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State and Configuration Feedback for Almost Global Tracking of Simple Mechanical Systems on a General Class of Lie Groups

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***Abstract*— We present a general intrinsic tracking controller design for fully-actuated simple mechanical systems, when the configuration space is one of a general class of Lie groups. We show that if a suitable error function can be found, then a general smooth and bounded reference trajectory may be tracked asymptotically from almost every initial condition, with locally exponential convergence. Such functions may be shown to exist on any compact Lie group, or on any product of a compact Lie group and $\mathcal{R}^n$. In the case of compact Lie groups, we show that the full-state feedback law composed with an exponentially convergent velocity estimator, also converges globally for almost every initial tracking error. We explicitly compute these controllers on $SO(3)$, and simulate their performance for the axisymmetric top problem.**

I. Introduction

Formally, a holonomic simple mechanical system consists of (i) a smooth manifold, corresponding to the configuration space of the system, (ii) a smooth Lagrangian corresponding to kinetic energy minus potential energy and (iii) a set of external forces or one-forms. When some of these forces may be used for control, we refer to a simple mechanical control system [5], [8]. In many systems of practical and theoretical interest, the configuration space of the system may be given the structure of a Lie group. Examples include underwater vehicles, satellites, surface vessels, airships, hovercrafts, robots, and MEMS [2], [3], [5], [6], [9].

The stabilization of a desired equilibrium of a simple mechanical system by means of a suitable choice of a potential function is extensively treated by Koditschek [8]. Bullo and Murray use the Riemannian structure of the configuration manifold of a fully-actuated simple mechanical control system to derive a full-state, feedback-plus-feedforward controller that tracks a general reference trajectory [5]. Significant progress in geometric control has been made by specializing results from the general Riemannian framework to Lie groups. However this approach may fail to fully exploit the additional structure available in the latter case. In fact, the group structure may be used to transform the trajectory tracking problem into the better understood problem of stabilizing the identity element. This is not possible in a general Riemannian setting.

The chief difficulty in the general tracking problem lies in defining the tracking error, and tracking error dynamics. When the configuration space is a Lie group we have a well-defined natural notion of error dynamics. Given two elements of the group, g and h, define the *configuration error* to be gh^{-1}. The configuration error is also an element of the Lie group, and has no analog in the general Riemannian approach. The velocity error is naturally defined using left translation. The derivatives of the configuration error and the velocity error define the tracking error dynamics on the tangent space of the Lie group. For fully-actuated simple mechanical systems on Lie groups we show that there exists a feedback control that transforms the tracking error dynamics to a simple mechanical system, with damping and potential energy arbitrarily assignable using additional state feedback. On a general Riemannian manifold the absence of the group operation precludes the definition of an intrinsic configuration error and associated tracking error dynamics on the state space. In this sense, the tracking problem on a Lie group is actually much more closely related to tracking on R^n than it is to the general case, for which the group operation is lacking.

Solution of the tracking problem is thus reduced to the task of stabilizing the identity element of the transformed system. A suitable state-feedback control is given by Koditschek [8], who shows that convergence is completely governed by the properties of the assigned potential energy function. To give "almost global" stability—that is, stability for all initial conditions in an open and dense subset of the state space—the potential energy should be a globally defined, smooth, proper Morse function, with a unique minimum at the identity. In the context of the tracking problem, we follow the nomenclature of Bullo and Murray [5], and call a function with these properties an *error function*. Suitable functions are constructed for Lie groups of practical interest by Koditschek (who calls them *navigation functions*) [8], and by Dynnikov and Vaselov [7]. A result of Morse [11] shows that such functions exist on any compact connected manifold. By a straightforward extension of these results, such functions also exist on any Lie group of the form $G\times\mathcal{R}^n$, where G is any compact connected Lie group. The presence of anti-stable equilibrium points, and saddle points and their stable manifolds, prevents the results from being truly global. We show that almost global stabilization of the identity element of the transformed system yields almost global tracking for the original system. We cite an observation of Koditschek [8] to show that, unless the Lie group is homeomorphic to $\mathcal{R}^n$, almost global tracking is the best one can hope for. If the Lie group is not of the form given above, the existence of a globally defined error function is, to our knowledge, an open question.

