The equation of l_1 is obtained as follows. Note that $f_1 = cf_3$ for some $c < 0$ on l_1. From this, we can obtain that the slope of l_1 is $k = \frac{x_2}{x_1} = c - 2 = \frac{1}{c+2}$, which can be solved to obtain $c = -\sqrt{5}$ and $k = -\sqrt{5} - 2$. Consequently, the equation of l_1 is $x_2 = kx_1 = -(\sqrt{5} + 2)x_1$.

Similarly, the equation of l_2 can be obtained (by noting that $f_2 = cf_3$ for some $c < 0$ on l_2) as $x_2 = (\sqrt{2} - 1)x_1$.

Assume we are given $x_0 = [1,\ 0.6]^T$ and $x_f = [1.5,\ 1.2]^T$ which are all in R_1. Given $\epsilon = 0.1$, we can design a switching law to make x_f ϵ-practical reachable from

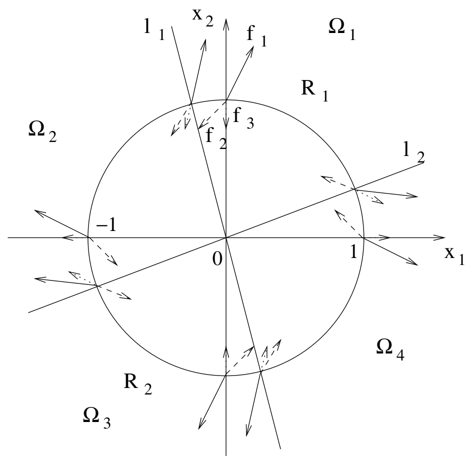

Fig. 2. The different patterns of vector fields, with f_1 denoted by solid vector, f_2 by dashed vector, and f_3 by dotted vector.

x_0 along the line segment between them. The switching law is designed based on the switching law in Section II. We can choose r_1 and a mentioned in the proof of Theorem 3 to be $r_1 = 0.04$ and $\|a\| = 0.02$. We choose the parameters mentioned in the switching law proposed in Section II to be $G_1 = 19.28$, $G_2 = 0.86$, $L_1 = 0.2$, $L_2 = 0.004$, and $\delta = 0.0028$. The state trajectory is shown in Fig. 3 and a zoomed-in view near x_0 is shown in Fig. 4. $\square$

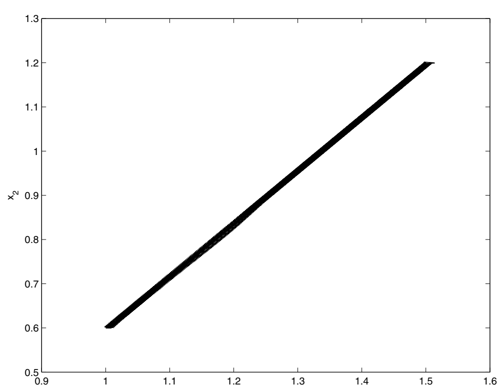

Fig. 3. The ϵ-practical reachable state trajectory along a line segment between $x_0 = [1,\ 0.6]^T$ and $x_f = [1.5,\ 1.2]^T$.

V. Conclusion

This paper reports some new results on practical stabilization and practical reachability problems of switched systems. We present new sufficient conditions for practical stabilizability which are geometrically more appealing and easier to check than conditions proposed in our previous papers. We then study the practical reachability problems in the light of our new practical stabilization results. It is shown that under appropriate conditions, practical reachability along some guiding paths can be obtained. Future research includes more detailed studies of the practical reachability of general classes of switched systems and characterization and computation of guidable regions.

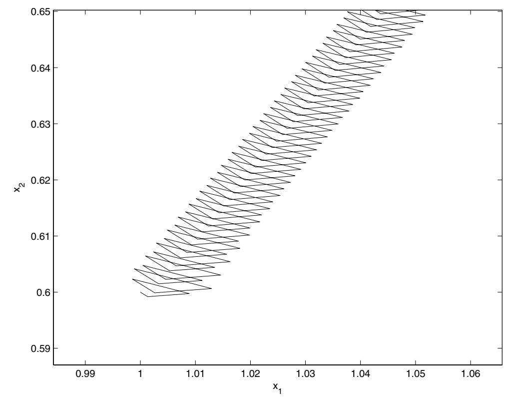

Fig. 4. A zoomed-in view of the state trajectory near $x_0 = [1,\ 0.6]^T$.

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