Implementation of full state-feedback requires both configuration and velocity measurements. In application it is not unusual for only one of these to be available for measurement. In some cases it is the velocities, while in others it is the configuration [2], [13], [12]. This paper is concerned with the latter case, where dynamic estimation of the velocities is necessary. We have shown in [10] that a "separation principle" applies for the dynamic configuration feedback tracking control obtained by composing the full state-feedback compensator with an exponetially convergent velocity observer. In a recent paper Aghannan and Rouchon gives such an intrinsic observer that provides locally exponentially convergent estimates of the velocities of a simple mechanical system on a Riemannian manifold, given configuration variable measurements [1]. In this paper we use this observer to implement the full state-feedback controller without velocity measurements. In section II we briefly present notation and review mathematical background for simple mechanical control systems on Lie groups. Section III-A presents an intrinsic, globally valid, full state-feedback tracking control for any fully-actuated simple mechanical system on a very wide class of Lie groups. This tracking control guarantees almost-globally asymptotic stability with locally exponential convergence to an arbitrary twice differentiable configuration reference signal. No invariance properties are required of the kinetic energy, potential energy, or external forces. To the best of our knowledge its the first time such a general result has been reported. In Section III-B we present the dynamic output feedback controller. In Section III-D we explicitly derive the full-state and dynamic output feedback controllers on the Lie group $SO(3)$ and the effectiveness of the dynamic output feedback is demonstrated using simulation of the axisymmetric top problem.

II. Mathematical Background

This section briefly describes the notations and a few geometric notions that will be employed in the rest of the paper. Let G be a connected finite dimensional Lie group and let $\mathcal{G} \simeq T_eG$ be its Lie algebra. The left translation of $\zeta \in \mathcal{G}$ to T_gG will be denoted $g \cdot \zeta = DL_g\, \zeta$. The adjoint representation $DL_g \cdot DR_{g^{-1}}$ will be denoted Ad_g. The Lie bracket on $\mathcal{G}$ for any two $\zeta, \eta \in \mathcal{G}$ will be denoted $[\zeta, \eta] = \mathrm{ad}_\zeta\, \eta$ and the dual of the ad operator will be denoted ad^*. Any smooth vector field $X(g)$ on G has the form $g \cdot \zeta(g)$ for some smooth $\zeta(g) \in \mathcal{G}$. Let $\{e_i\}$ be any basis for the Lie algebra $\mathcal{G}$ and let $\{E_i(g) = g \cdot e_i\}$ be the associated left invariant basis vector field on G. Now $[e_i, e_j] = C_{ij}^k e_k$, where C_{ij}^k are the structure constants of the Lie algebra $\mathcal{G}$ ($C_{ij}^k = -C_{ji}^k$), and $[E_i, E_j] = C_{ij}^k E_k$.

A. The Riemannian Structure

For each $g \in G$, $I(g) : \mathcal{G} \mapsto \mathcal{G}^*$ is an isomorphism such that the relation $\langle\langle \zeta\, ,\, \eta \rangle\rangle_{\mathcal{G}} = \langle I(g)\zeta\, ,\, \eta\rangle$ for $\zeta, \eta \in \mathcal{G}$ defines an inner product on $\mathcal{G}$. Here $\langle\cdot,\cdot\rangle$ denotes the usual pairing between a vector and a co-vector. Identifying $\mathcal{G}^*$ and $\mathcal{G}$ with $\mathcal{R}^n$, let $I_{ij}(g)$ and $I^{ij}(g)$ be the matrix representations of $I(g)$ and $I^{-1}(g)$ respectively. $I(g)$ is symmetric and positive definite. If $I(g)$ is globally smooth then such an $I(g)$ induces a unique metric on G by the relation $\langle\langle g \cdot \zeta\, ,\, g \cdot \eta\rangle\rangle = \langle I(g)\zeta\, ,\, \eta\rangle$. Further, it follows that every metric has such an associated family of isomorphisms. If the metric is left-invariant then I is a constant and any constant symmetric positive definite matrix induces a left-invariant metric on G.

In the remainder of the section we present expressions for the Levi-Civita connection and the Riemannian curvature corresponding to left-invariant metrics. We intensionally avoid the use of coordinate-frame fields to facilitate the coordinate-free expressions developed in section III-C. Associated with any metric there exists a unique connection that is torsion free and metric called the Levi-Civita connection. For a vector field $X = X^k E_k$ and a vector $v = v^k E_k$ the Levi-Civita connection is given by

$$\nabla_v X = (dX^k(v) + \omega_{ij}^k(g) v^i X^j) E_k\,, \tag{1}$$

where $\omega_{ij}^k(g)$ are the connection coefficients in the frame $\{E_k\}$. For left-invariant metrics the connection coefficients turn out to be constants, given by

$$\omega_{ij}^k = \frac{1}{2}\left(C_{ij}^k - I^{ks}(I_{ir}C_{js}^r + I_{jr}C_{is}^r)\right). \tag{2}$$

Since in general the E_k are not coordinate vector fields, ω_{ij}^k, are not the Christoffel symbols. The corresponding coefficients of the Riemannian curvature two-forms R_{jab}^k are also constant and can be shown to be,

$$R_{jab}^k = (-\omega_{rj}^k C_{ab}^r + 2\omega_{ar}^k \omega_{bj}^r). \tag{3}$$

The Riemannian curvature is then

$$R(\zeta,\eta)\xi = \{R_{jab}^k \xi^j (\zeta^a\eta^b - \zeta^b\eta^a) - \omega_{ij}^k C_{ab}^i \zeta^a \eta^b \xi^j\} e_k \tag{4}$$

B. Simple Mechanical Control Systems on Lie Groups

A simple mechanical control system evolving on a Lie group G equipped with a metric $<<\cdot\,,\cdot>>$ is defined as a system with kinetic energy $E(\dot{g}) = \frac{1}{2} << \dot{g}\,, \dot{g} >>$, conservative plus dissipative forces $f(g,\zeta) \in \mathcal{G}^*$ and a set of linearly independent forces $u_i f^i(g) \in \mathcal{G}^*$ for $i = 1, \cdots, m$, [5]. The scalar functions $u_i \in \mathcal{R}$ are the controls. If $m = n = dim(G)$ the system is said to be fully actuated.

Let $I(g) : \mathcal{G} \mapsto \mathcal{G}^*$ be the isomorphism associated with the kinetic energy metric; $<< g\cdot\zeta\, , g\cdot\eta >>= I(g)\zeta \cdot \eta$ for $\zeta, \eta \in \mathcal{G}$. Then the Euler-Lagrange equations of motion of the system are

$$\dot{g} \;=\; g \cdot \zeta\,, \tag{5}$$

$$\dot{\zeta} \;=\; \tilde{f}(g,\zeta) + I^{-1}(g)\left(f(g,\zeta) + \sum_i^m u_i f^i(g)\right), \tag{6}$$

where $\tilde{f}(g,\zeta) = -\omega^k_{ij}(g)\zeta^i\zeta^j e^k$. If the kinetic energy metric is left invariant then I is a constant and $\tilde{f}(g,\zeta) = I^{-1}ad^*_\zeta I\zeta$.

III. Intrinsic Tracking for Simple Mechanical Systems

Let $(g_r(t),\zeta_r(t)) \in G \times \mathcal{G}$ be a desired bounded and smooth reference trajectory to be tracked by (5)–(6). We introduce the *configuration error*, defined as

$$e(t) = g_r(t)g^{-1}(t). \tag{7}$$

This object is intrinsic and globally defined. Most importantly, it is itself an element of the configuration space. Differentiating (7) and setting $\eta_e = Ad_g(\zeta_r - \zeta)$ the error dynamics are computed to be

$$\dot{e} = e \cdot \eta_e, \tag{8}$$

$$\dot{\eta}_e = Ad_g\left(\dot{\zeta}_r - \dot{\zeta} + [\zeta,\zeta_r]\right), \tag{9}$$

where $\dot{\zeta}$ is given by (6). Observe that these dynamics are defined on $TG \simeq G \times \mathcal{G}$ as well. As we now show, through a suitable choice of controls, the dynamics of the configuration error may be given the form of a fully-actuated simple mechanical system with arbitrarily assignable potential energy and damping.

A. Full State Feedback Tracking

Let $B = I^{-1}(g)[f^1(g)\;\; f^2(g)\;\; \cdots\;\; f^n(g)]$. Substituting

$$u = B^{-1}\Big(\dot{\zeta}_r - \tilde{f}(g,\zeta) - I^{-1}f(g,\zeta) + [\zeta,\zeta_r] - Ad_{g^{-1}}\nu\Big) \tag{10}$$

in equation (9), we have the transformed error dynamics

$$\dot{e} = e \cdot \eta_e, \tag{11}$$

$$\dot{\eta}_e = \nu, \tag{12}$$

where $\nu \in \mathcal{G}$. These transformations reduce the problem of stably tracking the reference input to the problem of stabilizing $(id,0)$ of (11) – (12). The error dynamics (11)–(12) are those of a fully-actuated simple mechanical control system on G. This is a key observation, since it has been shown that the stability of simple mechanical systems is completely determined by the nature of the potential energy. In particular, it is shown by Koditschek [8] that any given point of a compact configuration space with or without boundary may be made an almost globally stable equilibrium by the appropriate choice of an potential energy function. Thus the task is now to assign a suitable potential energy, $F : G \mapsto \mathcal{R}$, such that the equilibrium $(id,0)$ is almost globally stable. The assigned potential energy function plays a role analogous to that of the norm in the case of tracking on R^n. The value of the potential energy for the configuration error is therefore a measure of the size of that error, and we refer to it as the *error function*. Observe that the convergence properties are completely determined by the properties of the error function, and are independent of the specifics of the simple mechanical system. Thus for any given Lie group, solution of the tracking problem is essentially reduced to the topological question of finding an appropriate error function for that space.

Let $\tilde{I}$ be any symmetric positive definite matrix. This induces an inner product $<< \cdot\,,\, \cdot >>_{\mathcal{G}}$ on $\mathcal{G}$ and a left invariant metric on G. This metric need not be the kinetic energy metric of the mechanical system under consideration. Let $\zeta_e = e^{-1} \cdot grad\, F$, where $< dF, e \cdot \eta > = << e^{-1} grad\, F, \eta >>_{\mathcal{G}}$ and

$$\nu = -\zeta_e - k\,\eta_e, \tag{13}$$

where k is a positive constant, yields the following error dynamics:

$$\dot{e} \;=\; e \cdot \eta_e, \tag{14}$$

$$\dot{\eta}_e \;=\; -\zeta_e - k\,\eta_e. \tag{15}$$

In summary, our approach implements a two-part composite control, in which the first component (10) is used to give the configuration error dynamics the structure of a simple mechanical control system, and the second component (13) is used to assign potential energy F and damping to the transformed system. The combined control is

$$u = B^{-1}\Big(Ad_{g^{-1}}\zeta_e + k(\zeta_r - \zeta) + \dot{\zeta}_r - \tilde{f}(g,\zeta) - I^{-1}f(g,\zeta) + [\zeta,\zeta_r]\Big), \tag{16}$$

We now state the properties required by the assigned potential energy function F for almost-global tracking. A function with only non-degenerate critical points is called *Morse*.

Definition 1: A smooth, proper Morse function $F : G \mapsto \mathcal{R}$, bounded below by zero, and with a unique minimum at the identity is called an *error function*.

A result of Morse [11] shows that such functions exist on any compact connected manifold. By a straightforward extension of these results, such functions also exist on any Lie group of the form $G \times \mathcal{R}^n$, where G is any compact connected Lie group. In [10] we have proved the following theorem:

Theorem 1: If $F : G \mapsto \mathcal{R}$ is an *error function* then the fully-actuated simple mechanical control system (5)–(6) with the control (16) almost globally tracks any smooth trajectory $(g_r(t),\zeta_r(t))$ with local exponential convergence.

Koditschek [8] points out that, unless the configuration manifold is homeomorphic to $\mathcal{R}^n$, global stability is impossible, so almost global stability is the best possible outcome in general. A *perfect* Morse function has exactly as many critical points as the homology of the underlying manifold requires. To minimize the number of unstable equilibrium points, whenever possible we use a perfect Morse function as the error function. Examples of perfect Morse functions on certain symmetric spaces, including $SO(n)$, $U(n)$, and $Sp(n)$, may be found in the literature [7]. Koditschek [8] gives an example of an error function that is a perfect Morse function on $SO(3)$, which we use in Section III-D.

B. Dynamic Output Feedback Tracking

The tracking control (16) involves both the configuration variable g and the velocity variable ζ. In this section we assume that only the configuration variables are available for measurement, and estimate the velocity. If ζ_o is the estimated value of ζ, the dynamic configuration-feedback tracking control obtained by composing a velocity observer with the state-feedback control (16) is

$$\begin{aligned} u &= B^{-1}(g)\Big(\mathrm{Ad}_{g^{-1}}\zeta_e + k(\zeta_r - \zeta_o) + \dot{\zeta}_r \\ &\quad -\tilde{f}(g,\zeta_o) - I^{-1}f(g,\zeta_o) + [\zeta_o,\zeta_r]\Big), \end{aligned} \tag{17}$$

where g is measured. The following theorem provides a separation principle for this dynamic configuration-feedback control and was proved in [10]

Theorem 2: Consider the fully actuated simple mechanical system (5)–(6) on a compact and connected Lie group G where the external forces are of the form $f(g,\zeta) = f^c(g)+f^d(g,\zeta)$ with $f^d(g,\zeta)$ linear in ζ. Then the dynamic configuration-feedback control (17) composed with any locally exponentially convergent velocity observer almost-globally tracks an arbitrary bounded once-differentiable reference trajectory $(g_r(t), \zeta_r(t))$ for sufficiently small initial observer errors.

One such velocity observer for simple mechanical systems is presented in [1], for the case $f(g,\zeta) = f^c(g)$, that is, in the absence of damping. It is shown in [9] that velocity dependent external forces do not in fact affect convergence. Because this locally exponentially convergent observer is also intrinsic [1], its use in conjunction with (17) yields a globally well defined configuration-feedback tracking control. As guaranteed by theorem 2 convergence is almost-globally asymptotic for sufficiently small initial observer errors.

C. Coordinate-Free Representation

For many cases of interest the formulation of this dynamic configuration-feedback control can be considerably simplified and expressed explicitly in a coordinate-free manner.

$$\dot{g_o} = g_o \cdot (\zeta_o - 2\alpha\zeta_{oe}), \tag{18}$$

$$\begin{aligned} \dot{\zeta}_o &= I^{-1}\left(\mathrm{ad}^*_{\zeta_o} I\zeta_o - \alpha(\mathrm{ad}^*_{\zeta_{oe}} I\zeta_o + \mathrm{ad}^*_{\zeta_o} I\zeta_o)\right) \\ &\quad + \alpha[\,\zeta_{oe}\,,\,\zeta_o] + \Gamma(S) - R(\zeta_o,\zeta_{oe})\zeta_o - \beta\zeta_{oe}, \end{aligned} \tag{19}$$

where α, β are positive constants, and the configuration error $\zeta_{oe} \in \mathcal{G}$ is defined by $\exp(\zeta_{oe}) = g^{-1}g_o$ for g_o and g sufficiently close. Here $S(g,\zeta_o,u) = f(g,\zeta_o) + \sum_i^m u_i(g,\zeta_o)f^i(g)$ and,

$$\Gamma(S) = (S^k - \omega^k_{ij}S^i\zeta^j_{eo})\, e^k. \tag{20}$$

The advantage of this formulation is that all the terms of the observer with the exception of the external forces S are independent of g. This leads to a compact and flexible representation that requires only changes to S and I to be adapted to different simple mechanical systems. Left-invariance of kinetic energy also allows the control (17) to be written as

$$\begin{aligned} u &= B^{-1}(g)\Big(\mathrm{Ad}_{g^{-1}}\zeta_e + k(\zeta_r - \zeta_o) + \dot{\zeta}_r \\ &\quad - I^{-1}(\mathrm{ad}^*_{\zeta_o} I\zeta_o + f(g,\zeta_o)) + [\zeta_o,\zeta_r]\Big), \end{aligned} \tag{21}$$

where now the inertial forces $\tilde{f}$ may be written in terms of ζ only. This is itself a significant simplification, and if in addition all external forces are also left-invariant then only the error feedback term $\mathrm{Ad}_{g^{-1}}\zeta_e$ is dependent on g. This last assumption is fairly common, see for example [5], [6].

D. The Rotation Group $SO(3)$

In this section we explicitly compute the state- and dynamic-feedback tracking controller for the example of a simple mechanical systems on the Lie group $SO(3)$, with left-invariant kinetic energy metric. These expressions can be readily adapted to a particular application by specifying the inertia tensor I, and the external forces $f(R,\zeta)$. We make this specialization for the axi-symmetric top, and simulate the resulting performance.

The three-dimensional rotation group, $SO(3)$, is the Lie group of matrices $R \in GL(3,\mathcal{R})$ that satisfy $R\,R^T = R^T R = id$ and $det(R) = 1$. The Lie algebra $so(3)$ of $SO(3)$ is the set of traceless skew symmetric three-by-three matrices. Note that $so(3) \simeq \mathcal{R}^3$ where the isomorphism is defined by the usual skew symmetrization. The adjoint representation $Ad_R : so(3) \mapsto so(3)$ is explicitly given by $Ad_R(\xi) = R\xi$, or $Ad_R(\hat{\xi}) = R\hat{\xi}R^T$, respectively. Define the isomorphism $I : so(3) \simeq \mathcal{R}^3 \mapsto so(3)^* \simeq \mathcal{R}^3$ by the positive definite matrix I. This induces a left invariant metric on $SO(3)$ by the relation, $<< R\cdot\xi\,,\,R\cdot\psi >> = << \xi\,,\,\psi >>_{so(3)} = I\xi\cdot\psi$, for any two elements $R\cdot\xi, R\cdot\psi \in T_R SO(3)$.

From (5) – (6), a simple mechanical control system on $SO(3)$ with left-invariant kinetic energy takes the form,

$$\dot{R} = R\,\hat{\zeta}, \tag{22}$$

$$\dot{\zeta} = I^{-1}\left(I\zeta \times \zeta + f(R,\zeta) + \sum_i^m u_i f^i(R,\zeta)\right) \tag{23}$$

Let $(R_r(t), \zeta_r(t))$ where $\dot{R}_r(t) = R(t)\hat{\zeta}_r(t)$ is a reference trajectory to be tracked by (22)–(23). The intrinsic tracking error $e(t) \in G$ is given by $e(t) = R_r(t)R^T(t)$. Let $F(e)$ an error function and let $\zeta_e = e^T grad\,F(e)$ with respect to the left-invariant kinetic energy metric induced by some three-by-three positive definite matrix $\tilde{I}$. The exact choice of $\tilde{I}$ is up to the designer.

Consider the function $F(e) = \frac{1}{2}trace\{K(\tilde{I} - e)\}$, where K is a symmetric, positive definite three-by-three matrix. It is shown in [8] that F is a Morse function with four critical points and a unique minima at the identity. It can also be shown that $\zeta_e = \tilde{I}^{-1}\Omega_e$, where $\hat{\Omega}_e = (Ke - e^T K^T)$. This implies that the tracking control (16)

$$
\begin{aligned}
u &= B^{-1}(R)\Big(R^T\zeta_e + k(\zeta_r - \zeta) + \dot{\zeta}_r - I^{-1}(I\zeta \times \zeta \\
&\quad + f(R)) + \zeta \times \zeta_r\Big), \qquad (24)
\end{aligned}
$$

achieves almost global tracking with local exponential convergence. It is pointed out in [7], [8] that any Morse function on $SO(3)$ has at least four critical points. Thus this F is a perfect Morse function on $SO(3)$, and has the fewest possible unstable equilibria.

The intrinsic observer (18) – (19) takes the form,

$$
\begin{aligned}
\dot{R}_o &= R_o \cdot (\zeta_o - 2\alpha\zeta_{eo}), \qquad (25)\\
\dot{\zeta}_o &= I^{-1}\left(I\zeta_o \times \zeta_o - \alpha(I\zeta_o \times \zeta_{eo} + I\zeta_{eo} \times \zeta_o)\right)\\
&\quad + \alpha\,\zeta_{eo} \times \zeta_o + \Gamma(S, \zeta_{eo}) + R_c(\zeta_o, \zeta_{eo})\zeta_o - \beta\zeta_{eo}, \qquad (26)
\end{aligned}
$$

where ζ_{eo} satisfies $exp(\zeta_{eo}) = R^T R_o$ and is given by, $\zeta_{oe} = \frac{\psi_o}{2\sin\psi_o}(R^T R_o - R_o^T R)$, where, $\cos\psi_o = (tr(R^T R_o) - 1)/2$, for $|\psi_o| < \pi$. The parallel transport term $\Gamma(S)$ is calculated from (20) where $S(R, \zeta_o) = f(R, \zeta_o) + \sum_i^m u_i f^i(R)$ and the curvature term $R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta}$ is calculated from (4). With this observer the tracking feedback (24) can be implemented with ζ replaced with ζ_o and achieves almost global tracking with only the measurement of the configuration R.

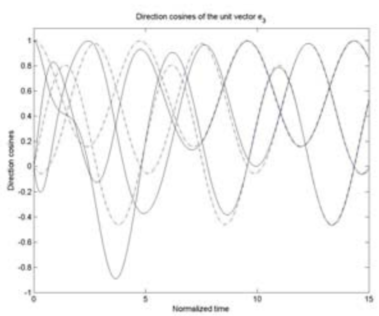

Fig. 1. With dynamic output feedback the direction cosines of the unit vector e_3. The axi-symmetric top values are the solid lines while the dotted lines are the reference values.

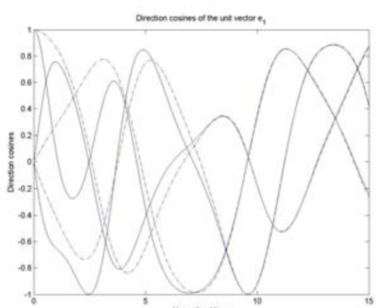

Fig. 2. With dynamic output feedback the direction cosines of the unit vector e_1. The axi-symmetric top values are the solid lines while the dotted lines are the reference values.

1) Simulation Results: In this section we apply the dynamic output feedback law (17) to a simulation of a simple mechanical control system in $SO(3)$. We consider the classical problem of a axisymmetric top in a gravitational field. Let $P = \{P_1, P_2, P_3\}$ be an inertial frame fixed at the fixed point of the top and let $e = \{e_1, e_2, e_3\}$ be a body-fixed orthonormal frame with the origin coinciding with that of P. At $t = 0$, the two frames coincide. Let the coordinates of a point p in the inertial frame P be given by x, and in the body frame e be given by X. The coordinates are related by $x(t) = R(t)X$ where $R(t) \in SO(3)$. Let $-P_3$ be the direction of gravity and let I be the inertia matrix of the axisymmetric top about the fixed point. The kinetic energy of the top is $K = I\zeta \cdot \zeta/2$, where ζ is the body angular velocity and the potential energy is $U(R) = mgl\,Re_3 \cdot P_3$. Here m is the mass of the top, g is the gravitational constant, l is the distance along the e_3 axis to the center of mass. For simplicity we assume the top to be symmetric about the e_3 axis, so $I = diag(I_1, I_1, I_3)$. The generalized potential forces $f(R)$ in the body frame are $< f(R), \zeta >= - < dU, R \cdot \zeta >= -mgl\,R\hat{\zeta}e_3 \cdot P_3$ for any $\zeta \in so(3)$, which yields $f(R) = mgl\ R^T P_3 \times e_3$. For convenience, let the desired reference trajectory $(R_r(t), \zeta_r(t))$ be generated by a simple mechanical system without external forces. It is not necessary for our results that the trajectory correspond to such a system.

In these simulations top parameters are $I_1 = I_2 = 1, I_3 = 2, mgl = 1$. The initial body angular velocity is $\zeta(0) = [1.3\ \ 1.2\ \ 1.1]$, and the initial configuration corresponds to a $\pi/2$ radian rotation about the $[1\ 1\ 1]^T$ axis. The reference trajectory $(R_r(t), \zeta_r(t))$ is generated by a model simple mechanical system without external forces. for the initial conditions $\zeta_r(0) = [-.8\ \ -.3\ \ -.5]$ and $R_r(0) = id$. The simulation results shown in Figure-1 and Figure-2 correspond to the dynamic output feedback with $\alpha = \beta = 10$. The initial observer velocity is zero and the initial observer configuration corresponding to a $0.9\pi/2$ radian rotation about the $[1\ 1\ 1]^T$ axis.

IV. Conclusions

We have presented an intrinsic methodology for designing tracking controllers for fully-actuated, simple mechanical control systems on a general class of Lie groups. We derived a state feedback controller giving almost global tracking, with local exponential convergence. Except on R^n, and spaces homeomorphic to R^n, global tracking is impossible, so this is the best stability result that can be obtained in general. When combined with a previously described locally convergent velocity estimator, the resulting configuration feedback system is also almost globally convergent with respect to tracking. State- and configuration-feedback controllers were explicitly computed for the Lie group $SO(3)$, and applied to the simulated control of an axisymmetric heavy top.

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